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Hochschild duality, localization, and smash products

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Abstract

In this work we study the class of algebras satisfying a duality property with respect to Hochschild homology and cohomology, as in [Proc. Amer. Math. Soc. 126 (1998) 1345–1348]. More precisely, we consider the class of algebras A such that there exists an invertible bimodule U and an integer number d with the property $H^{\bullet}(A, M) \cong H_{d-\bullet}(A, U \otimes_A M)$, for all A-bimodules M. We show that this class is closed under localization and under smash products with respect to Hopf algebras satisfying also the duality property.

We also illustrate the subtlety on dualities with smash products developing in detail the example $S(V) \# G$, the crossed product of the symmetric algebra on a vector space and a finite group acting linearly on V.

Keywords: Hochschild homology and cohomology; Duality; Localization; Smash products

Introduction

The aim of this work is to study the class of algebras satisfying a duality property with respect to Hochschild homology and cohomology, as in [4]. More precisely, we consider the class of algebras A such that there exists an invertible bimodule U and an integer

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number d with the property $H^\bullet(A, M) \cong H_{d-\bullet}(A, U \otimes_A M)$, for all A-bimodules M. We show that this class is closed under localization (Theorem 6) and under smash products (Theorem 17). By localization we mean an algebra morphism $A \to B$ with the following two properties: $B \otimes_A B \cong B$ as B-bimodule, and $B \otimes_A - \otimes_A B$ is exact. For smash product, the philosophy is the following: take A an algebra in this class with dualizing bimodule U, and H a Hopf algebra with dualizing bimodule H, then $A \# H$ has dualizing bimodule $U \# H$ (see Remark 16 for the definition of $U \# H$).

There is a subtlety on dualities with smash products, so the last section is devoted to develop the simplest example illustrating this: the algebra $S(V) \# G$, the crossed product of the symmetric algebra on a vector space, and a finite group acting linearly on V. Given an algebra A with dualizing module $U_A \cong A$ and a Hopf algebra with dualizing bimodule isomorphic to H, Theorem 17 says that $A \# H$ has a dualizing bimodule isomorphic to $U_A \# H$. The subtlety is that, even though the bimodule $U_A \cong A$ as A-bimodule, it may happen that $U_A \ncong A$ as H-module, and so $U_A \# H \ncong A \# H$ as $A \# H$-bimodule. In the example of $S(V)$ and $G \subset \mathrm{GL}(V)$, we show that the condition for $U_{S(V)} \cong S(V)$ as G-modules is that $G \subset \mathrm{SL}(V)$, and consequently, homology and cohomology will differ. In order to illustrate the duality, we compute the cohomology of this example in two different ways.

The example of Section 3 was motivated by a question of Paul Smith, whether the methods used in [1] would apply to $S(V) \# G$. The answer to that question is yes, and this calculation has also motivated Section 2.

General notations

Fix a field k of characteristic zero, unadorned $\otimes$ and Hom will denote $\otimes_k$ and Hom_k. If X is a graded vector space and $n \in \mathbb{Z}$, we will denote $X[n]$ the same vector space but with its degree shifted by n. For example, if X is nonzero only in degree zero, then $X[n]$ is nonzero only in degree n.

For any k-algebra B and k-symmetric bimodule M, the Hochschild homology and cohomology of B with coefficients in M are $\mathrm{Tor}_\bullet^{B^e}(B, M)$ and $\mathrm{Ext}_{B^e}^\bullet(B, M)$, respectively, where $B^e = B \otimes B^{\mathrm{op}}$; they are denoted $H_\bullet(B, M)$ and $H^\bullet(B, M)$. In the special case where $M = B$, we will also write $HH_\bullet(B) := H_\bullet(B, B)$ and $HH^\bullet(B) := H^\bullet(B, B)$.

The word "module" will mean "left module." All modules will be k-symmetric, so that B-bimodules is the same as B^e-modules. A B-bimodule P is called *invertible* if there exists another bimodule Q such that $P \otimes_B Q \cong B$ and $Q \otimes_B P \cong B$. The set of isomorphism classes of invertible B-bimodules which are k-symmetric is denoted by $\mathrm{Pic}_k(B)$.

Finally, in Section 3 there is some abuse of notation with the symbol det. Sometimes it denotes the usual determinant function, and some other times it denotes the 1-dimensional representation of $\mathrm{GL}(V)$, or its restriction to some $G \subset \mathrm{GL}(V)$. The meaning will be clear from the context.

The duality theorem of Van den Berg

In [4], the author proves a theorem relating the Hochschild homology and cohomology of a certain class of algebras. We will state this theorem in a way convenient for our purposes:

Theorem 1 [4, Theorem 3]. *Let A be a k-algebra which admits a finitely generated projective A^e-resolution (for instance, this is the case if A^e is noetherian). The following conditions are equivalent*:

(1) *There exists an invertible A-bimodule U_A, and an integer d such that $H^\bullet(A, M) \cong H_{d-\bullet}(A, U_A \otimes_A M)$ for all A^e-modules M.*
(2) *The projective dimension of A as A^e-module is finite, and $\mathrm{Ext}^n_{A^e}(A, A^e) = 0$ for all $n \geqslant 0$ except for $n = d$ where $U_A := \mathrm{Ext}^n_{A^e}(A, A^e)$ is an invertible A^e-module.*

1. Localization

The general framework of this section is the following: $A \to B$ is a k-algebra map such that

- The multiplication map induces an isomorphism of B^e-modules $B \otimes_A B \cong B$.
- The functors $B \otimes_A -$ and $- \otimes_A B$ are exact.

We look for conditions on B which, together with the assumption that A satisfies Van den Bergh's theorem, allow us to conclude that so does B.

Lemma 2. *Let $U \in \mathrm{Pic}(A)$ and $A \to B$ be such that $B \otimes_A B \cong B$. If $U \otimes_A B \cong B \otimes_A U$ as A^e-modules, then*

- *$B \otimes_A U \cong B \otimes_A U \otimes_A B$ as $B \otimes A^{\mathrm{op}}$-modules*;
- *$U \otimes_A B \cong B \otimes_A U \otimes_A B$ as $A \otimes B^{\mathrm{op}}$-modules*; *and*
- *$B \otimes_A U$ is a B^e-module in a natural way, $B \otimes_A U \in \mathrm{Pic}(B)$, its inverse is $B \otimes_A U^{-1} \otimes_A B$, and $U^{-1} \otimes_A B \cong B \otimes_A U^{-1}$ as A^e-modules.*

Proof. The first isomorphism is the composition

$$B \otimes_A (U \otimes_A B) \cong B \otimes_A (B \otimes_A U) = (B \otimes_A B) \otimes_A U \cong B \otimes_A U.$$

The second one is similar.

Now let U^{-1} be the inverse of U in $\mathrm{Pic}(A)$, so that $U \otimes_A U^{-1} \cong U^{-1} \otimes_A U \cong A$. Let us see that $B \otimes_A U^{-1} \otimes_A B$ is the inverse of $B \otimes_A U$:

$$\begin{aligned}
&(B \otimes_A U) \otimes_B \big(B \otimes_A U^{-1} \otimes_A B\big) \\
&\quad \cong (U \otimes_A B) \otimes_B B \otimes_A U^{-1} \otimes_A B \cong U \otimes_A B \otimes_A U^{-1} \otimes_A B \\
&\quad \cong B \otimes_A U \otimes_A U^{-1} \otimes_A B \cong B \otimes_A A \otimes_A B \cong B \otimes_A B \cong B,
\end{aligned}$$

and

$$\begin{aligned}&\big(B\otimes_A U^{-1}\otimes_A B\big)\otimes_B (B\otimes_A U)\\&\quad\cong B\otimes_A U^{-1}\otimes_A B\otimes_A U\cong B\otimes_A U^{-1}\otimes_A U\otimes_A B\\&\quad\cong B\otimes_A A\otimes_A B\cong B\otimes_A B\cong B. \qquad \square\end{aligned}$$

A bimodule U such that there is an isomorphism $B\otimes_A U\cong U\otimes_A B$ of A^e-modules will be said to *commute with* B.

Example 3. Let $g\in \mathrm{Aut}_k(A)$ be such that it admits an extension $\tilde g\in \mathrm{Aut}_k(B)$, i.e., $\tilde g(a)=g(a)$ for all $a\in A$. Then the element $Ag\in \mathrm{Pic}(A)$ commutes with B. In particular, $U=A$ commutes with B.

Proof. Let g be such an element and consider $Ag\in \mathrm{Pic}(A)$. There is an isomorphism of $B\otimes A^{\mathrm{op}}$-modules,

$$B\otimes_A Ag\to B\tilde g, \quad b\otimes ag\mapsto ba\tilde g.$$

On the other hand, one can define an isomorphism of $A\otimes B^{\mathrm{op}}$-modules

$$Ag\otimes_A B\to B\tilde g, \quad ag\otimes b\tilde g\mapsto a\tilde g(b)\tilde g.$$

In particular, $Ag\otimes_A B$ and $B\otimes_A Ag$ are isomorphic as A^e-modules. $\square$

Example 4. Let $g\in \mathrm{Aut}_k(A)$ be such that there exists no element $\tilde g\in \mathrm{Aut}_k(B)$ extending it. Then the bimodule Ag does not commutes with B.

Proof. Assume $B\otimes_A Ag\cong Ag\otimes_A B$ as A^e-modules. From Lemma 2 it follows that $B\otimes_A Ag\in \mathrm{Pic}(B)$. But, as a left B-module, $B\otimes_A Ag\cong B$, and it is well known that if an element $U\in \mathrm{Pic}(B)$ is such that ${}_BU\cong {}_BB$, then it is of the form $B\alpha$ for some $\alpha\in \mathrm{Aut}_k(B)$, the automorphism α being defined up to inner automorphism. In particular, for $a\in A$ one has that $g(a)=u\alpha(a)u^{-1}$ for some $u\in \mathcal U(B)$. Denoting $\tilde g:=u\alpha(-)u^{-1}$, we see that we have found an automorphism extending g, thus a contradiction. $\square$

Remark 5. Let $A\to B$ be such that $B\otimes_A B\cong B$. If M is a left B-module, then $M\cong B\otimes_A M$ as a left B-module. If N is another left B-module, then $\mathrm{Hom}_B(M,N)=\mathrm{Hom}_A(M,N)$.

Proof. Using the hypothesis on B, we see that

$$M\cong B\otimes_B M\cong (B\otimes_A B)\otimes_B M\cong B\otimes_A (B\otimes_B M)\cong B\otimes_A M;$$

it follows then that

$$\mathrm{Hom}_B(M,N)\cong \mathrm{Hom}_B(B\otimes_A M,N)\cong \mathrm{Hom}_A(M,N). \qquad \square$$

Theorem 6. *Let $A \in VdB(d)$ with dualizing bimodule U, and $A \to B$ be a morphism of k-algebras such that*

(1) *the functors $B \otimes_A -$ and $- \otimes_A B$ are exact*;
(2) *the canonical map induced by multiplication $B \otimes_A B \to B$ is an isomorphism*; *and*
(3) $B \otimes_A U \cong U \otimes_A B$ *as A^e-modules.*

Then $B \in VdB(d)$ with dualizing bimodule $B \otimes_A U \cong B \otimes_A U \otimes_A B$.

Notice that if $U = A$, then condition (3) is automatically satisfied, and the dualizing bimodule associated to B is B.

Proof. By Theorem 1, it is enough to show that the projective dimension of B as B^e-module is finite, that B admits a resolution by means of finitely generated B^e-projectives, and that $\mathrm{Ext}^d_{B^e}(B, B^e) = B \otimes_A U \otimes_A B$ and it vanishes elsewhere.

Let $P_\bullet$ be a finite resolution of A as A^e-modules, with P_n projective and finitely generated as A^e-modules. Since $B \otimes_A -$ and $- \otimes_A B$ are exact, the complex $B \otimes_A P_\bullet \otimes_A B$ is a resolution of $B \otimes_A A \otimes_A B \cong B$, and so B also has a finite resolution. The bimodules $B \otimes_A P_n \otimes_A B$ are clearly B^e-finitely generated and projective.

In order to compute $\mathrm{Ext}^\bullet_{B^e}(B, B^e)$, one can use this particular resolution, and consequently

$$\mathrm{Ext}^\bullet_{B^e}\big(B, B^e\big) = H^\bullet\big(\mathrm{Hom}_{B^e}\big(B \otimes_A P_\bullet \otimes_A B, B^e\big)\big) \cong H^\bullet\big(\mathrm{Hom}_{A^e}\big(P_\bullet, B^e\big)\big).$$

We claim that if P is A^e-projective finitely generated, then

$$\mathrm{Hom}_{A^e}\big(P_\bullet, B^e\big) \cong B \otimes_A \mathrm{Hom}_{A^e}\big(P_\bullet, A^e\big) \otimes_A B.$$

For that, consider the class of A^e-modules P such that $\mathrm{Hom}_{A^e}(P, B^e) \cong B \otimes_A \mathrm{Hom}_{A^e}(P_\bullet, A^e) \otimes_A B$. This class is closed under direct summands and finite sums, so it is enough to show our claim that the module A^e is in it, and that is clear. Using this isomorphism, one gets

$$H^\bullet\big(\mathrm{Hom}_{A^e}\big(P_\bullet, B^e\big)\big) \cong H^\bullet\big(B \otimes_A \mathrm{Hom}_{A^e}\big(P_\bullet, B^e\big) \otimes_A B\big)$$

and by flatness this is the same as

$$B \otimes_A H^\bullet(\mathrm{Hom}_{A^e}(P_\bullet, B^e)) \otimes_A B = B \otimes_A U[d] \otimes_A B. \qquad \square$$

Example 7. One can take $A = A_1(k) = k\{x, y\}/\langle [x, y] = 1\rangle$ and $B = k\{x, x^{-1}, y\}/\langle [x, y] = 1\rangle$. This example is a particular case of the following:

Example 8 (*Normal localization*). Let A be an algebra and $x \in A$ such that the set $\{1, x, x^2, x^3, \ldots\}$ satisfies the Ore conditions. Take $B = A[x^{-1}]$. If M is a right A-module, then as $k[x]$ modules we have an isomorphism $M \otimes_A B \cong M \otimes_{k[x]} k[x, x^{-1}]$.

This shows that $A \to B$ is flat. It is also clear that $B \otimes_A B \cong B$, in the same way as $k[x^{\pm 1}] \otimes_{k[x]} k[x^{\pm 1}] \cong k[x^{\pm 1}]$.

Example 9. Another generalization of Example 7 is the following situation: let $\mathcal{O}(X)$ be the algebra of functions on an affine variety X, and let U be an affine open subset of X. Let $A = \mathrm{Diff}(X)$ be the algebra of algebraic differential operators on X and similarly $B = \mathrm{Diff}(U)$. Since $B = \mathcal{O}(U) \otimes_{\mathcal{O}(X)} \mathrm{Diff}(X)$, the map $A \to B$ is flat, and $B \otimes_A B = B$. If A satisfies the theorem of Van den Bergh, then so it does B.

In the next section, we will study the behavior of the duality property with respect to smash products.

2. Smash products

In this section H is a Hopf algebra such that $H \in VdB(d)$ with dualizing bimodule $U_H = H$, $A \in VdB(d')$ is an H-module algebra with dualizing bimodule U_A, and $B := A \# H$. We will prove (see Theorem 17) that $B \in VdB(d + d')$, with dualizing bimodule $U_B = U_A \# H$ (see Remark 16 for the definition of $U \# H$).

Lemma 10. *If H is a Hopf algebra, then $H \in VdB(d)$ with dualizing bimodule H if and only if $\mathrm{Ext}^\bullet_H(k, M) \cong \mathrm{Tor}_{\bullet - d}(k, M)$ for all left H-modules M.*

Proof. Let M be a left H-module, then M_ε is the H^e-module with right action defined by $m.h := \varepsilon(h)m$ for all $m \in M$ and $h \in H$. If $H \in VdB(d)$, it follows that

$$\mathrm{Ext}^\bullet_H(k, M) = H^\bullet(H, M_\varepsilon) \cong H_{d-\bullet}(H, M_\varepsilon) = \mathrm{Tor}_{\bullet - d}(k, M).$$

On the other direction, if X is an H^e-module, then X^{ad} is the same underlying vector space but with left H action defined by $h \cdot_{\mathrm{ad}} x := h_1 x S(h_2)$. With this structure (see, for instance, [3]) one has

$$H^\bullet(H, X) = \mathrm{Ext}^\bullet_H\big(k, X^{\mathrm{ad}}\big) \cong \mathrm{Tor}_{\bullet - d}\big(k, X^{\mathrm{ad}}\big) \cong H_{d-\bullet}(H, M_\varepsilon) = \mathrm{Tor}_{\bullet - d}(k, M). \qquad \square$$

Example 11. Let G be a finite group such that $1/|G| \in k$. The Reynolds operator $e = (1/|G|)\sum_{g \in G} g$ induces an isomorphism $M_G \cong M^G$ for any G-module M. This implies that $k[G] \in VdB(0)$ with $U_{k[G]} = k[G]$. This example can be easily generalized in the following direction:

Example 12. Let H be a semisimple unimodular Hopf algebra, so that H admits a *central* integral $e \in H$ satisfying

$$he = \varepsilon(h)e, \qquad \varepsilon(e) = 1.$$

Then $H \in VdB(0)$ with $U_H = H$. It is known (see Radford [2, Theorem 4]) that the Drinfel'd double of a finite dimensional Hopf algebra is unimodular. If K is a finite dimensional

Hopf algebra and $D(K)$ is the Drinfel'd double, again by a result of Radford [2, Proposition 7] $D(K)$ is semisimple if and only if K is semisimple and cosemisimple. Taking $K = k[G]$ where G is a noncommutative group with $|G|^{-1} \in k$, we get $H := D(K)$ a noncommutative not cocommutative semisimple unimodular Hopf algebra.

Proof. Let H be a unimodular semisimple Hopf algebra, and let $e \in H$ be as above. We will show that $\operatorname{Hom}_H(k, M) \cong k \otimes_H M$. If M is a left H-module, then

$$\operatorname{Hom}_H(k, M) \cong \{m \in M \mid hm = \varepsilon(h)m\} =: M^H.$$

It is clear that every element of the form em belongs to M^H because

$$h(em) = (he)m = \varepsilon(h)em;$$

but if $m \in M^H$, then

$$em = \varepsilon(e)m = m,$$

so M^H coincides with the image of the multiplication by e. Let us consider the map

$$e : M \to M^H, \quad m \mapsto em.$$

The elements of the form $hm - \varepsilon(h)m$ belong to the kernel of this map, so it factors through $M_H := M/\langle hm - \varepsilon(h)m\rangle$. Now the map $M^H \to M_H$ defined by $m \mapsto \overline{m}$ defines an inverse, because in M_H, every element $m = \varepsilon(e)m$ is equivalent to em. We have shown that $H \in VdB(0)$. □

Example 13. The algebra $H = k[x]$ is a Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$. It belongs to the class $VdB(d)$ with $U_H = H$.

Proof. Write $k[x]^e = k[x] \otimes k[x] \cong k[x, y]$, and consider the Koszul resolution

$$0 \to k[x, y] \to k[x, y] \to k[x] \to 0,$$

where the first map is the multiplication by $(x - y)$ and the second map is the evaluation $x = y$. Applying the functor $\operatorname{Hom}_{k[x,y]}(-, k[x, y])$, one obtain the complex

$$0 \to \operatorname{Hom}_{k[x,y]}\big(k[x, y], k[x, y]\big) \to \operatorname{Hom}_{k[x,y]}\big(k[x, y], k[x, y]\big) \to 0,$$

where the map is again multiplication by $x - y$. This complex identifies with

$$0 \to k[x, y] \to k[x, y] \to 0$$

but notice that now the grading increases to the right, so the homology is $k[x, y]/(x - y) \cong k[x]$ in degree one, zero elsewhere, and we conclude that $k[x] \in VdB(1)$. □

Example 14. The algebra $k[x]$ admits a finitely generated $k[x]^e$-projective resolution; this fact implies a Künneth formula for Hochschild cohomology, and so the algebra $k[x_1, \dots, x_n] \in VdB(n)$, with $U_{k[x_1,\dots,x_n]} = k[x_1, \dots, x_n]$.

Example 15. The Hopf algebra $k[x_1^{\pm 1}, \dots, x_d^{\pm 1}] = k[\mathbb{Z}^n]$, belongs to the class $VdB(d)$, because as an algebra, it is a localization of $k[x_1, \dots, x_d]$. Also

$$U_{k[x_1^{\pm 1},\dots,x_d^{\pm 1}]} = U_{k[x_1,\dots,x_d]} \otimes_{k[x_1,\dots,x_d]} k\big[x_1^{\pm 1}, \dots, x_d^{\pm 1}\big] = k\big[x_1^{\pm 1}, \dots, x_d^{\pm 1}\big].$$

Remark 16. Let A be an H-module algebra and $U \in \mathrm{Pic}_k(A)$ such that U is also an H-module, with the compatibility property

$$h(aub) = h_1(a)h_2(u)h_3(b)$$

for all $a, b \in A$, $h \in H$, and $u \in U$. Let $U^{-1} := \mathrm{Hom}_A(U, A)$; this is also an H-module satisfying the same compatibility condition. If $U \# H$ is the abelian group $U \otimes H$ with $A \# H$-bimodule structure given by

$$(a \# h)(u \otimes k) := \big(ah_1(u) \otimes h_2 k\big), \qquad (u \otimes k)(a \# h) = \big(uk_1(a) \otimes k_2 h\big),$$

then $U \# H \in \mathrm{Pic}_k(A \# H)$, and its inverse is $U^{-1} \# H$. If M is left $A \# H$-module, then

$$(U \# H) \otimes_{A\#H} M \cong U \otimes_A M$$

as $A \# H$-modules, where the $A \# H$-module structure on $U \otimes_A M$ is the one induced by the obvious left A-structure and the diagonal H-structure.

Proof. We will only exhibit an isomorphism $U \# H \otimes_{A\#H} U^{-1} \# H \to A \# H$. Let us denote by $\langle\, ,\rangle$ the evaluation map $U \otimes_A U^{-1} \to A$; notice that $\langle\, ,\rangle$ is H-linear. For $u \in U$, $v \in U^{-1}$, h and $k \in H$, define

$$U \# H \otimes_{A\#H} U^{-1} \# H \to A \# H, \quad (u \otimes h) \otimes (v \otimes k) \mapsto \big\langle u, h_1(v)\big\rangle h_2 k.$$

Theorem 17. *Let $H \in VdB(d)$ be a Hopf algebra with $U_H = H$. If A is an H-module algebra with $A \in VdB(d)$, then $A \# H \in VdB(d + d')$ with $U_{A\#H} = U_A \# H$.*

Proof. Let B be $A \# H$. In [3], the author shows that, for a B-bimodule M, there is a spectral sequence converging to $H^\bullet(B, M)$ whose second term is $\mathrm{Ext}^p(k, H^q(A, M))$. Similarly, there is a spectral sequence with E^2 term equal to $\mathrm{Tor}_p^H(k, H_q(A, M))$ converging to $H_\bullet(B, M)$.

Now consider $M = B^e$, and let us compute $H^\bullet(B, B^e)$. First, one notes the following isomorphism of left A^e-modules:

$$B^e \cong A^e \otimes V,$$

where V is the vector space $H \otimes H$.

Using Stefan's spectral sequence, one has

$$E_2^{pq} = \mathrm{Ext}_H^p\big(k, H^q\big(A, B^e\big)\big) = \mathrm{Ext}_H^p\big(k, H^q\big(A, A^e \otimes V\big)\big).$$

Since $A \in VdB(d')$, it follows that

$$\begin{aligned} H^\bullet\big(A, A^e \otimes V\big) &\cong H_{d'-\bullet}\big(A, U \otimes_A A^e \otimes V\big) \cong H_{d'-\bullet}\big(A, U \otimes_A A^e\big) \otimes V \\ &\cong H^\bullet\big(A, A^e\big) \otimes V \cong U[d] \otimes V. \end{aligned}$$

This implies first that the spectral sequences degenerates at this step, and consequently, there is an isomorphism

$$H^\bullet\big(B, B^e\big) \cong \mathrm{Ext}_H^{*-d'}(k, U \otimes V).$$

Recall that $V = H \otimes H^{\mathrm{op}}$; we have to consider it as H-module with the adjoint action. Now we use the fact that $H \in VdB(d)$, with $U_H = H$, so $H^\bullet(H, X) \cong H_{d-\bullet}(H, X)$ for all H-bimodules X. In particular, for a left H-module X, one can consider the bimodule X_ε, and this gives the formula

$$\mathrm{Ext}_H^\bullet(k, X) = H^\bullet(H, X_\varepsilon) \cong H_{d-\bullet}(H, X_\varepsilon) = \mathrm{Tor}_{d-\bullet}^H(k, X).$$

This formula implies that

$$H^\bullet\big(B, B^e\big) \cong \mathrm{Ext}_H^{*-d'}(k, U_A \otimes V) \cong \mathrm{Tor}_{d'+d-\bullet}^H(k, U_A \otimes V).$$

On the other hand, $H_\bullet(B, U_A \otimes_A B^e) = H_\bullet(B, (U_A \# H) \otimes_B B^e)$ can be computed using a spectral sequence whose second term is

$$\begin{aligned} \mathrm{Tor}_\bullet^H\big(k, H_\bullet\big(A, U_A \otimes_A B^e\big)\big) &= \mathrm{Tor}_\bullet^H\big(k, H_\bullet\big(A, U \otimes_A \big(A^e \otimes V\big)\big)\big) \\ &= \mathrm{Tor}_\bullet^H(k, U_A \otimes V). \end{aligned}$$

This spectral sequence collapses giving an isomorphism

$$H_\bullet\big(B, U_A \otimes_A B^e\big) \cong \mathrm{Tor}_\bullet^H(k, U_A \otimes V).$$

In particular,

$$H^\bullet\big(B, B^e\big) \cong H_{d+d'-\bullet}\big(B, U \otimes_A B^e\big)$$

and

$$\begin{aligned} H^{d+d'}(B, B^e) &= H_0\big(B, U \otimes_A B^e\big) = H_0\big(B, (U \# H) \otimes_B B^e\big) \\ &= H_0\big(B, (U \# H) \otimes B\big) = U \# H. \qquad \square \end{aligned}$$

Corollary 18. *With the notations of the above theorem, assume $U = A$ as A-bimodules and H-modules, then*

$$H^{\bullet}(B, M) \cong H_{d+d'-\bullet}(B, M)$$

for all $A \# H$-bimodules M.

Example 19. Let $A \in VdB(d)$, $D \in \mathrm{Der}_k(A)$, and write the Ore extension $B = A[t, D]$. This algebra B coincides with $A \# k[t]$ where the $k[t]$-module action on A is given by $t.a = D(a)$, $B \in VdB(d + 1)$. For $A = k[x]$ and $D = \partial/\partial x$ one obtains the known result that $A_1(k) \in VdB(2)$.

Example 20. Let $0 \neq q \in k$, then $B = k\{x^{\pm 1}, y^{\pm 1}\}/\langle yx = qxy\rangle \in VdB(2)$. Indeed, this algebra is isomorphic to $k[x^{\pm 1}] \# k[y^{\pm 1}]$ where the H-module structure on $k[x^{\pm 1}]$ is given by $y.x = qx$.

Example 21. Let A be an algebra and G a finite group of automorphism of A. If $A \in VdB(d)$, then $A \# G \in VdB(d)$.

Warning: It can happen that A is such that $U_A \cong A$ as A-bimodule, but $U_A \ncong A$ as H-module. It is easy to show an example of this situation when $H = k[G]$.

One can first observe the following characterization of the $A^e \# G$-structures on a A-bimodule isomorphic to A:

Proposition 22. *Let U be an A^e-bimodule isomorphic to A. The set of all possible $A^e \# G$-module structures on U, modulo $A^e \# G$-isomorphism, is parametrized by $H^1(G, \mathcal{U}\mathcal{Z}(A))$, the first cohomology of G with coefficients in the* (*multiplicative*) *abelian group of units of the center of A.*

Proof. Fix an isomorphism $A \cong U$ and let u be the image of 1 in U. Hence $U = Au = uA$, and moreover, $au = ua$ for all $a \in A$. One has to define a G-action on U such that, for all $a, b \in A$ and $v \in U$, the following identity holds:

$$g(avb) = g(a)g(v)g(b).$$

Since the bimodule U is generated by u, it is clear that it is only necessary to define $g(u)$. The element $g(u)$ must belong to U, so it is of the form $a_g u$ for some a_g in A. But

$$au = ua$$

for all $a \in A$, and applying g, one obtains

$$ag(u) = g(u)a, \quad \forall a \in A;$$

and so

$$aa_g u = a_g ua = a_g au.$$

It follows that a_g must belong to the center of A. Also, every element of U is of the form

$$ag(u) = aa_g u,$$

so a_g must be a unit. We have then shown that the assignment $g \mapsto a_g$ must be a map from G into $\mathcal{U}(\mathcal{Z}(A))$.

If one wants associativity, the identity

$$g\bigl(h(u)\bigr) = (gh)(u), \quad \forall g, h \in G,$$

is required, so

$$g\bigl(h(u)\bigr) = g(a_h u) = g(a_h)a_g u = (gh)(u) = a_{gh} u.$$

But u is a basis of U with respect to the left A-structure, so

$$g(a_h)a_g = a_{gh}.$$

On the other hand, it is clear that an assignment $g \mapsto a_g$ from G into the units of center of A satisfying the above cocycle condition defines a G-action compatible with the A-bimodule structure.

Now assume that U has two G-actions that are isomorphic. Let us denote them by $g._1(u) = a_g u$, and $g._2(u) = b_g u$, and call U_1 and U_2 the bimodule U with the first and the second G-structure, respectively.

If $\phi : U_1 \to U_2$ is an isomorphism of $A^e \# G$-modules, then the image of u is some element λu, where $\lambda \in A$. Moreover, λ is a unit because ϕ is an isomorphism, and $\lambda \in \mathcal{Z}(A)$ because ϕ is A^e-linear.

Now G-linearity means that

$$\phi(g._1 u) = \phi(a_g u) = \lambda a_g u,$$

but also

$$\phi(g._1 u) = g._2 \phi(u) = g._2(\lambda u) = g(\lambda) g._2 u = g(\lambda) b_g u,$$

so we deduce

$$b_g = \lambda g\bigl(\lambda^{-1}\bigr) a_g,$$

and the two assignments differ by a coboundary. $\square$

Example 23. This proposition is more useful when the center of A is small. If A is a central k-algebra, G a finite group of k-linear automorphism, and $\chi : G \to k$ a character, let define A_χ with underlying A^e-module structure equal to A, and G-action given by $g.a = g(a)\chi(g)$. The above proposition tells us that all $A^e \# G$-module structures on the A^e-module A are of this type.

Despite Proposition 22, for an algebra $A \in VdB$, the dualizing bimodule U is a very particular one, namely $U_A = \operatorname{Ext}^d_{A^e}(A, A^e)$. The following is an example showing (without calculating $H^1(G, \mathcal{UZ}(A))$ that U is isomorphic to A as A^e bimodule, but not as G-module:

Example 24. Let V be a finite dimensional vector space, $A = S(V)$, and $G \subset \mathrm{GL}(V)$ a finite group. We claim that

$$\operatorname{Ext}^\bullet_{A^e}\big(A, A^e\big) = A \otimes \det{}^{-1}[d],$$

where $d = \dim(V)$, and $\det^{-1}$ is the dual of the determinant representation $\Lambda^d V$. Namely, $\det^{-1}$ is a one dimensional k-vector space, if $w \in \det^{-1}$ is a nonzero element, $g \in G$, and $a \in A$, then the G-action is given by

$$g(a \otimes w) = g(a) \det(g|_V)^{-1} \otimes w.$$

We conclude that $U_A \cong A$ as $A^e \# G$-modules if and only if $G \subset \mathrm{SL}(V)$.

Proof. Let $g \in G$, and choose a basis $\{x_1, \ldots, x_d\}$ of V which diagonalizes g. Notice that $S(V) = \bigotimes_{i=1} k[x_i]$, and this tensor product is g-equivariant with the diagonal action. The Künneth formula is g-equivariant, so we only need to prove the following lemma:

Lemma 25. *If $A = k[x]$ and g is the automorphism of A determined by $g(x) = \lambda x$, then $\operatorname{Ext}^\bullet_{A^e}(A, A^e) = A[1]$, and the action of g is given by multiplication by λ^{-1}.*

Proof of the lemma. It was shown in Example 13 that $k[x] \in VdB(1)$. Let us compute the g-action on

$$H^1\big(k[x], k[x, y]\big) = \operatorname{Der}\big(k[x], k[x, y]\big) / \operatorname{InnDer}\big(k[x], k[x, y]\big).$$

If $D : k[x] \to k[x, y]$ is a derivation, then D is determined by its value $D(x)$ on x, and this gives the isomorphism

$$\operatorname{Der}\big(k[x], k[x, y]\big) \cong k[x, y], \quad D \mapsto D(x). \tag{†}$$

If $p \in k[x, y]$, the inner derivation $[p, -]$ takes in x the value

$$[p, x] = p(x, y)y - xp(x, y) = (x - y)p(x, y).$$

This shows that, under the isomorphism (†), $\mathrm{InnDer} \cong (x-y)k[x,y]$, obtaining

$$H^1\big(A, A^e\big) = \mathrm{Der}\big(A, A^e\big) / \mathrm{InnDer}\big(A, A^e\big) \cong \frac{k[x,y]}{(x-y)k[x,y]} \cong k[x].$$

In order to compute the action of g on H^1, we recall that, if D is a derivation, then $g.D = g \circ D \circ g^{-1}$, so

$$(g.D)(x) = g\big(D\big(g^{-1}x\big)\big) = g\big(D\big(\lambda^{-1}x\big)\big) = \lambda^{-1} g\big(D(x)\big),$$

and if $D(x) \in k$ (this is always the case modulo an inner derivation), we get

$$(g.D)(x) = \lambda^{-1} D(x). \qquad \square$$

Turning back to the example $A = S(V)$ and $G \subset \mathrm{GL}(V)$ a finite subgroup, we see that $S(V) \# G \in VdB(\dim(V))$ but $U_{S(V)\#G} \cong S(V) \# G$ if and only if $G \subset \mathrm{SL}(V)$. This example shows a situation where $H^\bullet(B, M) = H_{d-\bullet}(B, U \otimes_B M)$ with $U \neq B$. In particular, $H^\bullet(B) \cong H_\bullet(B, U)$, which needs not be equal to $H_{d-\bullet}(B)$, and in fact it is different.

3. The example $S(V) \# G$

We finish with a computation of the homology and cohomology of $S(V) \# G$.

Let k be a field, V a finite dimensional k-vector space, G a finite subgroup of $\mathrm{GL}(V, k)$, $A = S(V)$, and we will assume that $1/|G| \in k$. For simplicity we will also assume that k has a primitive $|G|$-th root of 1. This condition is not really necessary because of the following reason: consider ξ a primitive $|G|$-root of unity in the algebraic closure of k and let K be $k(\xi)$ the field generated by k and ξ. One can view G inside $\mathrm{GL}(V \otimes K, K)$, and consider it acting on $A \otimes K = S_K(V \otimes K)$. A descend property of the Hochschild homology and cohomology with respect to this change of the base field assures that the dimension over K of the (co)homology of the extended algebra is the same as the dimension over k of the (co)homology of the original one.

If $g \in G$, $V^g = \{x \in V \mid g(x) = x\}$. As g-module, V^g admits a unique complement in V, we will call it V_g. We have $V = V^g \oplus V_g$ as g-modules, and this decomposition is canonical.

3.1. Homology of $S(V) \# G$

Theorem 26. *With the notations as in the above paragraph, denote $\langle G \rangle$ the set of conjugacy classes of G, and for $g \in G$ let $\mathcal{Z}_g$ be the centralizer of g in G, so that $\mathcal{Z}_g = \{h \in G \mid hg = gh\}$. The Hochschild homology of $S(V) \# G$ is given by*

$$H_n\big(S(V) \# G\big) = H_n\big(S(V), S(V) \# G\big)^G = \bigoplus_{\langle g \rangle \in \langle G \rangle} \big(S\big(V^g\big) \otimes \Lambda^n\big(V^g\big)\big)^{\mathcal{Z}_g},$$

where $\Lambda^n(V^g)$ is the homogeneous component of degree n of the exterior algebra on V^g.

Proof. With the hypothesis on the characteristic and the order of the group, the spectral sequence of [3] gives the following isomorphism:

$$H_n\big(S(V)\,\#\,G\big) = H_n\big(S(V), S(V)\,\#\,G\big)^G = \bigoplus_{\langle g\rangle\in\langle G\rangle} H_n\big(S(V), S(V)g\big)^{\mathcal{Z}_g},$$

valid for any k-algebra of the type $A\,\#\,G$. Since $V = V^g \oplus V_g$, it follows that

$$S(V) \cong S\big(V^g\big) \otimes S(V_g)$$

as algebras, and

$$S(V)g \cong S\big(V^g\big) \otimes S(V_g)g$$

as $S(V)$-bimodules. Using the Künneth formula, one gets

$$H_n\big(S(V), S(V)g\big)^{\mathcal{Z}_g} = \bigoplus_{p+q=n} \big(H_p\big(S\big(V^g\big)\big) \otimes H_q\big(S(V_g), S(V_g)g\big)\big)^{\mathcal{Z}_g}.$$

By the Hochschild–Kostant–Rosenberg theorem, or directly by computing using a Koszul type resolution, one sees that, if W is a finite dimensional k-vector space,

$$H_n\big(S(W)\big) = \Omega^n\big(S(W)\big) = S(W) \otimes \Lambda^n W.$$

The homology with coefficients is computed in the following lemma.

Lemma 27. $H_\bullet(S(V_g), S(V_g)g) = k[0]$ *with trivial* $\mathcal{Z}_g$*-action.*

Proof. Let $h \in \mathcal{Z}_g$. One can diagonalize simultaneously h and g in V_g. If $\{x_1, \dots, x_k\}$ is a basis of eigenvectors of both h and g, then the algebra $S(V_g)$ is isomorphic to

$$k[x_1, \dots, x_k] = \bigotimes_{i=1}^{k} k[x_i] \quad \text{and} \quad S(V_g)g = k[x_1, \dots, x_k]g = \bigotimes_{i=1}^{k} k[x_i]g_i,$$

where g_i acts on x_i by multiplication of the corresponding eigenvalue of g. Notice also that h acts on each x_i by multiplication by some λ_i', because x_i is also an eigenvector of h.

Using Künneth formula again, one gets

$$H_\bullet\big(S(V_g), S(V_g)g\big) = \bigotimes_i H_\bullet\big(k[x_i], k[x_i]g_i\big).$$

Let us now make the explicit computation for the algebra $k[x]$, g acting by $x \mapsto \lambda x$, and h acting by $x \mapsto \lambda' x$.

Consider, as in Example 13, the resolution of $k[x]$ as $k[x]$-bimodule

$$0 \to k[x] \otimes k[x] \to k[x] \otimes k[x] \to k[x] \to 0.$$

Here the first morphism is given by $p \otimes q \mapsto px \otimes q - p \otimes xq$ and the second one is the multiplication map.

By tensoring with $k[x]g$ over $k[x]^e$, one gets the complex

$$0 \to k[x]g \to k[x]g \to 0$$

with differential

$$pg \mapsto pgx - xpg = px(\lambda - 1)g,$$

whose homology is $H_\bullet(k[x].k[x]g)$. The fact that $\lambda \neq 1$ implies that the differential is injective and the image equals $xk[x]g$, so $H_1 = 0$ and $H_0 = k$. It is clear that h acts trivially on H_0, and the proof of the lemma is complete. □

The sum

$$H_n\big(S(V), S(V)g\big)^{\mathcal{Z}_g} = \bigoplus_{p+q=n} \big(H_p\big(S\big(V^g\big)\big) \otimes H_q\big(S(V_g), S(V_g)g\big)\big)^{\mathcal{Z}_g}$$

reduces to

$$H_n\big(S(V), S(V)\big)^{\mathcal{Z}_g} = \big(S\big(V^g\big) \otimes \Lambda^n\big(V^g\big)\big)^{\mathcal{Z}_g}$$

and the proof of the theorem is finished. □

Example 28. Let $k = \mathbb{C}$, $V = \mathbb{C}^2$, G a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$. Then

$$\begin{aligned}
H_0\big(S(V) \# G\big) &= S(V)^G \oplus \mathbb{C}^{\#\{\langle g\rangle \neq 1\}},\\
H_1\big(S(V) \# G\big) &= \big(S(V) \otimes V\big)^G,\\
H_2\big(S(V) \# G\big) &= \big(S(V) \otimes \Lambda^2(V)\big)^G = S(V)^G,\\
H_n\big(S(V) \# G\big) &= 0, \quad \forall n > 2.
\end{aligned}$$

3.2. Cohomology: direct computation

The formula

$$H^n\big(S(V) \# G\big) = H^n\big(S(V), S(V) \# G\big)^G = \bigoplus_{\langle g\rangle \in \langle G\rangle} H^n\big(S(V), S(V)g\big)^{\mathcal{Z}_g}$$

is also valid. Using $S(V) = S(V^g) \otimes S(V_g)$, and Künneth formula, one gets

$$H^n\big(S(V),S(V)g\big)^{\mathcal{Z}_g}=\bigoplus_{p+q=n}\big(H^p\big(S\big(V^g\big),S\big(V^g\big)\big)\otimes H^q\big(S\big(V_g\big),S\big(V_g\big)g\big)\big)^{\mathcal{Z}_g}$$
$$=\bigoplus_{p+q=n}\big(S\big(V^g\big)\otimes\Lambda^p\big(\big(V^g\big)^\bullet\big)\otimes H^q\big(S(V_g),S(V_g)g\big)\big)^{\mathcal{Z}_g}.$$

Here we have used the isomorphism

$$H^\bullet\big(S(W),S(W)\big)=\Lambda^\bullet_{S(W)}\operatorname{Der}\big(S(W)\big)=S(W)\otimes\Lambda^\bullet W^*.$$

Now we need the analogue of Lemma 27 for cohomology, whose proof is the same as that of Lemma 25.

Lemma 29. *Let $A=k[x]$, g,h the automorphisms determined by $g(x)=\lambda x$ and $h(x)=\mu x$, with $\lambda\neq 1$. Then $H^\bullet(A,Ag)=k[1]$, and the action of h is given by multiplication by μ^{-1}.*

Corollary 30. *If we denote by $d_g=\dim_k(V_g)$, then*

$$H^\bullet\big(S(V_g),S(V_g)g\big)=\det|_{V_g}^{-1}[d_g].$$

This is an isomorphism of $\mathcal{Z}_g$-modules.

Proof. From the fact that g and h commute, one can choose a basis $\{x_1,\dots,x_n\}$ of eigenvectors of both g and h. The corollary follows from Künneth formula, and the lemma above applied to $S(V)=\bigotimes_{i=1}^n k[x_i]$. $\square$

We have obtained the following formula:

Theorem 31.

$$H^\bullet(S(V)\#G)=\bigoplus_{\langle g\rangle\in\langle G\rangle}\big(S\big(V^g\big)\otimes\Lambda^\bullet\big(\big(V^g\big)^\bullet\big)\otimes\det|_{V_g}^{-1}[d_g]\big)^{\mathcal{Z}_g}.$$

3.3. *Cohomology: computation using duality*

Using Theorem 17 for $H=k[G]$ (see Example 24), we know that

$$H^\bullet(A\#G)=H^\bullet\big(A\#G,(U_A\#G)\otimes_{A\#G}A\#G\big)=H_{d-\bullet}(A\#G,U_A\#G)$$
$$=H_{d-\bullet}\big(A\#G,\big(A\otimes\det^{-1}\big)\#G\big).$$

Using Stefan's spectral, this is the same as

$$H_{d-\bullet}\big(A,\big(A\otimes \det{}^{-1}\big)\# G\big)^G=\bigoplus_{\langle g\rangle\in\langle G\rangle} H_{d-\bullet}\big(A,\big(A\otimes \det{}^{-1}(V)\big).g\big)^{\mathcal{Z}_g}$$
$$=\bigoplus_{\langle g\rangle\in\langle G\rangle}\big(H_{d-\bullet}(A,A.g)\otimes \det{}^{-1}(V)\big)^{\mathcal{Z}_g}.$$

Now the same techniques of writing $V=V^g\oplus V_g$ apply, and we obtain

$$\bigoplus_{\langle g\rangle\in\langle G\rangle}\big(H_{d-\bullet}(A,A.g)\otimes \det{}^{-1}\big)^{\mathcal{Z}_g}=\bigoplus_{\langle g\rangle\in\langle G\rangle}\big(H_{d-\bullet}\big(S\big(V^g\big)\big)\otimes \det{}^{-1}\big)^{\mathcal{Z}_g}$$
$$=\bigoplus_{\langle g\rangle\in\langle G\rangle}\big(S\big(V^g\big)\otimes \Lambda^{d-\bullet}\big(V^g\big)\otimes \det{}^{-1}\big)^{\mathcal{Z}_g}.$$

The difference between this formula and that of Theorem 31, having det or $\det|_{V_g}$ is explained by the fact that in Theorem 31, one has also $\Lambda^\bullet((V^g)^*)$, while here one has $\Lambda^{d-\bullet}(V^g)$. The multiplication map induces a morphism of $\mathcal{Z}_g$-modules

$$\Lambda^\bullet\big(V^g\big)\otimes \Lambda^{\dim(V^g)-\bullet}(V^g)\to \Lambda^{\dim(V^g)}V^g=\det|_{V^g},$$

and as a consequence one has an isomorphism of $\mathcal{Z}_g$-modules

$$\Lambda^\bullet\big(V^g\big)^*\cong \Lambda^{\dim(V^g)-\bullet}\big(V^g\big)\otimes \det|_{V^g}^{-1}.$$

So we get the same after noticing that $\det=\det|_{V^g}\otimes\det|_{V_g}$.

Example 32. Let $k=\mathbb{C}$, $V=\mathbb{C}^2$, G a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. In this case, homology and cohomology is the same:

$$\begin{aligned}
&H^0\big(S(V)\# G\big)=S(V)^G,\\
&H^1\big(S(V)\# G\big)=\big(S(V)\otimes V\big)^G,\\
&H^2\big(S(V)\# G\big)=S(V)^G\oplus \mathbb{C}^{\#\{\langle g\rangle\neq 1\}},\\
&H^n\big(S(V)\# G\big)=0,\quad \forall n>2.
\end{aligned}$$

Example 33. Let $G=C_2=\{1,t\}$ the cyclic group of order two. Let k be a field of $\mathrm{ch}(k)\neq 2$, $A=k[x]$ with t acting on A by $x\mapsto -x$. Using Theorem 26, one gets

$$\begin{aligned}
&H_0(A\# G)=A^G\oplus k=k\big[x^2\big]\oplus k,\\
&H_1(A\# G)=(A\otimes k.dx)^G=k\big[x^2\big]x\,dx,\\
&H_n(A\# G)=0,\quad \forall n>1.
\end{aligned}$$

On the other hand,

$$H^0(A\,\#\,G) = A^G = k\big[x^2\big],$$
$$H^1(A\,\#\,G) = (A\otimes k.\partial_x)^G \oplus \left(\frac{\mathrm{Der}(A,At)}{\mathrm{InnDer}(A,At)}\right)^{C_2} = k\big[x^2\big]x\partial_x \oplus 0,$$
$$H^n(A\,\#\,G) = 0, \quad \forall n > 1.$$

In this example, homology and cohomology are not the same. The cohomology is $k[x^2]$-free, while the homology has torsion.

In the above example, we see that the cohomology is a "part" of the homology. The same phenomenon happens in the following:

Example 34. Let $W = k^n$, consider S_n acting on W by permutation of the coordinates, and let

$$V = \big\{(1,1,\dots,1)\big\}^{\perp} := \left\{(x_1,\dots,x_n)\in W \;\Big|\; \sum_{i=1}^{n} x_i = 0\right\}.$$

We claim that

$$H^\bullet\big(S(V)\,\#\,S_n\big) = H^\bullet\big(S(V), S(V)\,\#\,A_n\big)^{S_n},$$

where A_n denote as usual the subgroup of even permutations.

In fact, we can prove an analogous formula in the following general setting:

Example 35. Let $G\subset \mathrm{GL}(V)$ be a finite subgroup, $S := G\cap \mathrm{SL}(V) = \mathrm{Ker}(\det : G\to k^\times)$, and $C := \det(G)\subset k^\times$. Then

$$H_\bullet\big(S(V)\,\#\,G\big) = \bigoplus_{w\in C}\left(\bigoplus_{\langle g\rangle\in\langle G\rangle,\,\det(g)=w} H_\bullet\big(S(V), S(V)g\big)^{\mathcal{Z}_g}\right),$$

and each of this summands is nonzero, while in cohomology, there are only the terms corresponding to $w = 1$:

$$H^\bullet\big(S(V)\,\#\,G\big) = \bigoplus_{\langle g\rangle\in\langle G\rangle,\,\det(g)=1} H^\bullet\big(S(V), S(V)g\big)^{\mathcal{Z}_g}.$$

In particular,

$$H^\bullet\big(S(V)\,\#\,G\big) = H^\bullet\big(S(V), S(V)\,\#\,S\big)^G \quad \text{and} \quad H^\bullet\big(S(V)\,\#\,G\big) \neq H_{d-\bullet}\big(S(V)\,\#\,G\big).$$

Proof. The formula for the homology is just noticing that the set $\langle G\rangle$ can be split into smaller pieces, parametrized by the values of the determinant. To see that each summand is nonzero, we make them explicit. Using Theorem 26, we know that

$$H_\bullet\big(S(V),S(V)g\big)^{\mathcal{Z}_g}=\big(S\big(V^g\big)\otimes\Lambda^\bullet V^g\big)^{\mathcal{Z}_g}.$$

Even if $V^g=0$, one always has the element $1\in(S(V^g)\otimes\Lambda^\bullet V^g)^{\mathcal{Z}_g}$.

The interesting part is the formula for the cohomology. Recall from the duality formula that

$$H^\bullet\big(S(V),S(V)g\big)\cong\det{}^{-1}\otimes H_{d-\bullet}\big(S(V),S(V)g\big).$$

If one shows that $H_\bullet(S(V),S(V)g)$ is a trivial g-module, then, for $\det(g)\neq 1$ we will have

$$\begin{aligned}\big(\det{}^{-1}\otimes H_{d-\bullet}\big(S(V),S(V)g\big)\big)^{\mathcal{Z}_g}&\subseteq\big(\det{}^{-1}\otimes H_{d-\bullet}\big(S(V),S(V)g\big)\big)^g\\&=\big(\det{}^{-1}\big)^g\otimes H_{d-\bullet}\big(S(V),S(V)g\big)\\&=0.\end{aligned}$$

So let us see that $H_\bullet(S(V),S(V)g)$ has trivial g-action. For that, write $V=V^g\oplus V_g$, then $H_\bullet(S(V),S(V)g)\cong H_\bullet(S(V^g))\otimes H_\bullet(S(V),S(V)g))$. Clearly $H_\bullet(S(V^g))$ is a trivial g-module, and $H_\bullet(S(V),S(V)g))$ also has trivial g-action in virtue of Lemma 27. □

Remark 36. The equality between homology and cohomology depends not only on G, but on the representation. For example, given an arbitrary finite subgroup $G\subset\mathrm{GL}(V)$, we can consider the action on V and on V^*, and G will act symplectically on $W=V\oplus V^*$. In this case we have

$$G\hookrightarrow\mathrm{Sp}(W)\subset\mathrm{SL}(W),$$

so that

$$H^\bullet\big(S(W)\#G\big)=H_{\dim(W)-\bullet}\big(S(W)\#G\big).$$

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