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Journal of Algebra 261 (2003) 54–101
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Theory of braided Hopf crossed products ☆

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Received 24 January 2002

Communicated by Susan Montgomery

Abstract

We define a type of crossed product over braided Hopf algebras, which generalizes the ones introduced by Blattner, Cohen, and Montgomery, and Doi and Takeuchi, and we study some of its properties. For instance, we prove Maschke's Theorem for these new crossed products and we construct a natural Morita context which extends the one obtained by Cohen, Fischman, and Montgomery.

Keywords: Hopf algebra; Crossed products

1. Introduction

The classical notion of Hopf crossed product was introduced in [BCM,DT] as a natural generalization of group crossed product to the context of Hopf algebras. These algebras are constructed in the following way: given a Hopf algebra H, an algebra A, a weak action $h \otimes a \mapsto h \cdot a$ of H on A, and a cocycle $f : H \otimes H \to A$, the crossed product $A \#_f H$ is the vector space $A \otimes H$ endowed with the multiplication

$$(a \# h)(b \# l) = \sum_{(h)(l)} a(h_{(1)} \cdot b) f(h_{(2)} \otimes l_{(1)}) \# h_{(3)} l_{(2)}.$$

☆ Supported by UBACYT X193, CONICET, and Fundación Antorchas.
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0021-8693/03/$ – see front matter
doi:10.1016/S0021-8693(02)00546-X

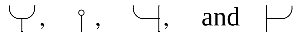
To ensure that $A \#_f H$ is an associative unitary algebra, the action and the cocycle must verify suitable conditions. In [GG] we proposed a generalization of this concept introducing a map $s : H \otimes A \to A \otimes H$, compatible with the algebraic structures of H and A. When s is the flip, we recover the notion introduced in [BCM,DT]. In this paper we extend the definition given in [GG] by allowing H to be a braided Hopf algebra, and we study some properties of the resulting algebras. Our aim is not to obtain the all-embracing notion of crossed product, but to introduce the type of crossed product which includes the classical type, the automorphism Ore extensions type, and the one considered in [B,M], all that in order to show that many of results for standard Hopf crossed products, appearing in Montgomery's book [Mo], remain valid in this new context.

This paper is organized as follows: in Section 2 we recall a very general definition of crossed product given by [Br] and review some results needed later. In Section 3 we recall the definitions and basic properties of braided bialgebras and Hopf algebras given in [T1]. In Section 4 we introduce the notion of transposition and we study its properties. In Section 5 we adapt to our context the notions of H-comodule and H-module algebras. In Section 6 we generalize the concepts of weak and true actions of a Hopf algebra on an algebra and, for the latter, we introduce the associated smash products. In Section 7 we study the ring of invariants of such an action. In Section 8 we generalize the Morita equivalence considered in [CFM] to our settings. This is one of our main results. In Section 9 we introduce the crossed products to be studied in this paper. In Section 10 we give intrinsic characterizations of these crossed products as a sort of cleft extensions and normal Galois extensions. For the case considered in [B,M], the first characterization was obtained in [AV]. In Section 11 we show that our crossed products satisfy Maschke's Theorem. Finally, in Section 12, we determine necessary and sufficient conditions for two crossed products to be equivalent in a natural sense.

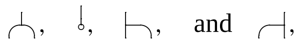
Some, but not all, of the questions studied in this paper were considered in [GG], for the case where H is a standard Hopf algebra.

In this article we work in the category of vector spaces over a field k. Then we assume implicitly that all the maps are k-linear. The tensor product over k is denoted by $\otimes$, without any subscript. Given a vector space V and $n \geqslant 1$, sometimes we let V^n denote the n-fold tensor product $V \otimes \cdots \otimes V$. Given vector spaces U, V, W, and a map $f : V \to W$, we write $U \otimes f$ for $\mathrm{id}_U \otimes f$ and $f \otimes U$ for $f \otimes \mathrm{id}_U$. We assume that the reader is familiar with the notions of algebra, coalgebra, module, and comodule. Unless otherwise explicitly established, we assume that the algebras are associative unitary and the coalgebras are coassociative counitary. Given an algebra A and a coalgebra C, we let $\mu : A \otimes A \to A$, $\eta : k \to A$, $\Delta : C \to C \otimes C$, and $\epsilon : C \to k$ denote the multiplication, the unit, the comultiplication, and the counit, respectively, specified with subscript if necessary. Moreover, given k-vector spaces V and W, we let $\tau : V \otimes W \to W \otimes V$ denote the flip $\tau(v \otimes w) = w \otimes v$.

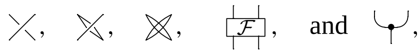
Some of the results of this paper are valid in the context of monoidal categories. In fact, in this article we use the well known graphic calculus for monoidal and braided categories. As usual, morphisms will be composed from up to down and tensor products will be represented by horizontal concatenation in the corresponding order. The identity map of a vector space will be represented by a vertical line. Given an algebra A, the diagrams



stand for the multiplication map, the unit, the action of A on a left A-module, and the action of A on a right A-module, respectively. Given a coalgebra C, the comultiplication, the counit, the coaction of C on a right C-comodule, and the coaction of C on a left C-module will be represented by the diagrams

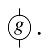


respectively. The maps c, s, χ, $\mathcal{F}$, and f, which appear in Definitions 3.1, 4.1, 2.2, and Proposition 9.1, will be represented by the diagrams



respectively. The inverse maps of c and s (when s is bijective) will be represented by

Finally, any other map $g: V \to W$ will be geometrically represented by the diagram



Let V, W be vector spaces and let $c: V \otimes W \to W \otimes V$ be a map. If V is an algebra, then we say that c is compatible with the algebra structure of V if $c(\eta \otimes W) = W \otimes \eta$ and $c(\mu \otimes W) = (W \otimes \mu)(c \otimes V)(V \otimes c)$. If V is a coalgebra, then we say that c is compatible with the coalgebra structure of V if $(W \otimes \epsilon)c = \epsilon \otimes W$ and $(W \otimes \Delta)c = (c \otimes V)(V \otimes c)(\Delta \otimes W)$. Finally, if W is an algebra or a coalgebra, then we introduce the notion that c is compatible with the structure of W in an obvious way.

2. Brzeziński's crossed products

In this section we recall a very general definition of crossed product, introduced in [Br], and its basic properties. For the proofs we refer to [Br,BD].

Throughout this section A is a unitary algebra and V is a vector space equipped with a distinguished element $1 \in V$.

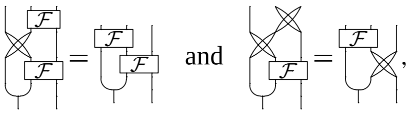
Definition 2.1. Given maps $\chi: V \otimes A \to A \otimes V$ and $\mathcal{F}: V \otimes V \to A \otimes V$, we let $A \# V$ denote the algebra (in general, non-associative and non-unitary) whose underlying vector space is $A \otimes V$ and whose multiplication map is given by

$$\mu_{A\#V} := (\mu \otimes V)(\mu \otimes \mathcal{F})(A \otimes \chi \otimes V).$$

The element $a \otimes v$ of $A \# V$ will usually be written $a \# v$. The algebra $A \# V$ is called a *crossed product* if it is associative with $1 \# 1$ as identity.

Definition 2.2. Let $\chi : V \otimes A \to A \otimes V$ and $\mathcal{F} : V \otimes V \to A \otimes V$ be maps.

(1) χ is a twisting map if it is compatible with the algebra structure of A and $\chi(1 \otimes a) = a \otimes 1$.
(2) $\mathcal{F}$ is normal if $\mathcal{F}(1 \otimes v) = \mathcal{F}(v \otimes 1) = 1 \otimes v$, and it is a cocycle which satisfies the twisted module condition if



where  $= \chi$ and  $= \mathcal{F}$.

More precisely, the first equality says that $\mathcal{F}$ is a cocycle and the second that $\mathcal{F}$ satisfies the twisted module condition.

Theorem 2.3 (T. Brzeziński). *The algebra $A \# V$ is a crossed product iff χ is a twisting map and $\mathcal{F}$ is a normal cocycle satisfying the twisted module condition.*

Note that the multiplication of a crossed product verifies

$$\mu_{A\#V}(a \# 1 \otimes b \# v) = ab \# v. \tag{1}$$

Conversely, it is easy to check that each associative multiplication that satisfies (1) is the multiplication of a crossed product. The twisting map χ and the cocycle $\mathcal{F}$ are given by $\chi(v \otimes a) = (1 \otimes v)(a \otimes 1)$ and $\mathcal{F}(v \otimes w) = (1 \otimes v)(1 \otimes w)$.

Next, we mention a few results.

Proposition 2.4. *Let $A \# V$ be a crossed product and let B be an algebra. Given an algebra morphism $\alpha : A \to B$ and a map $\beta : V \to B$ such that*

(1) $\beta(1) = 1$,
(2) $\mu(\alpha \otimes \beta)\chi = \mu(\beta \otimes \alpha)$,
(3) $\mu(\alpha \otimes \beta)\mathcal{F} = \mu(\beta \otimes \beta)$,

there exists a unique algebra morphism $\gamma : A \# V \to B$ verifying $\gamma(a \# 1) = \alpha(a)$ and $\gamma(1 \# v) = \beta(v)$. Conversely, if $\gamma : A \# V \to B$ is an algebra morphism, then the maps $\alpha(a) := \gamma(a \# 1)$ and $\beta(v) := \gamma(1 \# v)$ verify (1)–(3).

Proposition 2.5. *An algebra B is isomorphic to a crossed product $A \# V$ iff there are maps $A \xrightarrow{\alpha} B \xleftarrow{\beta} V$ such that α is a morphism of algebras, $1 = \beta(1)$, and $\mu(\alpha \otimes \beta) : A \otimes V \to B$ is an isomorphism of vector spaces.*

Example 2.6 (Twisted tensor products). Let B be an algebra, $\chi : B \otimes A \to A \otimes B$ a twisting map, and $\mathcal{F} : B \otimes B \to A \otimes B$ the *trivial cocycle* $\mathcal{F}(v \otimes v') = 1 \otimes vv'$. It is immediate that $\mathcal{F}$ is normal and verifies the cocycle condition. Moreover, the twisted module condition reduces to $(A \otimes \mu)(\chi \otimes B)(B \otimes \chi) = \chi(\mu \otimes A)$. Hence, χ is a twisting map in the sense of [CSV], and (B, A, χ) is a matched pair of algebras. The crossed products $A \otimes_\chi B$ constructed from this type of data are called twisted tensor products or matched products. These algebras, which are a direct generalization of the tensor products, were studied in [CSV,Ta].

We finish this section by introducing a definition which we will need later.

Definition 2.7. Let $\chi : V \otimes A \to A \otimes V$ be a twisting map. A subalgebra A' of A is stable under χ if $\chi(V \otimes A') \subseteq A' \otimes V$.

3. Braided bialgebras and braided Hopf algebras

Below, we recall the concepts of braided bialgebra and Hopf algebra following the presentation given in [T1].

Definition 3.1. A braided bialgebra is a vector space H, endowed with an algebra structure, a coalgebra structure, and a bijective Yang–Baxter operator $c \in \mathrm{End}_k(H^2)$ (called the braid of H) such that: c is compatible with the algebra and coalgebra structures of H, η is a coalgebra morphism, ϵ is an algebra morphism, and $\Delta\mu = (\mu \otimes \mu)(H \otimes c \otimes H)(\Delta \otimes \Delta)$. Moreover, if there exists a map $S : H \to H$ which is the inverse of the identity in the monoid $\mathrm{Hom}_k(H, H)$ with the convolution product, then we say that H is a braided Hopf algebra and we call S the antipode of H.

Usually H denotes a braided bialgebra (understanding the structure maps), and c denotes its braid.

Remark 3.2. Assume that H is an algebra and a coalgebra and $c \in \mathrm{Aut}_k(H^2)$ is a solution of the Yang–Baxter equation, which is compatible with the algebra and coalgebra structures. Then H is a braided bialgebra with braid c iff $\Delta : H \to H \otimes_c H$ and $\epsilon : H \to k$ are morphisms of algebras.

Definition 3.3. Let H and L be braided bialgebras. A map $g : H \to L$ is a morphism of braided bialgebras if it is a morphism of algebras and coalgebras, and $c(g \otimes g) = (g \otimes g)c$.

Let H be a braided Hopf algebra. It was proved in [T1, Proposition 5.5] that c commutes with S (that is $c(S \otimes H) = (H \otimes S)c$ and $c(H \otimes S) = (S \otimes H)c$). Moreover, it is also true that $S\eta = \eta$, $S\mu = \mu(S \otimes S)c$, $\epsilon S = \epsilon$, and $\Delta S = c(S \otimes S)\Delta$. Finally, if L is another braided Hopf algebra and $g : H \to L$ is a morphism of bialgebras, then $gS = Sg$.

Example 3.4. Let V be a vector space and $c : V \otimes V \to V \otimes V$ a bijective Yang–Baxter operator. Let $T(V)$ be the tensor algebra of V. It is clear that there exists a unique Yang–

Baxter operator $c : T(V) \otimes T(V) \to T(V) \otimes T(V)$ extending $c : V \otimes V \to V \otimes V$ and compatible with the algebra structure of $T(V)$. Now, from the universal property of $T(V)$ it follows that there are unique algebra maps $\Delta : T(V) \to T(V) \otimes_c T(V)$ and $\epsilon : T(V) \to k$ such that $\Delta(v) = 1 \otimes v + v \otimes 1$ and $\epsilon(v) = 0$ for all $v \in V$. It is easy to check that $T(V)$, endowed with these structure maps, is a braided Hopf algebra with braid c and antipode map S determined by $S(v) = -v$ for all $v \in V$ and $S(xy) = \mu_{T(V)}((S \otimes S)(c(x \otimes y)))$. We let $T_c(V)$ denote this braided Hopf algebra.

Example 3.5. Let H be a braided bialgebra. If $I \subseteq H$ is an ideal, a coideal, and $c(I \otimes H + H \otimes I) \subseteq I \otimes H + H \otimes I$, then the quotient vector space H/I has a unique braided bialgebra structure such that the canonical projection $\pi : H \to H/I$ is a braided bialgebra map. Moreover, if H is a braided Hopf algebra and $S(I) \subseteq I$, then H/I is a braided Hopf algebra.

Example 3.6. Let $T_c(V)$ as in Example 3.4. If $c^2 = \mathrm{id}_V$, then the ideal of $T(V)$ generated by $v \otimes w - c(v \otimes w)$, for all $v, w \in V$, satisfies the conditions in Example 3.5. The quotient braided Hopf algebra is called the symmetric algebra of V in respect to c, and is denoted $S_c(V)$.

Example 3.7. Let $q \in k$. Recall that the r-dimensional quantum affine space $k_q[x_1, \dots, x_r]$ is the k-algebra generated by variables $x_1, \dots, x_r$ and relations $x_j x_i = q x_i x_j$, for $i < j$. This algebra is a braided Hopf algebra via the comultiplication Δ, the counit ϵ, and the braid c determined by $\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1$, $\epsilon(x_i) = 0$, and

$$c(x_j \otimes x_i) = \begin{cases} q x_i \otimes x_j & \text{if } i < j, \\ x_i \otimes x_i & \text{if } i = j, \\ q^{-1} x_i \otimes x_j & \text{if } i > j, \end{cases}$$

respectively. In fact, this is a particular case of Example 3.6. It is easy to check that

$$x_1^{m_1} \cdots x_r^{m_r} x_1^{n_1} \cdots x_r^{n_r} = q^{\sum_{i<j} n_i m_j} x_1^{m_1+n_1} \cdots x_r^{m_r+n_r},$$

$$\Delta\left(x_1^{m_1} \cdots x_r^{m_r}\right) = \sum_{h_1+l_1=m_1,\dots,h_r+l_r=m_r} q^{\sum_{i<j} h_j l_i} \prod_{v=1}^{r} \binom{m_v}{l_v} x_1^{h_1} \cdots x_r^{h_r} \otimes x_1^{l_1} \cdots x_r^{l_r},$$

$$c\left(x_1^{m_1} \cdots x_r^{m_r} \otimes x_1^{n_1} \cdots x_r^{n_r}\right) = q^{\sum_{i<j} n_i m_j - \sum_{i>j} n_i m_j} x_1^{n_1} \cdots x_r^{n_r} \otimes x_1^{m_1} \cdots x_r^{m_r}.$$

Finally, the antipode is the map $S(x_1^{m_1} \cdots x_r^{m_r}) = (-1)^{m_1+\cdots+m_r} x_1^{m_1} \cdots x_r^{m_r}$. This braided Hopf algebra is a particular example of the quantum linear spaces studied in [AS].

Remark 3.8. If H is a braided bialgebra, then $H \otimes H$ is an algebra $H \otimes_c H$, with unit $\eta \otimes \eta$ and multiplication map $(\mu \otimes \mu)(H \otimes c \otimes H)$, and it is a coalgebra $H \otimes^c H$, with counit $\epsilon \otimes \epsilon$ and comultiplication map $(H \otimes c \otimes H)(\Delta \otimes \Delta)$. Moreover, if c is involutive then $H \otimes H$, endowed with these structures, is a braided bialgebra $H \otimes_c^c H$ with braid $(H \otimes c \otimes H)(c \otimes c)(H \otimes c \otimes H)$. Finally, if c is involutive and H is a braided Hopf algebra, then $H \otimes_c^c H$ also is with antipode $S \otimes S$.

Remark 3.9. Let H be a braided Hopf algebra. A direct computation shows that $\widetilde{H} = (H, \mu\tau, \eta, \tau\Delta, \epsilon, S)$ is a braided Hopf algebra with braid $\tilde{c} := \tau c \tau$, where $\tau : H \otimes H \to H \otimes H$ denotes the flip.

Remark 3.10. If H is a braided bialgebra, then $H_c^{\mathrm{op}} := (H, \mu c^{-1}, \eta, \Delta, \epsilon)$ and $H_c^{\mathrm{cop}} := (H, \mu, \eta, c^{-1}\Delta, \epsilon)$ are braided bialgebras, with braid c^{-1}. By combining these constructions we obtain the braided bialgebras $H_c^{\mathrm{op\,cop}} := (H, \mu c^{-1}, \eta, c\Delta, \epsilon)$ and $H_c^{\mathrm{cop\,op}} := (H, \mu c, \eta, c^{-1}\Delta, \epsilon)$, with braid c. Moreover, if S is an antipode for H, then S is also an antipode for $H_c^{\mathrm{op\,cop}}$ and $H_c^{\mathrm{cop\,op}}$, and if S is bijective, then the composition inverse of S is an antipode for H_c^{op} and H_c^{cop}. For the proof see [AG, Proposition 2.2.4].

3.1. *Rigid braided Hopf algebras*

In this subsection we recall the definition and some properties of rigid braided Hopf algebras, that we will need later.

Let H be a finite-dimensional braided Hopf algebra, $\mathrm{ev} : H^* \otimes H \to k$ the evaluation map $\mathrm{ev}(\phi \otimes h) := \phi(h)$, where $\phi \in H^*$ and $h \in H$, and $\mathrm{coev} : k \to H \otimes H^*$ the coevaluation map $\mathrm{coev}(1) := \sum_i e_i \otimes e_i^*$, where $\{e_i\}$ and $\{e_i^*\}$ are bases of H and H^*, respectively, mutually dual. Lyubashenko [L1] has introduced the map $c^\flat : H^* \otimes H \to H \otimes H^*$, defined by $c^\flat := (\mathrm{ev} \otimes H \otimes H^*)(H^* \otimes c \otimes H^*)(H^* \otimes H \otimes \mathrm{coev})$.

Definition 3.11. A finite-dimensional braided Hopf algebra H is called rigid if the map $c^\flat$ is bijective.

Let H be a rigid braided Hopf algebra.

Theorem 3.12 [T2, Theorem 4.1]. *The antipode S is bijective.*

Definition 3.13. An element $t \in H$ is a left integral if $ht = \epsilon(h)t$ for all $h \in H$, and it is a right integral if $th = \epsilon(h)t$ for all $h \in H$. Let $\int_H^l$ denote the set of left integrals and let $\int_H^r$ denote the set of right integrals.

Theorem 3.14 [L2, Theorem 1.6], [FMS, Corollary 5.8], [T2, Theorem 4.6], [D1, Theorem 3]. *The sets $\int_H^l$ and $\int_H^r$ are one-dimensional vector subspaces of H.*

Theorem 3.15 [T1, Section 7]. *The sets $\int_H^l$ and $\int_H^r$ verify*

$$c\left(\int_H^l \otimes H\right) = H \otimes \int_H^l, \qquad c\left(H \otimes \int_H^l\right) = \int_H^l \otimes H,$$

$$c\left(\int_H^r \otimes H\right) = H \otimes \int_H^r, \quad \text{and} \quad c\left(H \otimes \int_H^r\right) = \int_H^r \otimes H.$$

Corollary 3.16. *There exist unique isomorphisms of braided Hopf algebras*

$$f_H^l : H \to H \quad \text{and} \quad f_H^r : H \to H$$

such that $c(h \otimes t) = t \otimes f_H^l(h)$ *and* $c(u \otimes h) = f_H^r(h) \otimes u$ *for all* $t \in \int_H^l \setminus \{0\}$, $u \in \int_H^r \setminus \{0\}$, *and* $h \in H$.

Corollary 3.17. *Let* $t \in \int_H^l$. *Since*

$$S(t)S\big(f_H^l(h)\big) = \mu(S \otimes S)c(h \otimes t) = S(ht) = \epsilon(h)S(t) = \epsilon\big(S\big(f_H^l(h)\big)\big)S(t),$$

we have $S(\int_H^l) = \int_H^r$. *In a similar way, it can be proved that* $S(\int_H^r) = \int_H^l$. *Moreover, using Corollary* 3.16*, it can be proved that* $\int_H^l = \int_{H_c^{\mathrm{cop}}}^r$ *and* $\int_H^r = \int_{H_c^{\mathrm{cop}}}^l$.

Remark 3.18. Using that c and S are compatible, $S(\int_H^l) = \int_H^r$, and $S(\int_H^r) = \int_H^l$, it is easy to see that $c(t \otimes h) = f_H^r(h) \otimes t$ and $c(h \otimes u) = u \otimes f_H^l(h)$, for all $t \in \int_H^l \setminus \{0\}$, $u \in \int_H^r \setminus \{0\}$, and $h \in H$. It follows easily that there exists $q \in k \setminus \{0\}$ such that $c(t \otimes t) = qt \otimes t$ and $c(u \otimes u) = qu \otimes u$.

Let t be a non-zero left integral. There is an algebra map $\alpha : H \to k$ such that $th = \alpha(h)t$ for all $h \in H$. This map α is called the modular function. From Corollary 3.17 and Remark 3.18 it follows that if u is a non-zero right integral, then $hu = \alpha(S(f_H^l(h)))u$.

Theorem 3.19 [T2, Section 7]. *We have* $(\alpha \otimes H)c = H \otimes \alpha$ *and* $(H \otimes \alpha)c = \alpha \otimes H$.

Let H be a rigid braided Hopf algebra and let $\phi : H^* \otimes H^* \to (H \otimes H)^*$ be the isomorphism defined by $\phi(f \otimes g)(x \otimes y) = g(x)f(y)$. Then H^* is a braided Hopf algebra, with the structure defined by $c_{H^*} := \phi^{-1}c^*\phi$, $\mu_{H^*} := \Delta^*\phi c_{H^*}^{-1}$, $\Delta_{H^*} := c_{H^*}\phi^{-1}\mu^*$, $\eta_{H^*} := \epsilon^*$, $\epsilon_{H^*} := \eta^*$, and $S_{H^*} := S^*$. For a proof of this assertion see [AG, Lemma 2.2.3].

Theorem 3.20 (Maschke's Theorem). *A rigid braided Hopf algebra* H *is semisimple iff there exists* $t \in \int_H^l$ *such that* $\epsilon(t) \neq 0$.

Using this theorem, it is easy to see that if H is semisimple, then $\int_H^l = \int_H^r$ and the maps f_H^r and f_H^l of Corollary 3.16 are the identity maps.

4. Transpositions

Let H be a braided bialgebra and let A be an algebra. In this section we introduce a particular type of twisting maps of H on A, called transpositions, which are compatible with the braided bialgebra structure of H.

4.1. Definition, basic properties, and examples

Definition 4.1. A transposition of H on A is a twisting map $s : H \otimes A \to A \otimes H$ satisfying the equation $(s \otimes H)(H \otimes s)(c \otimes A) = (A \otimes c)(s \otimes H)(H \otimes s)$ (compatibility of s with c) and compatible with the bialgebra structure of H. When H is a braided Hopf algebra we also require that $s(S \otimes A) = (A \otimes S)s$. By Proposition 4.3 below, this condition is automatically verified when s is a bijective map.

Remark 4.2. Using the compatibility of s with c, it is easy to check that s is a transposition of H on A iff it is a transposition of H_c^{op} on A and that this happens iff s is a transposition of H_c^{cop} on A. Moreover, a direct computation shows that, if s is a bijective map, then s is a transposition of H on A iff $\tilde{s}^{-1} := \tau s^{-1} \tau$ is a transposition of $\widetilde{H}$ on A^{op}, where $\widetilde{H}$ is the braided Hopf algebra of Remark 3.9 and A^{op} is the opposite algebra of A.

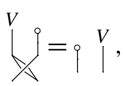
The following result is a variant of [T1, Proposition 5.5] and its proof is identical.

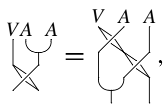
Proposition 4.3. *Let H be a braided Hopf algebra, A a vector space, and let $s : H \otimes A \to A \otimes H$ be a bijective map. If s is compatible with the bialgebra structure of H, then $s(S \otimes A) = (A \otimes S)s$.*

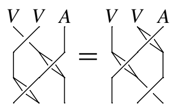
The following result generalizes [GG, Proposition 2.1.5], and it can be proved in the same way.

Proposition 4.4. *Let* $\operatorname{Prim} H$ *be the space of primitive elements in H. For each transposition $s : H \otimes A \to A \otimes H$, it is true that* $s(\operatorname{Prim} H \otimes A) \subseteq A \otimes \operatorname{Prim} H$.

Example 4.5. Let $T_c(V)$ be as in Example 3.4. A map $s : V \otimes A \to A \otimes V$ extends to a transposition of $T_c(V)$ on A iff

(i) 

, (ii) 

, and (iii) 

.

If $c^2 = \mathrm{id}$, the same fact is valid for the quantum symmetric algebra $S_c(V)$.

Remark 4.6. In relation to Example 4.5 it is useful to note that:

(1) If A is the tensor algebra $T_c(W)$, then each map $s : V \otimes W \to W \otimes V$ verifying $(s \otimes V)(V \otimes s)(c \otimes W) = (W \otimes c)(s \otimes V)(V \otimes s)$ extends to a unique map $s : V \otimes A \to A \otimes V$ satisfying conditions (i)–(iii).
(2) Assume that $c_W^2 = \mathrm{id}$. If A is the symmetric algebra $S_c(W)$, then each map $s : V \otimes W \to W \otimes V$ verifying $(s \otimes V)(V \otimes s)(c \otimes W) = (W \otimes c)(s \otimes V)(V \otimes s)$ and $(c_W \otimes V)(W \otimes s)(s \otimes W) = (W \otimes s)(s \otimes W)(V \otimes c_W)$ extends to a unique map $s : V \otimes A \to A \otimes V$ satisfying conditions (i)–(iii).

Example 4.7. Let $k_q[\partial/\partial \mathbf{X}] = k_q[\partial/\partial x_1, \dots, \partial/\partial x_r]$ and $k_q[\mathbf{X}] = k_q[x_1, \dots, x_r]$ be two r-dimensional quantum affine spaces (see Example 3.7) and let $p \in k \setminus \{0\}$. Remark 4.6(2) implies that the map $s : \langle \partial/\partial \mathbf{X} \rangle \otimes \langle \mathbf{X} \rangle \to \langle \mathbf{X} \rangle \otimes \langle \partial/\partial \mathbf{X} \rangle$, defined by

$$s\left(\frac{\partial}{\partial x_j} \otimes x_i\right) = \begin{cases} q^{-1} x_i \otimes \dfrac{\partial}{\partial x_j} & \text{if } i < j, \\ p x_i \otimes \dfrac{\partial}{\partial x_j} & \text{if } i = j, \\ q x_i \otimes \dfrac{\partial}{\partial x_j} & \text{if } i > j, \end{cases}$$

determines a transposition of $k_q[\partial/\partial \mathbf{X}]$ on $k_q[\mathbf{X}]$.

Example 4.8. Let A and B be algebras and let $A \otimes_\chi B$ be a twisted tensor product (see Example 2.6). Let $s_A : H \otimes A \to A \otimes H$ and $s_B : H \otimes B \to B \otimes H$ be transpositions. It is easy to check that if

$$(\chi \otimes H)(B \otimes s_A)(s_B \otimes A) = (A \otimes s_B)(s_A \otimes B)(H \otimes \chi),$$

then $(A \otimes s_B)(s_A \otimes B)$ is a transposition of H on $A \otimes_\chi B$.

Example 4.9. Let H be a braided bialgebra and let $s : H \otimes A \to A \otimes H$ be a transposition. Assume that the braid c of H is involutive. Using Corollary 4.21 below, it is easy to check that $(s \otimes H)(H \otimes s)$ is a transposition of $H \otimes_c^c H$ on A.

4.2. Transpositions of groups

The aim of this section is to characterize the transpositions of a group algebra $k[G]$ on an algebra A. We begin by noting that each map $s : k[G] \otimes A \to A \otimes k[G]$ determines univocally maps $\alpha_x^y : A \to A$ $(x, y \in G)$, by $s(x \otimes a) = \sum_{y \in G} \alpha_x^y(a) \otimes y$.

Proposition 4.10. *s is a transposition iff, for all $x, y, z \in G$, the following conditions hold*:

(1) α_x^v $(v \in G)$ *is a complete family of orthogonal idempotents,*
(2) $\alpha_1^1 = \mathrm{id}$,
(3) $\alpha_{xy}^z = \sum_{vw=z} \alpha_x^v \alpha_y^w$,
(4) $\alpha_x^x(1) = 1$,
(5) $\alpha_x^y(ab) = \sum_{v \in G} \alpha_x^v(a) \alpha_v^y(b)$.

Proof. It is easy to see that s is a transposition iff $\alpha_x^y = \alpha_{x^{-1}}^{y^{-1}}$ and $\alpha_x^v \alpha_y^w = \alpha_y^w \alpha_x^v$, for all $x, y, v, w \in G$, and properties (1)–(5) are verified. Hence, we must prove that (1)–(5) imply the two equalities. From (1)–(3) we have

$$\alpha_x^y = \sum_{v \in G} \alpha_x^y \alpha_x^v \alpha_{x^{-1}}^{v^{-1}} = \alpha_x^y \alpha_{x^{-1}}^{y^{-1}} = \sum_{v \in G} \alpha_x^v \alpha_{x^{-1}}^{v^{-1}} \alpha_{x^{-1}}^{y^{-1}} = \alpha_{x^{-1}}^{y^{-1}}.$$

This proves the first equality. In order to prove the second equality, it suffices to verify by (1) that $\alpha_y^{w'}\alpha_x^v\alpha_y^w = 0$ if $w \neq w'$. Since $\alpha_y^w = (\alpha_y^w)^2$ and $\alpha_x^{v'}$ ($v' \in G$) is a complete family of orthogonal idempotents, it suffices to check that $\alpha_x^{v'}\alpha_y^{w'}\alpha_x^v\alpha_y^w(a) = 0$ for each $v' \in G$, $a \in \operatorname{Im}\alpha_y^w$. Let $z' = v'w'$ and $z = vw$. First suppose $z \neq z'$. Then we have

$$0 = \alpha_{xy}^{z'}\alpha_{xy}^{z}(a) = \left(\sum_{r's'=z'} \alpha_x^{r'}\alpha_y^{s'}\right)\left(\sum_{rs=z} \alpha_x^{r}\alpha_y^{s}\right)(a) = \sum_{r's'=z'} \alpha_x^{r'}\alpha_y^{s'}\alpha_x^v\alpha_y^w(a).$$

Because the images of the maps $\alpha_x^{r'}$ ($r' \in G$) form a direct sum, from the above equations it follows that $\alpha_x^{v'}\,\alpha_y^{w'}\,\alpha_x^v\,\alpha_y^w(a) = 0$, as desired. Now, suppose $z = v'w' = vw$. Then we have

$$\alpha_x^v\alpha_y^w(a) = \sum_{rs=z} \alpha_x^r\alpha_y^s(a) = \alpha_{xy}^z(a) = \left(\alpha_{xy}^z\right)^2(a) = \sum_{r's'=z} \alpha_x^{r'}\alpha_y^{s'}\alpha_x^v\alpha_y^w(a).$$

Because the images of the maps $\alpha_x^{r'}$ ($r' \in G$) form a direct sum and $v' \neq v$, the last equations imply $\alpha_x^{v'}\alpha_y^{w'}\alpha_x^v\alpha_y^w(a) = 0$. □

Proposition 4.11. *Let $s : k[G] \otimes A \to A \otimes k[G]$ be a map. Given $x, y \in G$, we write $A_x^y = \{a \in A\colon\ s(x \otimes a) = a \otimes y\}$. We have that s is a transposition iff the family $A_x^y(s)$ verifies*:

(1) $\bigoplus_{y \in G} A_x^y = A$, *for all* $x \in G$,
(2) $A_1^1 = A$,
(3) $A_{xy}^z = \bigoplus_{vw=z} A_x^v \cap A_y^w$, *for all* $x, y, z \in G$,
(4) $1 \in A_x^x$, *for all* $x \in G$,
(5) *if* $a \in A_x^y$ *and* $b \in A_y^z$, *then* $ab \in A_x^z$, *for all* $x, y, z \in G$.

Proof. It is immediate that if s is a transposition, then (1), (2), (4), (5) are verified. It is easy to check that the maps $\alpha_x^v\alpha_y^w$ ($v, w \in G$) are a complete family of orthogonal idempotents. So,

$$\bigoplus_{z \in G} A_{xy}^z = A = \bigoplus_{v,w \in G} A_x^v \cap A_y^w.$$

Since $A_x^v \cap A_y^w \subseteq A_{xy}^{vw}$, also (3) is valid. To prove the inverse assertion it suffices to check that the idempotents $\alpha_x^y \in \operatorname{End}_k(A)$, associated with the decompositions $A = \bigoplus_{y \in G} A_x^y$, satisfy the properties enunciated in Proposition 4.10. We leave this task to the reader. □

Remark 4.12. By Proposition 4.11(3) it is easy to prove by induction on n that for each finite family $x_1, \dots, x_n$ of elements of G,

$$A = \bigoplus_{y_1,\dots,y_n \in G} A_{x_1}^{y_1} \cap \cdots \cap A_{x_n}^{y_n}.$$

Proposition 4.13. *A transposition $s : k[G] \otimes A \to A \otimes k[G]$ is bijective iff $A = \bigoplus_{x\in G} A_x^y$ for each $y \in G$. In this case*

$$s^{-1}(a \otimes x) = \sum_{y\in G} y \otimes \alpha_y^x(a) \quad \textit{for all } x \in G,\ a \in A.$$

Proof. It is immediate that s is bijective iff, for each $y \in G$ and $a \in A$, there exist unique $x_1, \dots, x_n \in G$ and $a_1, \dots, a_n \in A$ such that $s(x_i \otimes a_i) = a_i \otimes y$ and $a_1 + \cdots + a_n = a$. That is, iff $A = \bigoplus_{x\in G} A_x^y$ for all $y \in G$. The last assertion can be easily checked. □

Theorem 4.14. *Let G be a finitely generated group and let A be a k-algebra. Each transposition $s : k[G] \otimes A \to A \otimes k[G]$ determines an $\mathrm{End}(G)^{\mathrm{op}}$-gradation $A = \bigoplus_{\zeta\in \mathrm{End}(G)} A_\zeta$ on A, by*

$$A_\zeta = \bigcap_{x\in G} A_x^{\zeta(x)} = \bigl\{a \in A\colon\ s(x \otimes a) = a \otimes \zeta(x) \textit{ for all } x \in G\bigr\}.$$

The map defined in this way is bijective. Moreover, a transposition s is invertible iff $A_\zeta = 0$ for all $\zeta \in \mathrm{End}(G) \setminus \mathrm{Aut}(G)$.

Proof. Fix a transposition $s : k[G] \otimes A \to A \otimes k[G]$. Let $x_1, \dots, x_n$ be a set of generators of G. By Proposition 4.11, for each $\zeta \in \mathrm{End}(G)$, the homogeneous component of degree ζ of A is

$$A_\zeta = \bigcap_{x\in G} A_x^{\zeta(x)} = A_{x_1}^{\zeta(x_1)} \cap \cdots \cap A_{x_n}^{\zeta(x_n)}.$$

By Remark 4.12, in order to check that $A = \bigoplus_{\zeta\in \mathrm{End}(G)} A_\zeta$ it suffices to verify that if $A_{x_1}^{y_1} \cap \cdots \cap A_{x_n}^{y_n} \neq 0$ then there exists an endomorphism ζ of G verifying $\zeta(x_i) = y_i$. Let a be a non-zero element of $A_{x_1}^{y_1} \cap \cdots \cap A_{x_n}^{y_n}$ and let $K \subseteq G$ be the set of all $x \in G$ such that $a \in A_x^y$ for some (necessarily unique) $y \in G$. By Proposition 4.11(2)–(3) and the fact that $A_x^y = A_{x^{-1}}^{y^{-1}}$ for all $x, y \in G$, K is a subgroup of G. Since $x_1, \dots, x_n \in K$, we have $K = G$. We define ζ by $\zeta(x) := y$ if $a \in A_x^y$. It is easy to see that $\zeta \in \mathrm{End}(G)$. By Proposition 4.11(4)–(5), $A = \bigoplus_{\zeta\in \mathrm{End}(G)} A_\zeta$ is an $\mathrm{End}(G)^{\mathrm{op}}$-graded algebra. Conversely, given an $\mathrm{End}(G)^{\mathrm{op}}$-gradation $A = \bigoplus_{\zeta\in \mathrm{End}(G)} A_\zeta$ of A, the formula $s(x \otimes a) = a \otimes \zeta(x)$ if $a \in A_\zeta$ defines a transposition. The last assertion can be easily checked. □

4.3. Transpositions and integrals

The relation between integrals and transpositions is given in the following result.

Theorem 4.15. *Let H be a rigid braided Hopf algebra, A an algebra, and s a bijective transposition of H on A. There is a unique automorphism of algebras $g_s : A \to A$ such that $s(t \otimes a) = g_s(a) \otimes t$ for all left or right integrals $t \in H$. Moreover, we have*

$$s\big(f_H^r \otimes g_s\big) = \big(g_s \otimes f_H^r\big)s \quad \textit{and} \quad s\big(f_H^l \otimes g_s^{-1}\big) = \big(g_s^{-1} \otimes f_H^l\big)s,$$

where f_H^r and f_H^l are the maps introduced in Corollary 3.16.

Proof. It is clear that if there exists a map as g_s, then it is unique. Let $t \in H$ be a left integral. By Remark 3.18 and the compatibility of s with ϵ and μ_H, we have:

$$\begin{aligned}\epsilon(h)s(t \otimes a) &= \epsilon\big(f_H^r(h)\big)s(t \otimes a) = s\big(f_H^r(h)t \otimes a\big)\\ &= (A \otimes \mu)(s \otimes H)(H \otimes s)(c \otimes A)(t \otimes h \otimes a)\\ &= (A \otimes \mu)(A \otimes c)(s \otimes H)(H \otimes s)(t \otimes h \otimes a), \qquad (2)\end{aligned}$$

where f_H^r is the map introduced in Corollary 3.16. Let us write $s^{-1}(a \otimes h) = \sum_j h_j \otimes a_j$. By replacing $\sum_j t \otimes h_j \otimes a_j$ by $t \otimes h \otimes a$ in (2), we obtain:

$$(A \otimes \mu)(A \otimes c)(s \otimes H)(t \otimes a \otimes h) = \sum_j \epsilon(h_j)s(t \otimes a_j) = \epsilon(h)s(t \otimes a),$$

where the last equality is valid because $(\epsilon \otimes A)s^{-1} = A \otimes \epsilon$. Let us write $s(t \otimes a) = \sum_i a_i \otimes t_i$, where a_i are linearly independent. We have proved that

$$\sum_i a_i \otimes \epsilon(h)t_i = \sum_i a_i \otimes \mu c(t_i \otimes h) \quad \text{for all } h \in H.$$

Since the elements a_i are linearly independent, this equality implies that each t_i is a right integral of $H_c^{\mathrm{cop\,op}}$, and a left integral of H (Corollary 3.17). Hence, there exist elements λ_i in k such that $t_i = \lambda_i t$. So, $s(t \otimes a) = g_s(a) \otimes t$ with $g_s(a) = \sum_i \lambda_i a_i$. Since s is compatible with the algebra structure of A, the map g_s is an endomorphism of algebras. Since $S(\int_H^l) = \int_H^r$ and $s(S \otimes A) = (A \otimes S)s$, we also have $s(u \otimes a) = g_s(a) \otimes u$, when u is a right integral. A similar argument applied to the transposition $\tilde{s}^{-1}$, introduced in Remark 4.2, proves that g_s is bijective. Finally, if $s(f_H^l(h) \otimes a) = \sum_j a_j \otimes h_j$ and $s(h \otimes g_s(a)) = \sum_k a'_k \otimes h'_k$, then

$$\begin{aligned}\sum g_s(a_j) \otimes t \otimes h_j &= (s \otimes H)(H \otimes s)(c \otimes A)(h \otimes t \otimes a)\\ &= (A \otimes c)(s \otimes H)(H \otimes s)(h \otimes t \otimes a)\\ &= \sum a'_k \otimes t \otimes f_H^l\big(h'_k\big).\end{aligned}$$

Hence $(g_s \otimes H)s(f_H^l \otimes A) = (A \otimes f_H^l)s(H \otimes g_s)$. In a similar way we can see that $s(f_H^r \otimes g_s) = (g_s \otimes f_H^r)s$. □

Remark 4.16. Let $s : H \otimes A \to A \otimes H$ be a bijective transposition. If H is a rigid semisimple braided Hopf algebra, then $g_s = \mathrm{id}_A$. In fact, let t be a non-zero integral of H

and let $a \in A$ be arbitrary. Since $\epsilon(t)a = \epsilon(t)g_s(a)$ and by Maschke's Theorem $\epsilon(t) \neq 0$, we obtain $g_s(a) = a$.

Proposition 4.17. *Let H be a rigid braided Hopf algebra, $\alpha : H \to k$ the modular function of H, A an algebra, and $s : H \otimes A \to A \otimes H$ a bijective transposition. We have $(A \otimes \alpha)s = \alpha \otimes A$.*

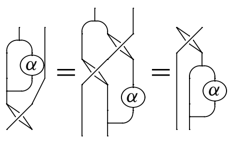
Proof. Let t be a non-zero left integral of H. Write $s(h \otimes a) = \sum a_i \otimes h_i$. Since s is compatible with the multiplication of H and, by Theorem 4.15, there is an automorphism of algebras $g_s : A \to A$ such that $s(t \otimes a) = g_s(a) \otimes t$ for all $a \in A$, we have

$$\alpha(h)g_s(a) \otimes t = s\big(\alpha(h)t \otimes a\big) = s(th \otimes a) = \sum g_s(a_i) \otimes th_i = \sum \alpha(h_i)g_s(a_i) \otimes t.$$

Hence, $\alpha(h)a = \sum \alpha(h_i)a_i$. □

Corollary 4.18. *Let H be a rigid braided Hopf algebra, $\alpha : H \to k$ the modular function of H, A an algebra, and $s : H \otimes A \to A \otimes H$ a bijective transposition. Then we have $(A \otimes \alpha \rightharpoonup (\,))s = s(\alpha \rightharpoonup (\,) \otimes A)$.*

Proof. By Proposition 4.17 and the compatibility of s with Δ, we have



as desired. □

4.4. A technical property

We finish this section by proving an elementary result which determines sufficient conditions for two diagrams to represent the same map. We will use this result repeatedly in the following sections.

Recall that the Braid Group $\mathbb{B}_r$ is the group defined by generators $\tau_1, \dots, \tau_{r-1}$ and relations

$$\tau_i \tau_j = \tau_j \tau_i \quad \text{if } |i - j| \geqslant 2, \tag{3}$$

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}. \tag{4}$$

The symmetric group S_r is the group defined by generators $\sigma_1, \dots, \sigma_{r-1}$ and relations (3), (4), and $(\sigma_i)^2 = 1$ $(1 \leqslant i < r)$. We see the elements of S_r as functions in the usual way. The canonical map $\phi : \mathbb{B}_r \to S_r$ is the morphism defined by $\phi(\tau_i) = \sigma_i$. For each $\tau \in \mathbb{B}_r$, the permutation associated with τ is by definition $\phi(\tau)$.

An element $\tau \in \mathbb{B}_r$ is simple if there exist $\tau_{i_1}, \dots, \tau_{i_n}$ such that $\tau = \tau_{i_1} \cdots \tau_{i_n}$ and $\sigma_{i_1} \cdots \sigma_{i_n}$ is a reduced expression of $\phi(\tau)$ (i.e., for each pair of indices $p < q$, there exists $0 \leqslant s \leqslant n$ such that $\sigma_{i_j} \cdots \sigma_{i_n}(p) < \sigma_{i_j} \cdots \sigma_{i_n}(q)$ for all $j > s$ and $\sigma_{i_j} \cdots \sigma_{i_n}(p) > \sigma_{i_j} \cdots \sigma_{i_n}(q)$ for all $j \leqslant s$). Such an expression of τ is called simple. A simple expression $\tau_{i_1} \cdots \tau_{i_n}$ of τ is normal if

$$\phi(\tau_{i_1} \cdots \tau_{i_j})(i) < \phi(\tau_{i_1} \cdots \tau_{i_j})(i+1) \quad \text{for all } j \text{ and each } i > i_j.$$

For example, by (4), $\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$. Both expressions are simple. However, the first is normal, while the second is not.

Proposition 4.19. *Each simple element $\tau \in \mathbb{B}_r$ has a unique normal expression.*

Proof. *Existence*. We are going to prove that if τ has a simple expression $\tau = \tau_{i_1} \cdots \tau_{i_n}$, then there exists a normal expression $\tau = \tau_{j_1} \cdots \tau_{j_n}$. By inductive hypothesis we can assume that $\tau_{i_1} \cdots \tau_{i_{n-1}}$ is a normal expression. Let m be the maximum of all the indices i such that $\phi(\tau)(i) > \phi(\tau)(i+1)$. It is clear that $i_n \leqslant m$. We divide the proof into three cases.

Case 1 $(m = i_n)$. In this case it is easy to see that $\tau_{i_1} \cdots \tau_{i_{n-1}}\tau_{i_n}$ is a normal expression of τ.

Case 2 $(m > i_n + 1)$. In this case, by using that $\tau_{i_1} \cdots \tau_{i_{n-1}}$ is a normal expression, it can be proved that $i_{n-1} = m$ and that

$$\tau_{i_1} \cdots \tau_{i_{n-1}}\tau_{i_n} = \tau_{i_1} \cdots \tau_{i_{n-2}}\tau_{i_n}\tau_{i_{n-1}}.$$

By inductive hypothesis, we can write $\tau_{i_1} \cdots \tau_{i_{n-2}}\tau_{i_n}$ in a normal form $\tau_{j_1} \cdots \tau_{j_{n-1}}$. This finishes the proof, since the resultant expression $\tau = \tau_{j_1} \cdots \tau_{j_{n-1}}\tau_{i_{n-1}}$ of τ verifies the conditions of Case 1.

Case 3 $(m = i_n + 1)$. In this case, by using that $\tau_{i_1} \cdots \tau_{i_{n-1}}$ is a normal expression, it can be proved that $i_{n-1} = m$ and that there exists $l \geqslant 1$ such that $i_{n-s} = i_n + s$ for $1 \leqslant s \leqslant l$ and $i_{n-l-1} = i_n$. So,

$$\begin{aligned} \tau_{i_1} \cdots \tau_{i_{n-1}}\tau_{i_n} &= \tau_{i_1} \cdots \tau_{i_{n-l-2}}\tau_{i_n+l} \cdots \tau_{i_n+2}\tau_{i_n}\tau_{i_n+1}\tau_{i_n} \\ &= \tau_{i_1} \cdots \tau_{i_{n-l-2}}\tau_{i_n+l} \cdots \tau_{i_n+2}\tau_{i_n+1}\tau_{i_n}\tau_{i_n+1}, \end{aligned}$$

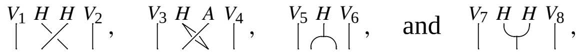
where the last equality follows from (4). To finish, it suffices to note that the last expression is normal.

Uniqueness. Suppose $\tau_{j_1} \cdots \tau_{j_{n'}} = \tau_{i_1} \cdots \tau_{i_n}$ are two normal expressions of τ. Then by definition $j_{n'} = m = i_n$. The proof can be finished immediately by induction on $\min(n, n')$. □

Corollary 4.20. *Let τ and τ' be two simple elements of $\mathbb{B}_r$. If $\phi(\tau) = \phi(\tau')$ then $\tau = \tau'$.*

Proof. Let $\tau = \tau_{i_1} \cdots \tau_{i_n}$ and $\tau' = \tau_{j_1} \cdots \tau_{j_{n'}}$ be the normal expressions of τ and τ', respectively. We note that $j_{n'} = m = i_n$, where m is as in the proof of Proposition 4.19. The proof can be finished immediately by induction on $\min(n, n')$. □

Next, we consider diagrams obtained by composition of morphisms of the form



where $V_1, \dots, V_7$ are tensor products of A and H. We enumerate the vertices at the top and the bottom of such a diagram D from left to right. Let l be a line that joins the top with the bottom of D. Let $t(l)$ and $b(l)$ denote the vertex at the top and at the bottom of l, respectively. Given vertices i at the top and j at the bottom of D, let $n_D(i, j)$ denote the cardinal of the set of descending lines that join i with j.

For instance, for the diagrams representing c and s, we have $n_D(1,1) = n_D(2,2) = 0$ and $n_D(1,2) = n_D(2,1) = 1$. We say that a *diagram D is admissible if two descending lines l and l' in D cross* (by means of c, s, or a multiplication followed by a comultiplication) *at most once*, and if such a crossing occurs, then $t(l) \neq t(l')$ and $b(l) \neq b(l')$.

Corollary 4.21. *Let D_1 and D_2 be two admissible diagrams. If*

(1) *D_1 and D_2 have the same domain and the same codomain,*
(2) *$n_{D_1}(i, j) = n_{D_2}(i, j)$ for each top vertex i and each bottom vertex j,*

then the maps represented by D_1 and D_2 coincide.

Proof. Let ϕ_1 and ϕ_2 be the maps represented by D_1 and D_2, respectively. By the compatibility of Δ and μ with c and s, and the fact that $\Delta\mu = (\mu \otimes \mu)\Delta_{H \otimes^c H}$, we can replace D_1 and D_2 by admissible diagrams that represent the same maps, but that have, from up to down, first the comultiplications, then the braiding maps, and finally the multiplications. Hence, $\phi_1 = \phi_1^M \phi_1^D \phi_1^C$ and $\phi_2 = \phi_2^M \phi_2^D \phi_2^C$, where ϕ_1^M, ϕ_2^M consist of multiplications, ϕ_1^D, ϕ_2^D are made out of braiding maps, and ϕ_1^C, ϕ_2^C consist of comultiplications. We claim that $\phi_1^C = \phi_2^C$. In fact $\sum_j n_{D_l}(i, j)$ is the number of comultiplications that occurs in the vertex i of the top of diagram D_l for $l = 1, 2$. Since $n_{D_1}(i, j) = n_{D_2}(i, j)$ for all i, j, the claim follows from coassociativity of comultiplication. Similarly, $\phi_1^M = \phi_2^M$. Finally, the fact that $\phi_1^D = \phi_2^D$ follows from Corollary 4.20. □

5. *H*-comodule algebras and *H*-module algebras

Let H be a braided bialgebra. The aim of this section is to adapt to our context the notions of H-comodule algebra and H-module algebra. Although in this section the maps s (sometimes adorned with a subscript) are not necessarily transpositions, we represent them geometrically by the same diagrams as true transpositions.

An H-braided space (V,s) is a vector space V endowed with a map $s: H\otimes V\to V\otimes H$ which is compatible with the bialgebra structure of H and verifies

$$(s\otimes H)(H\otimes s)(c\otimes V)=(V\otimes c)(s\otimes H)(H\otimes s)$$

(compatibility of s with the braid). When H is a braided Hopf algebra we also require that $s(S\otimes V)=(V\otimes S)s$. A map $g: V\to V'$ is said to be an homomorphism of H-braided spaces, from (V,s) to (V',s'), if $(g\otimes H)s=s'(H\otimes g)$. We let $\mathcal{B}_H$ denote the category of all H-braided spaces. It is easy to check that this is a monoidal category with

- unit (k,τ), where $\tau: H\otimes k\to k\otimes H$ is the flip;
- tensor product $(V,s_V)\otimes(U,s_U)=(V\otimes U,s_{V\otimes U})$, where $s_{V\otimes U}: H\otimes V\otimes U\to V\otimes U\otimes H$ is the map $s_{V\otimes U}:=(V\otimes s_U)(s_V\otimes U)$; and
- the usual associativity and unit constraints.

Note that (H,c) is a coalgebra object and an algebra object in $\mathcal{B}_H$. Hence, one can consider left and right (H,c)-modules and (H,c)-comodules in this monoidal category. To abbreviate we will say that (V,s) is a right H-comodule, to mean that it is a right (H,c)-comodule in $\mathcal{B}_H$ and that (V,s) is a left H-module, to mean that it is a left (H,c)-module in $\mathcal{B}_H$. Note that if (V,s) is a right H-comodule, then V is a right H-comodule in the standard sense. Similarly for left H-modules.

For instance, when H is a standard bialgebra and $s: H\otimes V\to V\otimes H$ is the flip, then (V,s) is a right H-comodule iff V is a right standard H-comodule, and (V,s) is a left H-module iff V is a left standard H-module.

Let V be a right standard H-comodule with coaction ν. Recall that $v\in V$ is coinvariant if $\nu(v)=v\otimes 1$. As usual, V^{coH} denotes the set of coinvariants of V.

Remark 5.1. If (V,s) is a right H-comodule, then V^{coH} is stable under s (that is, $s(H\otimes V^{\mathrm{coH}})\subseteq V^{\mathrm{coH}}\otimes H$). So, (V^{coH},s) is an H-braided space.

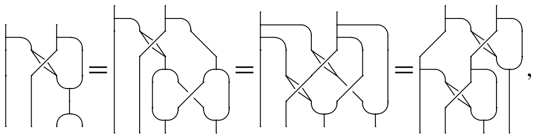
Given right H-comodules (V,s_V) and (U,s_U) with coactions ν_V and ν_U respectively, we let $\nu_{V\otimes U}$ denote the codiagonal coaction

$$\nu_{V\otimes U}:=(V\otimes U\otimes\mu)(V\otimes s_U\otimes H)(\nu_V\otimes\nu_U).$$

In the following proposition we show, in particular, that $(V,s_V)\otimes(U,s_U)$ is a right H-comodule via $\nu_{V\otimes U}$.

Proposition 5.2. *The category $(\mathcal{B}_H)^H$ of right H-comodules in $\mathcal{B}_H$, endowed with the usual associativity and unit constraints, is monoidal.*

Proof. First note that (k,τ), endowed with the trivial coaction, is an H-comodule. This is the unit of $(\mathcal{B}_H)^H$. Next, we prove that the tensor product of two H-comodules (V,s_V) and (U,s_U) is an H-comodule. It is easy to check that $\nu_{V\otimes U}$ is counitary. So, we must prove only that $\nu_{V\otimes U}$ is a morphism in $\mathcal{B}_H$ and that $\nu_{V\otimes U}$ is coassociative. We check the second assertion and leave the first one to the reader. We have



where the first equality follows from the fact that H is a braided Hopf algebra, the second one follows from the compatibility of s_U with the comultiplication of H and the coassociativity of ν_V and ν_U, and the third one follows from the fact that ν_U is a morphism in $\mathcal{B}_H$. Finally, it is immediate that the usual associativity and unit constrains are morphisms in $(\mathcal{B}_H)^H$. □

Definition 5.3. We say that (V, s) is a right H-comodule algebra if it is an algebra in $(\mathcal{B}_H)^H$.

For instance, (k, τ) and (H, c) are right H-comodule algebras and $\eta : (k, \tau) \to (H, c)$ is an H-comodule algebra homomorphism.

Remark 5.4. Let (B, s) be a right H-comodule, where B is an algebra. It is easy to see that (B, s) is a right H-comodule algebra iff s is a transposition and the coaction $\nu : B \to B \otimes_s H$ is an algebra homomorphism, where $B \otimes_s H$ is the algebra mentioned in Example 2.6.

The following result will be used in Section 10 in order to prove that a crossed product $B \#_f H$ with invertible cocycle is a free right B-module.

Proposition 5.5. *Assume that H is a braided Hopf algebra. Let $s : H \otimes B \to B \otimes H$ be an bijective transposition. By Remarks* 3.9 *and* 3.10*, $\widetilde{H_c^{\mathrm{cop}}}$ is a braided Hopf algebra with braid $\widetilde{c^{-1}} = \tau c^{-1}\tau$. Let $\tilde{s}^{-1} : \widetilde{H_c^{\mathrm{cop}}} \otimes B^{\mathrm{op}} \to B^{\mathrm{op}} \otimes \widetilde{H_c^{\mathrm{cop}}}$ be the transposition $\tilde{s}^{-1} = \tau s^{-1}\tau$ introduced in Remark* 4.2*. If (B, s) is a right H-comodule algebra with coaction ν, then $(B^{\mathrm{op}}, \tilde{s}^{-1})$ is a right $\widetilde{H_c^{\mathrm{cop}}}$-comodule algebra with coaction $\tilde{\nu} := \tau s^{-1}\nu$.*

This result generalizes [GG, Proposition 4.4], and can be proved similarly.

Given right H-modules (V, s_V) and (U, s_U), with actions ρ_V and ρ_U, respectively, we let $\rho_{V\otimes U}$ denote the diagonal action

$$\rho_{V\otimes U} := (\rho_V \otimes \rho_U)(H \otimes s_V \otimes U)(\Delta \otimes V \otimes U).$$

In the following proposition we show, in particular, that $(V, s_V) \otimes (U, s_U)$ is a left H-module via $\rho_{V\otimes U}$.

Proposition 5.6. *The category ${}_H(\mathcal{B}_H)$ of left H-modules in $\mathcal{B}_H$, endowed with the usual associativity and unit constraints, is monoidal.*

The proof is similar to that of Proposition 5.2.

Definition 5.7. We say that (V, s) is a left H-module algebra if it is an algebra in ${}_H(\mathcal{B}_H)$.

Note that if (B, s) is an H-module algebra, then s is a transposition.

6. Weak s-actions and smash products

Let H be a braided bialgebra, A an algebra, and $s : H \otimes A \to A \otimes H$ a transposition. In this section we generalize the classical concept of smash product $A \# H$ to our setting. The twisting map χ involved in the construction of smash product has the form $\chi = (\rho \otimes H)(H \otimes s)(\Delta \otimes A)$, where ρ is a map from $H \otimes A$ to A. We begin by determining the hypothesis χ must satisfy in order for the map ρ to exist. Note that ρ is uniquely determined by $\rho = (A \otimes \epsilon)\chi$.

(Proposition 6.1, Theorem 6.3, Proposition 6.4, and Proposition 6.5 below are direct generalizations of Proposition 3.2, Theorem 3.4, Lemma 3.5, and Proposition 3.6 of [GG]. All the proofs given there work in our setting.)

Proposition 6.1. *Let $\chi : H \otimes A \to A \otimes H$ be a map. The following assertions are equivalent*:

(1) *There is an arrow $\rho : H \otimes A \to A$ such that $\chi = (\rho \otimes H)(H \otimes s)(\Delta \otimes A)$.*
(2) $(\chi \otimes H)(H \otimes s)(\Delta \otimes A) = (A \otimes \Delta)\chi$.

An object (V, s) in $\mathcal{B}_H$, endowed with a map $\rho : H \otimes V \to V$, is said to be a weak left H-module in $\mathcal{B}_H$, or simply a weak left H-module, if

(1) $\rho(1 \otimes a) = a$, for all $a \in A$,
(2) $s(H \otimes \rho) = (\rho \otimes H)(H \otimes s)(c \otimes A)$.

The category ${}_{\mathrm{w}.H}(\mathcal{B}_H)$ of weak left H-modules in $\mathcal{B}_H$ becomes a monoidal category in the same way as ${}_H(\mathcal{B}_H)$. A weak left H-module algebra (A, s) is by definition an algebra in this category.

Remark 6.2. (A, s) is a weak H-module algebra iff A is an usual algebra, s is a transposition of H on A, and the structure map ρ satisfies the following conditions:

(1) $\rho(H \otimes \mu) = \mu(\rho \otimes \rho)(H \otimes s \otimes A)(\Delta \otimes A \otimes A)$,
(2) $\rho(h \otimes 1) = \epsilon(h)1$, for all $h \in H$,
(3) $\rho(1 \otimes a) = a$, for all $a \in A$,
(4) $s(H \otimes \rho) = (\rho \otimes H)(H \otimes s)(c \otimes A)$.

Let A be an algebra and $s : H \otimes A \to A \otimes H$ a transposition. A map $\rho : H \otimes A \to A$ is said to be a weak s-action of H on A if it satisfies the conditions of the above remark. An s-action is a weak s-action which satisfies

(5) $\rho(H \otimes \rho) = \rho(\mu \otimes A)$.

Note that (A, s) is an H-algebra in ${}_H(\mathcal{B}_H)$ via $\rho : H \otimes A \to A$ iff A is an usual algebra, s is a transposition of H on A, and ρ is an s-action of H on A.

Theorem 6.3. *Let $\rho : H \otimes A \to A$ be a map. The map $\chi : H \otimes A \to A \otimes H$, defined by $\chi := (\rho \otimes H)(H \otimes s)(\Delta \otimes A)$, is a twisting map of H on A iff ρ satisfies the first three conditions of Remark* 6.2. *More precisely, condition* (1) *happens iff χ is compatible with μ_A, condition* (2) *happens iff $\chi(h \otimes 1) = 1 \otimes h$ for all $h \in H$, and condition* (3) *happens iff $\chi(1 \otimes a) = a \otimes 1$, for all $a \in A$.*

In the rest of this section $\chi : H \otimes A \to A \otimes H$ is the twisting map associated with a weak s-action $\rho : H \otimes A \to A$. Note that χ is a map in $\mathcal{B}_H$ since, by Remark 3.2(4), so is ρ.

Proposition 6.4. *Let $T = (H^2 \otimes s \otimes H)(H^3 \otimes s)(\Delta_{H \otimes^c H} \otimes A)$. The twisting map χ satisfies*:

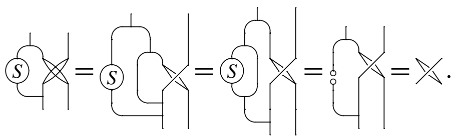
$$(\chi \otimes H)(H \otimes \chi) = \big(\rho \otimes H^2\big)\big(H \otimes \rho \otimes H^2\big)T \quad \textit{and}$$
$$\chi(\mu \otimes A) = (\rho \otimes \mu)\big(\mu \otimes A \otimes H^2\big)T.$$

Proposition 6.5. *The map ρ is an s-action iff the twisting map χ satisfies the equation*

$$\chi(\mu \otimes A) = (A \otimes \mu)(\chi \otimes H)(H \otimes \chi).$$

Hence, if ρ is an s-action, then (H, A, χ) forms a matched pair of algebras in $\mathcal{B}_H$.

Remark 6.5.1. If ρ is an s-action, then s is univocally determined by the formula $s = (\rho \otimes H)(S \otimes \chi)(\Delta \otimes A)$. Indeed, in this case we have



Definition 6.6. Let $\rho : H \otimes A \to A$ be an s-action. We define the smash product $A \# H$ as the matched product associated with the matched pair of algebras (H, A, χ). By Proposition 6.5 and Example 2.6, $A \# H$ is an associative algebra with identity $1 \# 1$.

It is easy to see that $A \simeq A \# 1$ and $H \simeq 1 \# H$. For this reason we frequently abbreviate the element $a \# h$ by ah.

Remark 6.7. When ρ is the trivial action $h \otimes a \mapsto \epsilon(h)a$, then the algebra $A \# H$ (Definition 6.6) is called the twisted tensor product by s of A with H, and is denoted $A \otimes_s H$. This notation is coherent with that in Remark 3.8.

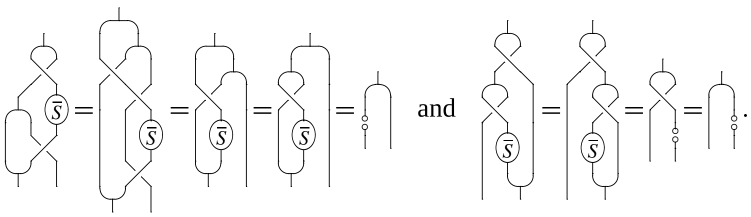
Lemma 6.8. *Let H be a braided Hopf algebra with bijective antipode S. Let $\overline{S}$ be the composition inverse of S. Then*

$$(\mu \otimes H)(H \otimes c)\big(\Delta \otimes \overline{S}\big)\, c^{-1}\Delta = (\eta\epsilon \otimes H)\Delta \quad \textit{and}$$
$$(H \otimes \mu)\big(H \otimes \overline{S} \otimes H\big)\big(c^{-1} \otimes H\big)(\Delta \otimes H)\, c^{-1}\Delta = (H \otimes \eta\epsilon)\Delta.$$

Proof. From $\Delta S = c(S \otimes S)\Delta$ it follows that

$$\mu\big(H \otimes \overline{S}\big)\, c^{-1}\Delta = \mu\big(\overline{S} \otimes H\big)\, c^{-1}\Delta = \eta\epsilon.$$

Using this, the compatibility of c^{-1} with Δ, the coassociativity of Δ and $c^{-1}\Delta$, and the fact that c commutes with S, we obtain

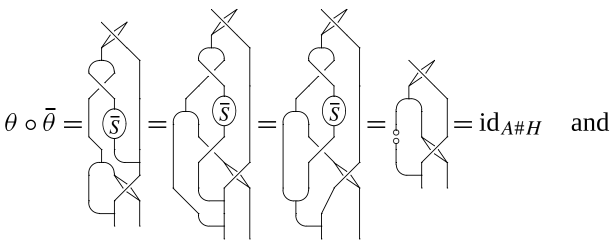
 □

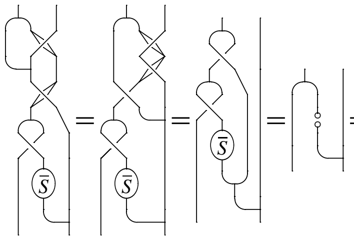
Proposition 6.9. *Let H be a braided Hopf algebra, $s : H \otimes A \to A \otimes H$ a transposition, ρ an s-action, and $A \# H$ the corresponding smash product. If the antipode S and the transposition s are bijective, then the map $\theta : H \otimes A \to A \# H$, defined by $\theta(h \otimes a) = ha$, is an isomorphism of right A-modules. Consequently, $A \# H$ is a free right A-module.*

Proof. Let $\overline{S}$ be the composition inverse of S. We assert that

$$\bar{\theta} := (H \otimes \rho)(H \otimes \overline{S} \otimes A)\big(c^{-1} \otimes A\big)(\Delta \otimes A)s^{-1}$$

is the inverse of θ. In fact, by Remark 6.2(3)–(5) and Lemma 6.8, we have



$\bar{\theta}\circ\theta =$  $= \mathrm{id}_{H\otimes A}$. □

7. The ring of invariants

Let H be a braided Hopf algebra, A an algebra, $s : H\otimes A\to A\otimes H$ a transposition, $\rho : H\otimes A\to A$ an s-action, and χ the twisting map associated with ρ. In this section we study the ring of invariants of H in A.

Definition 7.1. An element $a\in A$ is said to be invariant if $\rho(h\otimes a)=\epsilon(h)a$, for all $h\in H$. We let A^H denote the set of invariants of H in A.

Proposition 7.2. *An element $a\in A$ is invariant iff $\chi(h\otimes a)=s(h\otimes a)$, for all $h\in H$.*

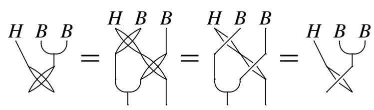
Proof. If $\chi(h\otimes a)=s(h\otimes a)$, then $\rho(h\otimes a)=(A\otimes\epsilon)\chi(h\otimes a)=(A\otimes\epsilon)s(h\otimes a)=\epsilon(h)a$. Conversely, if $\rho(h\otimes a)=\epsilon(h)a$ for all $h\in H$, then by Remark 6.2(4) and the fact that $(H\otimes\epsilon)c^{-1}=\epsilon\otimes H$,

$$\begin{aligned}\chi(h\otimes a) &= (\rho\otimes H)(H\otimes s)(\Delta\otimes A)(h\otimes a)\\ &= (\rho\otimes H)(H\otimes s)(c\otimes A)\big(c^{-1}\otimes A\big)(\Delta\otimes A)(h\otimes a)\\ &= s(H\otimes\rho)\big(c^{-1}\otimes A\big)(\Delta\otimes A)(h\otimes a)\\ &= s(h\otimes a),\end{aligned}$$

for all $h\in H$. □

Proposition 7.3. *It is true that A^H is a subalgebra of A.*

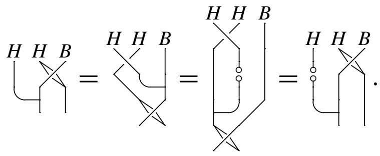
Proof. It is clear that $1\in A^H$ and A^H is closed by sums and action of scalars. Let $B=A^H$. The equalities



show that A^H is closed by products. □

Proposition 7.4. *It is true that $s(H\otimes A^H)\subseteq A^H\otimes H$. Consequently, the map $g_s : A\to A$, introduced in Theorem* 4.15, *verifies $g_s(A^H)\subseteq A^H$.*

Proof. Let $B = A^H$. By Remark 6.2(4) and the definition of A^H,



It follows easily that $s(H \otimes A^H) \subseteq A^H \otimes H$. □

Lemma 7.5. *Assume that H is a rigid braided Hopf algebra and let $t \in \int_H^l$. The map $\hat{t} : A \to A$, given by $\hat{t}(a) = t \cdot a$, is a right A^H-module map with values in A^H, where the dot denotes the s-action. Moreover, $t \cdot (ab) = g_s(a)(t \cdot b)$ for all $a \in A^H$ and $b \in A$, where $g_s : A \to A$ is the map introduced in Theorem* 4.15.

Proof. From Remark 6.2(5) and the fact that $t \in \int_H^l$, we obtain that $h \cdot (t \cdot a) = (ht) \cdot a = \epsilon(h)(t \cdot a)$ for all $h \in H$ and $a \in A$. This shows that the image of $\hat{t}$ is included in A^H. Let $a \in A$ and $b \in A^H$. For $h \in H$ and $a \in A$ we write $s(h \otimes a) = \sum_i a_i \otimes h_i$. By Remark 6.2(1), the fact that $b \in A^H$, and the compatibility of s with ϵ, we have

$$t \cdot (ab) = \sum_i (t_{(1)} \cdot a_i)(t_{(2)_i} \cdot b) = \big(t_{(1)}\epsilon(t_{(2)}) \cdot a\big)b = (t \cdot a)b.$$

The proof of the last assertion is similar. □

Definition 7.6. The map $\hat{t} : A \to A^H$ in Lemma 7.5 is called a *left trace function for H on A*.

(Items (1)–(4) of Theorem 7.7 below generalize Lemma 4.3.4, Corollary 4.3.5, Theorems 4.3.7 and 4.4.2 of [Mo], respectively. Moreover, Propositions 7.8 and 7.9 generalize Lemma 4.4.3 and Proposition 4.4.4 of [Mo]. All the proofs given there, except the one of Lemma 4.4.3, work in our setting.)

The unique point that requires some attention is the fact that the map $\mu : \mathcal{L}(W_{A\#H}) \to \mathcal{L}(V_{A^H})$ [Mo, p. 49] is A^H-linear. Next, we check this fact. Let $v \in V$, $h \in H$, and $a \in A^H$. Since $\chi(h \otimes a) = s(h \otimes a)$ and $(A \otimes \epsilon)s = \epsilon \otimes A$, we have

$$\begin{aligned} \mu(v \otimes_A ha) &= \mu\big(v \otimes_A s(h \otimes a)\big) = v \otimes_A (A \otimes \epsilon)\big(s(h \otimes a)\big) \\ &= v \otimes_A (\epsilon \otimes A)(h \otimes a) = \epsilon(h)va = \mu(v \otimes h)\,a. \end{aligned}$$

Theorem 7.7. *Assume that H is a rigid braided Hopf algebra and that $\hat{t} : A \to A^H$ is surjective. Let $c \in A$ such that $t \cdot c = 1$. The following facts hold*:

(1) *$e = tc$ is an idempotent of $A \# H$ verifying $e(A \# H)e = A^H e \simeq A^H$.*
(2) *If A is left or right Noetherian, then so is A^H.*

(3) *If A is left or right Noetherian and finitely generated as a k-algebra, then A^H is finitely generated as a k-algebra.*
(4) *If A is right Noetherian, then A is a right Noetherian A^H-module.*

Proposition 7.8. *Assume that H is a rigid braided Hopf algebra and that s is bijective. Let t be a non-zero left integral of H and let $\overline{S}$ be the composition inverse of S. Then, for all $a \in A$, $h \in H$,*

(1) *$hat = (h \cdot a)t$ and $tah = \sum_i t(\overline{S}(\alpha \rightharpoonup h)_i \cdot a_i)$, where $\alpha : H \to k$ is the modular function of H and*

$$\sum_i \overline{S}(\alpha \rightharpoonup h)_i \otimes a_i = s^{-1}\bigl(a \otimes \overline{S}(\alpha \rightharpoonup h)\bigr),$$

(2) *AtA is an ideal of $A \# H$.*

Proof. (2) follows immediately from (1). The first equality of (1) can be obtained arguing as in [Mo, Lemma 4.3.4]. We prove the second equality. Let $h \in H$ and $a \in A$. Write

$$s^{-1}(a \otimes h) = \sum_i h_i \otimes a_i \quad \text{and} \quad c^{-1}\bigl(h_{i_{(1)}} \otimes h_{i_{(2)}}\bigr) = \sum_j h_{i_{(2)_j}} \otimes h_{i_{(1)_j}}.$$

From the proof of Proposition 6.9, we obtain

$$a \# h = \sum_{i,j} \bigl(1 \# h_{i_{(2)_j}}\bigr)\bigl(\overline{S}\bigl(h_{i_{(1)_j}}\bigr) \cdot a_i \# 1\bigr).$$

Hence, from Theorem 3.19,

$$\begin{aligned}
(1 \# t)(a \# h) &= \sum_{i,j} (1 \# t)\bigl(1 \# h_{i_{(2)_j}}\bigr)\bigl(\overline{S}\bigl(h_{i_{(1)_j}}\bigr) \cdot a_i \# 1\bigr) \\
&= \sum_{i,j} \bigl(1 \# \alpha\bigl(h_{i_{(2)_j}}\bigr)t\bigr)\bigl(\overline{S}\bigl(h_{i_{(1)_j}}\bigr) \cdot a_i \# 1\bigr) \\
&= \sum_{i,j} (1 \# t)\bigl(\overline{S}\bigl(h_{i_{(1)}}\alpha\bigl(h_{i_{(2)}}\bigr)\bigr) \cdot a_i \# 1\bigr).
\end{aligned}$$

Put $(H \otimes s^{-1})(s^{-1} \otimes H)(A \otimes \Delta)(a \otimes h) = \sum_{i,j} h_{(1)_i} \otimes h_{(2)_j} \otimes a_{i_j}$. From the compatibility of s with Δ and $\overline{S}$, and from Proposition 4.17,

$$\begin{aligned}
(1 \# t)(a \# h) &= \sum_{i,j} (1 \# t)\bigl(\overline{S}\bigl(h_{(1)_i}\alpha\bigl(h_{(2)_j}\bigr)\bigr) \cdot a_{i_j} \# 1\bigr) \\
&= \sum_{i} (1 \# t)\bigl(\overline{S}\bigl(h_{(1)_i}\alpha\bigl(h_{(2)}\bigr)\bigr) \cdot a_i \# 1\bigr)
\end{aligned}$$

$$= \sum_i (1 \# t)\big(\overline{S}(\alpha \rightharpoonup h)_i \cdot a_i \# 1\big). \qquad \square$$

Proposition 7.9. *Assume that H is a rigid braided Hopf algebra. Let t be a non-zero left integral of H. The following assertions hold*:

(1) *For any $a \in A \cap AtA$, there exist $\{b_i\}, \{c_i\} \in A$ such that for all $d \in A$, $ad = \sum_{i=1}^n b_i \hat{t}(c_i d)$. Consequently, $aA \subseteq \sum_{i=1}^n b_i A^H$.*
(2) *If $AtA = A \# H$, then A is a finitely generated A^H-module.*
(3) *If $A \cap AtA$ contains a regular element of A, then A is a right A^H-submodule of a finite free A^H-module.*

8. A Morita context relating $A \# H$ and A^H

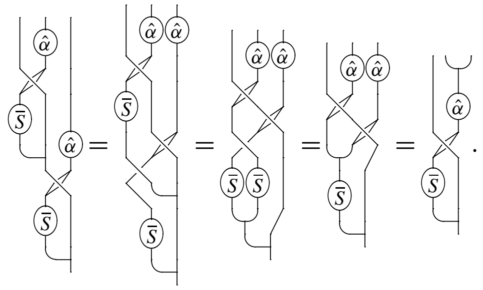
Let H be a rigid braided Hopf algebra, A an algebra, s a bijective transposition of H on A, and ρ an s-action of H on A. In this section we construct a Morita context between A^H and $A \# H$, generalizing the main result of [CFM]. It is immediate that A is a left A^H-module via $a \triangleright b := g_s^{-1}(a)b$, where $g_s : A \to A$ is the k-algebra isomorphism introduced in Theorem 4.15 and a right A^H-module by the right multiplication. Moreover, it is easy to check that A is a left $A \# H$-module via $(a \# h) \cdot b = a(h \cdot b)$, where $h \cdot b$ denotes $\rho(h \otimes b)$. With this action A becomes an $(A \# H, A^H)$-bimodule.

Proposition 8.1. *A is a right $A \# H$-module via $b \leftarrow (a \# h) = \sum_i \overline{S}((\alpha \rightharpoonup h)_i) \cdot (ba)_i$, where $\alpha : H \to k$ is the modular function of H, $\sum_i (\alpha \rightharpoonup h)_i \otimes (ba)_i = s^{-1}(ba \otimes \alpha \rightharpoonup h)$, and $\overline{S}$ is the composition inverse of S. Moreover, A is an $(A^H, A \# H)$-bimodule.*

Proof. First we prove that A is a right $A \# H$-module. For this it suffices to check that A is a right A-module, a right H-module, and that $(b \leftarrow h)a = b \leftarrow (ha)$, for all $a, b \in A$ and $h \in H$. The first assertion is evident. We check the others.

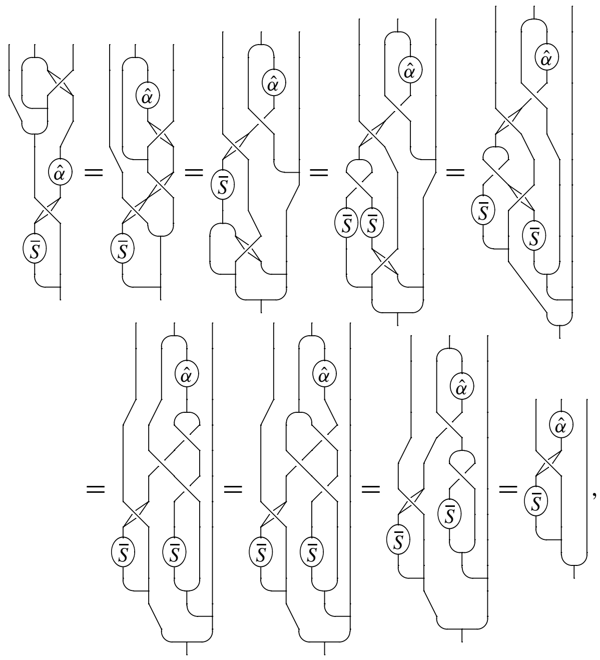
Step 1. *A is a right H-module via $b \leftarrow h = \sum_i \overline{S}((\alpha \rightharpoonup h)_i) \cdot b_i$.*

It is clear that $b \leftarrow 1 = b$. Now, we prove that $(b \leftarrow h) \leftarrow l = b \leftarrow (hl)$, for all $b \in A$ and $h, l \in H$. To abbreviate we write $\hat{\alpha}$ instead of $\alpha \rightharpoonup (\,)$. By Remark 6.2(4)–(5), the facts that $c^{-1}(\overline{S} \otimes H) = (H \otimes \overline{S})c^{-1}$, the map s is compatible with $\mu(\overline{S} \otimes \overline{S})c^{-1} = \overline{S}c\mu$, and $\hat{\alpha}$ is an algebra map, we have



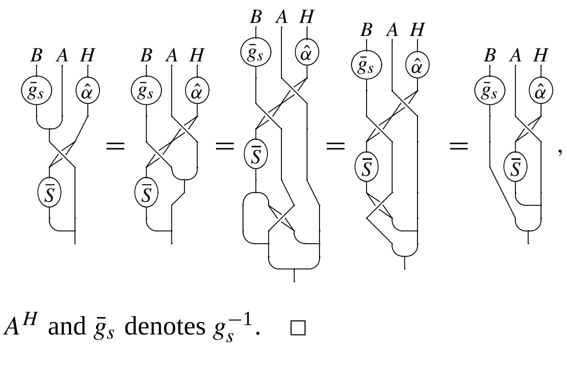
Step 2. *$b \leftarrow (ha) = (b \leftarrow h)a$, for all $a, b \in A$ and $h \in H$.*

In fact,



where the first equality follows from Corollary 4.18 and the compatibility of s with μ_A, the second one follows from Remark 6.2(1), (4), the third one from the fact that $\Delta\overline{S} = (\overline{S} \otimes \overline{S})c^{-1}\Delta$, the forth from the fact that $s(\overline{S} \otimes A) = (A \otimes \overline{S})s$ and from Remark 6.2(5), the fifth from the compatibility of s and c with $c^{-1}\,\Delta$ (see Remarks 3.10 and 4.2), the sixth from the fact that $(H \otimes \Delta)(H \otimes \hat{\alpha})\Delta = (\Delta \otimes \hat{\alpha})\Delta$, the seventh from the compatibility of c with Δ, and the eighth follows from the fact that $\mu(\overline{S} \otimes H)c^{-1}\Delta = \eta_H\epsilon_H$.

It remains to prove that the left A^H and the right $A \# H$ actions commute. It suffices to note that, by the compatibility of s with μ_A, Remark 6.2(1), Proposition 7.4, and the fact that $(\overline{S} \otimes A^H)s^{-1} = s^{-1}(A^H \otimes \overline{S})$, we have



where B denotes A^H and $\bar{g}_s$ denotes g_s^{-1}. □

Lemma 8.2. *Let $q \in k \setminus \{0\}$ such that $c(t \otimes t) = qt \otimes t$ for each $t \in \int_H^l$. Then it is true that $qt \cdot g_s(a) = g_s(t \cdot a)$, for all $a \in A$.*

Proof. Since by Remark 6.2(4), $c(t \otimes t) = qt \otimes t$, we have

$$g_s(t \cdot a) \otimes t = s(t \otimes t \cdot a) = (\cdot \otimes H)(H \otimes s)(c \otimes A)(t \otimes t \otimes a) = qt \cdot g_s(a) \otimes t.$$

The assertion follows immediately. □

In the proof of the next lemma we follow closely the arguments in [DNR, p. 224].

Lemma 8.3. *For each $t \in \int_H^l$ it is true that $S(t) = qt_{(1)}\alpha(t_{(2)})$, where $q \in k \setminus \{0\}$ is as in Lemma* 8.2.

Proof. Let $\varphi : H^* \to H$ be the map defined by $\varphi(h^*) = t_{(1)}h^*(t_{(2)})$. Since φ is bijective, there exists $T \in H^*$ such that $t_{(1)}T(t_{(2)}) = 1$. Applying ϵ we obtain $T(t) = 1$. Hence, we have

$$\begin{aligned} S(h) &= S(h_{(1)})\epsilon(h_{(1)})t_{(1)}T(t_{(2)}) = \sum_i S(h_{(1)})h_{(2)}t_{(1)_i}T(h_{(3)_i}t_{(2)}) \\ &= \sum_i \epsilon(h_{(1)})t_{(1)_i}T(h_{(2)_i}t_{(2)}) = \sum_i t_{(1)_i}T(h_i t_{(2)}), \end{aligned}$$

where $\sum_i l_i \otimes h_i$ denotes $c(h \otimes l)$. Consequently, by Corollary 3.17, Remark 3.18, and Theorem 3.19,

$$\begin{aligned} S(t) &= f_H^r(t_{(1)})T(tt_{(2)}) = f_H^r(t_{(1)})\alpha\big(f_H^r(t_{(2)})\big)T(t) = qt_{(1)}\alpha(t_{(2)})T(t) \\ &= qt_{(1)}\alpha(t_{(2)}). \qquad \square \end{aligned}$$

Recall that in a Morita context connecting two rings R and S consists of an (R, S)-bimodule M, an (S, R)-bimodule N, and two bimodule maps

$$[\,,\,] : N \otimes_R M \to S \quad \text{and} \quad (\,,\,) : M \otimes_S N \to R,$$

such that $m \cdot [n, m'] = (m, n) \cdot m'$ and $[n, m] \cdot n' = n \cdot (m, n')$, for all $m, m' \in M$ and $n, n' \in N$.

Theorem 8.4. *The bimodules $M = {}_{A^H} A_{A\#H}$ and $N = {}_{A\#H} A_{A^H}$, together with the maps*

$$\begin{aligned} &[\,,\,] : N \otimes_{A^H} M \to A \# H, \quad \textit{given by} \quad [a, b] = atb, \\ &(\,,\,) : M \otimes_{A\#H} N \to A^H, \quad \textit{given by} \quad (a, b) = t \cdot (ab), \end{aligned}$$

give a Morita context for A^H and $A \# H$.

Proof. By Proposition 7.2 and Theorem 4.15, $ta = g_s(a)t$ for all $a \in A^H$. It follows easily that $[\,,\,]$ is middle A^H-linear. Now, we prove that $[\,,\,]$ is an $A\,\#\,H$-bimodule map. Since $hbt = (h \cdot b)t$, for all $b \in A$ and $h \in H$, we have

$$(a\,\#\,h)[b,c] = (a\,\#\,h)btc = ahbtc = a(h\cdot b)tc = \bigl[a(h\cdot b), c\bigr] = \bigl[(a\,\#\,h)\cdot b, c\bigr].$$

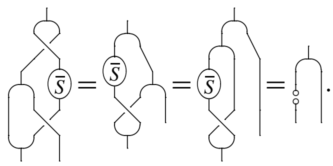
Thus, $[\,,\,]$ is a left $A\,\#\,H$-module map. Let us see that $[\,,\,]$ is a right $A\,\#\,H$-module map. Let $b \in A$ and $h \in H$. By Proposition 6.9,

$$bh = \sum_{i,j} h_{i\,(2)\,j}\bigl(\overline{S}\bigl(h_{i\,(1)\,j}\bigr)\cdot b_i\bigr),$$

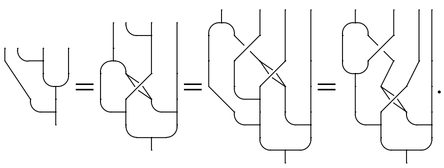
where $\sum_i h_i \otimes b_i = s^{-1}(b \otimes h)$ and $\sum_j h_{i\,(2)\,j} \otimes h_{i\,(1)\,j} = c^{-1}(h_{i\,(1)} \otimes h_{i\,(2)})$. Hence, by Theorem 3.19. the compatibility of s with Δ and Proposition 4.17,

$$\begin{aligned}
[a,b](c\,\#\,h) &= atbch = at\sum_{i,j} h_{i\,(2)\,j}\bigl(\overline{S}\bigl(h_{i\,(1)\,j}\bigr)\cdot (bc)_i\bigr)\\
&= a\sum_{i,j}\alpha\bigl(h_{i\,(2)\,j}\bigr)t\bigl(\overline{S}\bigl(h_{i\,(1)\,j}\bigr)\cdot (bc)_i\bigr) = \Bigl[a, \sum_i \alpha(h_{i\,(2)})\overline{S}(h_{i\,(1)})\cdot (bc)_i\Bigr]\\
&= \Bigl[a, \sum_i \overline{S}(\alpha \rightharpoonup h_i)\cdot (bc)_i\Bigr] = \Bigl[a, \sum_i \overline{S}\bigl((\alpha \rightharpoonup h)_i\bigr)\cdot (bc)_i\Bigr]\\
&= \bigl[a, b \leftarrow (c\,\#\,h)\bigr].
\end{aligned}$$

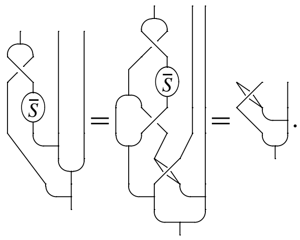
Now, we prove that $(\,,\,)$ is middle $A\,\#\,H$-linear. Since $(H \otimes \overline{S})c^{-1} = c^{-1}(\overline{S} \otimes H)$, Δ is compatible with c, and $\mu c^{-1}(\overline{S} \otimes H)\Delta = \epsilon\eta$, we have



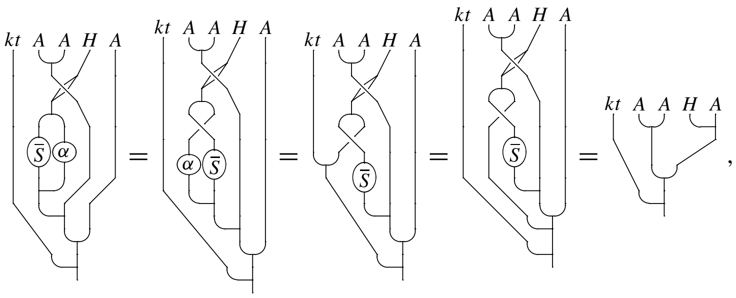
On the other hand, by (1), (4), (5) of Remark 6.2 we have



Thus,



Using that and Theorem 3.19, we obtain



which shows that $(c \leftarrow (a \# h), b) = (c, (a \# h) \cdot b)$, for all $a, b, c \in A$ and $h \in H$.

Let us see that $(,)$ is left and right A^H-linear. Let $a, b \in A$ and $c \in A^H$. By Remark 6.2(1), Proposition 7.4, and Theorem 4.15, we have

$$\begin{aligned}
(c \triangleright a, b) &= t \cdot \bigl(g_s^{-1}(c)ab\bigr) \\
&= \mu(\rho \otimes \rho)(H \otimes s \otimes A)\bigl(\Delta(t) \otimes g_s^{-1}(c) \otimes ab\bigr) \\
&= \mu(\rho \otimes \rho)(\eta\epsilon \otimes s \otimes A)\bigl(\Delta(t) \otimes g_s^{-1}(c) \otimes ab\bigr) \\
&= c\bigl(t \cdot (ab)\bigr) = c(a, b).
\end{aligned}$$

Similarly $(,)$ is right A^H-linear.

To finish the proof we must see that

$$a \leftarrow [b, c] = (a, b) \triangleright c \quad \text{and} \quad [a, b] \cdot c = a(b, c),$$

for all $a, b, c \in A$. Let $q \in k \setminus \{0\}$ defined by $c(t \otimes t) = qt \otimes t$. Using the compatibility of s^{-1} with S, Theorem 4.15, and Lemmas 8.2 and 8.3, we obtain

$$a \leftarrow [b, c] = (a \leftarrow b \# t) \leftarrow c \# 1 = q^{-1}\bigl(t \cdot g_s^{-1}(ab)\bigr)c = g_s^{-1}\bigl(t \cdot (ab)\bigr)c = (a, b) \triangleright c,$$

and from Remark 6.2(1)

$$[a,b]\cdot c = atb\cdot c = \sum_i \big((at_{(1)}\cdot b_i)\,\#\,t_{(2)_i}\big)\cdot c = \sum_i (at_{(1)}\cdot b_i)(t_{(2)_i}\cdot c)$$
$$= a\big(t\cdot(bc)\big) = a(b,c),$$

where $\sum_i t_{(1)}\otimes b_i\otimes t_{(2)_i} = \sum t_{(1)}\otimes s(t_{(2)}\otimes b)$. $\square$

Corollary 8.5. *If $\hat{t}: A\to A^H$ is surjective and $AtA = A\,\#\,H$, then $A\,\#\,H$ is Morita equivalent to A^H.*

9. Normal cocycles and crossed products

Let H be a braided bialgebra and A an algebra. In this section we introduce the notion of crossed product. The twisting map χ and the cocycle $\mathcal{F}$ involved in the construction of the crossed product have the form

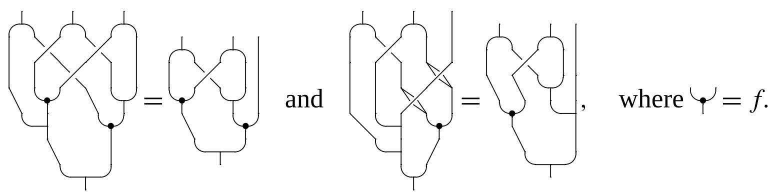
$$\chi = (\rho\otimes H)(H\otimes s)(\Delta\otimes A) \quad \text{and} \quad \mathcal{F} = (f\otimes\mu)\Delta_{H\otimes^c H},$$

where $s: H\otimes A\to A\otimes H$ is a transposition, ρ is a map from $H\otimes A$ to A, $H\otimes^c H$ is the coalgebra of Remark 3.8, and f is a map from H^2 to A. In Section 6 we have established what hypothesis χ must satisfy in order for the map ρ to exist, and we also proved that χ is a twisting map iff ρ satisfies the first three conditions of Remark 6.2. In this section we assume that ρ is a weak s action and $\chi = (\rho\otimes H)(H\otimes s)(\Delta\otimes A)$, and we study the relations between the maps F and f. We start by determining the conditions which F must satisfy in order for the map f to exist. Note that in this case f is uniquely determined by the formula $f = (A\otimes\epsilon)\mathcal{F}$. In a further step, we establish necessary and sufficient conditions on f for $\mathcal{F}$ to be a normal cocycle satisfying the twisted module condition in respect to χ.

Proposition 9.1. *Let $\mathcal{F}: H\otimes H\to A\otimes H$ be a map. There is a map $f: H\otimes H\to A$ such that $\mathcal{F} = (f\otimes\mu)\Delta_{H\otimes^c H}$ iff $(\mathcal{F}\otimes\mu)\Delta_{H\otimes^c H} = (A\otimes\Delta_H)\mathcal{F}$.*

The proof is left to the reader.

Definition 9.2. Let $f: H\otimes H\to A$ be a map. We say that f is normal if $f(1\otimes x) = f(x\otimes 1) = \epsilon(x)$ for all $x\in H$, and that f is a cocycle that satisfies the twisted module condition if



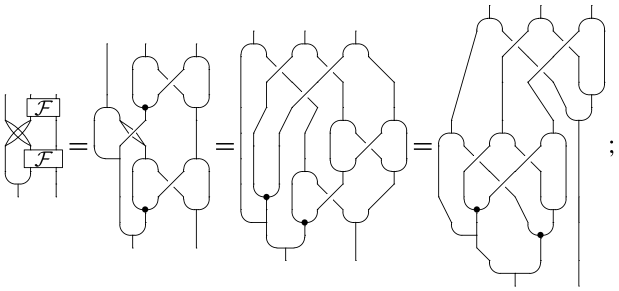
More precisely, the first equality is the cocycle condition and the second is the twisted module condition. Finally, we say that f is compatible with s if it is a map in $\mathcal{B}_H$, in other words, if

$$(f \otimes H)(H \otimes c)(c \otimes H) = s(H \otimes f).$$

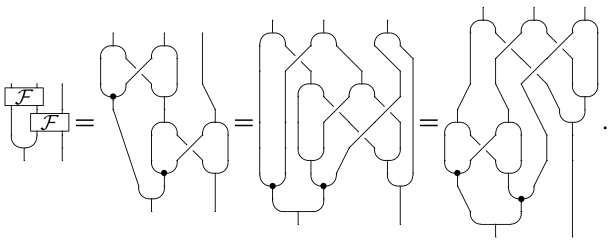
Let $f : H^2 \to A$ be a map and let $\mathcal{F} : H \otimes H \to A \otimes H$ be the map $\mathcal{F} := (f \otimes \mu)\, \Delta_{H \otimes^c H}$. It is immediate that $\mathcal{F}$ is a map in $\mathcal{B}_H$ iff f is compatible with s.

Theorem 9.3. *Let $f : H^2 \to A$ be a map and let $\mathcal{F} := (f \otimes \mu)\Delta_{H \otimes^c H}$. Assume that f is compatible with s. Then $\mathcal{F}$ is a normal cocycle satisfying the twisted module condition iff f is.*

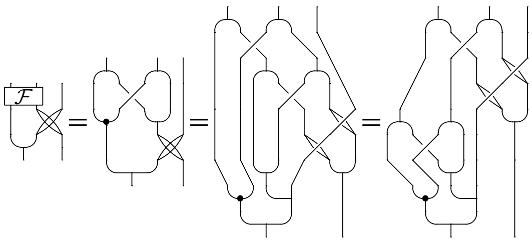
Proof. Clearly $\mathcal{F}$ is normal iff f is. Let us consider the cocycle conditions. By the compatibility of f with s, the fact that $\Delta\mu = (\mu \otimes \mu)(H \otimes c \otimes H)(\Delta \otimes \Delta)$, and Corollary 4.21, we have:



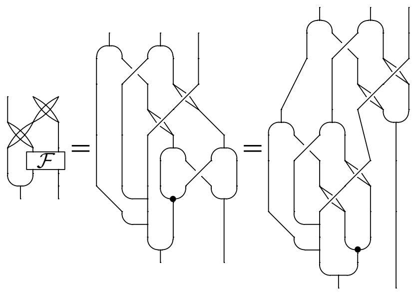
and by the fact that $\Delta\, \mu = (\mu \otimes \mu)(H \otimes c \otimes H)(\Delta \otimes \Delta)$, the compatibility of μ with c, and Corollary 4.21, we have



The above imply that the cocycle conditions on $\mathcal{F}$ and f are equivalent. To finish the proof, it remains to show that $\mathcal{F}$ satisfies the twisted module condition iff f does. To check this, we do not use the fact that f is compatible with s. From the second formula of Proposition 6.4 and Corollary 4.21, we obtain:



from the first formula of Proposition 6.4 and Corollary 4.21, we obtain:



From these equalities it follows easily that the twisted module conditions on $\mathcal{F}$ and f are equivalent. □

Definition 9.4. Let H be a braided bialgebra, A an algebra, s a transposition of H on A, $\rho : H \otimes A \to A$ a weak s-action, χ the twisting map associated with ρ, $f : H^2 \to A$ a normal cocycle compatible with s satisfying the twisted module condition, and $\mathcal{F} : H \otimes H \to A \otimes H$ the map $\mathcal{F} = (f \otimes \mu)\Delta_{H\otimes^c H}$. By definition, the crossed product associated with (s, ρ, f) is the algebra $A \#_f H$, constructed from χ and $\mathcal{F}$ in Definition 2.1. By Theorems 2.3, 6.3, and 9.3 we know that $A \#_f H$ is associative and unitary.

Next we consider several examples of crossed products constructed from data which satisfies the conditions of Definition 9.4.

Example 9.5. When H is a standard Hopf algebra and s is the flip, Definition 9.4 gives the classical crossed products introduced in [BCM,DT]. When H is a Hopf algebra in a braided category $\mathcal{C}$ whose underlying monoidal category is of vector spaces and s is the braid of $\mathcal{C}$, then we obtain the algebra crossed products underlying the bialgebra crossed products considered in [M].

Example 9.6. Suppose f is trivial (that is, $f(h \otimes l) = \epsilon(h)\epsilon(l)$ for all $h, l \in H$). Then f is automatically a normal cocycle and the crossed product condition holds iff ρ is an s-action. It follows that the crossed products with f trivial are the smash products introduced in Section 6. In [GG] we show that in this sense the Ore extensions $A[x, \alpha, \delta]$, with homeomorphism $\alpha : A \to A$ and an α-derivation $\delta : A \to A$ such that $\alpha\delta = \delta\alpha$, are smash products. Another example is the algebra of differential operators $D_{q,p}(\mathbf{X}, \partial/\partial\mathbf{X})$ ($p, q \in k \setminus \{0\}$), which is the algebra generated by the variables $x_1, \ldots, x_r, \partial/\partial x_1, \ldots, \partial/\partial x_r$, and the relations

$$x_j x_i = q x_i x_j, \qquad \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} = q \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}, \qquad \frac{\partial}{\partial x_j} x_i = q^{-1} x_i \frac{\partial}{\partial x_j} \quad \text{if } i < j,$$
$$\frac{\partial}{\partial x_i} x_i = p_i x_i \frac{\partial}{\partial x_i} + 1, \qquad \frac{\partial}{\partial x_j} x_i = q x_i \frac{\partial}{\partial x_j} \quad \text{if } i > j.$$

Let s be the transposition of $k_q[\partial/\partial\mathbf{X}]$ on $k_q[\mathbf{X}]$ (considered in Example 4.7), and for $1 \leqslant j \leqslant r$ let $\delta_j^{(p)} : k_q[\mathbf{X}] \to k_q[\mathbf{X}]$ be the map defined by

$$\delta_j^{(p)}\left(x_1^{n_1} \cdots x_r^{n_r}\right) = \begin{cases} [n_j]_p x_1^{n_1} \cdots x_j^{n_j - 1} \cdots x_r^{n_r} & \text{if } n_j > 0, \\ 0 & \text{if } n_j = 0, \end{cases}$$

where $[n]_p = 1 + n + \cdots + n^{p-1}$. It is easy to check that the formula

$$\rho\left(\frac{\partial}{\partial x_j} \otimes P\right) = \delta_j^{(p)}(P)$$

defines an s-action of $k_q[\partial/\partial\mathbf{X}]$ on $k_q[\mathbf{X}]$, and that $D_{q,p}(\mathbf{X}, \partial/\partial\mathbf{X})$ is isomorphic to the smash product constructed from these data.

Example 9.7. Suppose ρ is the trivial s-action $\rho(h \otimes a) = \epsilon(h)a$. Then the twisted module condition is satisfied iff $\mu(A \otimes f)(s \otimes H)(H \otimes s) = \mu(f \otimes A)$. If this equality is valid and f is a normal cocycle compatible with s (for the trivial action), then $A \#_f H$ is a crossed product denoted by $A_f^s[H]$. If f is also trivial, then $A^s[H] := A_f^s[H]$ equals $A \otimes_s H$.

Example 9.8. Let G be a finitely generated group, A a k-algebra, and $s : k[G] \otimes A \to A \otimes k[G]$ a bijective transposition. By Theorem 4.14 there is an $\mathrm{Aut}(G)^{\mathrm{op}}$-gradation $A = \bigoplus_{\zeta \in \mathrm{Aut}(G)} A_\zeta$ on A such that $s(x \otimes a) = a \otimes \zeta(x)$ for all $a \in A_\zeta$. It is easy to check that a map $g \otimes a \mapsto g \cdot a$ is a weak s-action of $k[G]$ on A iff

(1) $x \cdot (ab) = (x \cdot a)(\zeta(x) \cdot b)$, for $a \in A_\zeta$ and $x \in G$,
(2) $x \cdot 1 = 1$, for all $x \in G$,
(3) $1 \cdot a = a$, for all $a \in A$,
(4) $x \cdot a \ \in A_\zeta$, for all $x \in G$, $a \in A_\zeta$;

and that a map $f : k[G] \otimes k[G] \to A$ is a normal cocycle compatible with s, satisfying the twisted module condition, iff

(1) $\mathrm{Im}\, f \subseteq A_{\mathrm{id}}$,
(2) $f(x \otimes 1) = f(1 \otimes x) = 1$, for all $x \in G$,

(3) $xf(y \otimes z)f(x \otimes yz) = f(x \otimes y)f(xy \otimes z)$, for all $x, y, z \in G$,
(4) $(x \cdot (y \cdot a))f(\zeta(x) \otimes \zeta(y)) = f(x \otimes y)((xy) \cdot a)$, for all $x, y \in G$, $a \in A_\zeta$.

The multiplication map of the crossed product $A \#_f k[G]$, constructed from these data, is given by

$$(a \# x)(b \# y) = a(x \cdot b)f\big(\zeta(x) \otimes y\big) \# \zeta(x)y, \quad \text{if } b \in A_\zeta. \tag{5}$$

The group of automorphisms $\mathrm{Aut}(G)$ acts on G^{op} via $\zeta \cdot x = \zeta(x)$. Consider the semidirect product $G^{\mathrm{op}} \rtimes \mathrm{Aut}(G)$. From (5) it follows immediately that $A \#_f k[G]$ is a $(G^{\mathrm{op}} \rtimes \mathrm{Aut}(G))^{\mathrm{op}}$-graded algebra, with $A_\zeta \otimes x$ as the homogeneous component of degree (x, ζ).

We finish this section showing that if f is convolution invertible, then the map s univocally determined.

Theorem 9.9. *Let H be a braided Hopf algebra, A an algebra, $\chi : H \otimes A \to A \otimes H$ a twisting map, and $\mathcal{F} : H \otimes H \to A \otimes H$ a normal cocycle satisfying the twisted module condition. If there exists a map $s : H \otimes A \to A \otimes H$ compatible with the coalgebra structure of H, a map $\rho : H \otimes A \to A$ satisfying $\rho(1 \otimes a) = a$ for all $a \in A$, and a convolution invertible map $f : H \otimes^c H \to A$ such that*

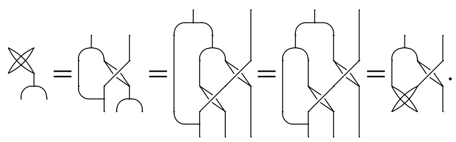
$$\chi = (\rho \otimes H)(H \otimes s)(\Delta \otimes A) \quad \text{and} \quad \mathcal{F} = (f \otimes \mu)\Delta_{H \otimes^c H},$$

then

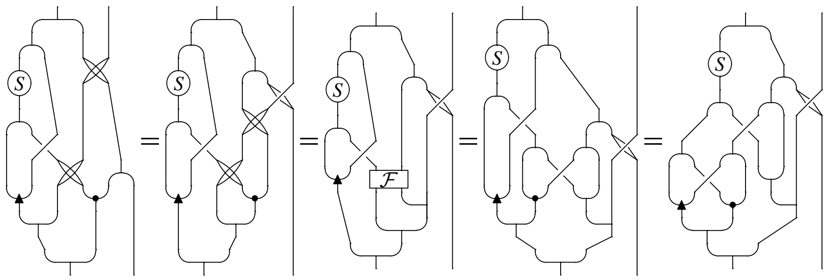
$$\rho = (A \otimes \epsilon)\chi, \qquad f = (A \otimes \epsilon)\mathcal{F}, \quad \text{and}$$

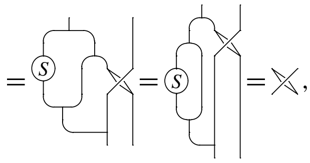
$$\begin{aligned} s = {} & \big(\mu^2 \otimes H\big)\big(f^{-1} \otimes A \otimes f \otimes H\big)(H \otimes H \otimes \chi \otimes \Delta)(H \otimes c \otimes A \otimes H) \\ & \times (\Delta \otimes H \otimes A \otimes H)(S \otimes H \otimes \chi)\big(\Delta^2 \otimes A\big). \end{aligned}$$

Proof. The formula for f is immediate, and using the compatibility of s with ϵ it is easy to check the formula for ρ. It remains to prove the assertion about s. By the compatibility of s with Δ, we have:



Using this, the twisted module condition of $\mathcal{F}$, Corollary 4.4.3 of [GG], and the fact that $\rho(1 \otimes a) = a$ for all $a \in A$, we obtain:



, where  $= f^{-1}$. □

10. Intrinsic characterizations

Let H be a braided Hopf algebra. In this section we adapt to our context the notions of cleft, H-Galois, and normal H-extensions. We also prove that the characterization of crossed product $A\,\#_f\, H$ with convolution invertible cocycle, as an H-Galois normal extension and as a cleft extension, remains valid in our setting. As in [S], we recall a well-known result [Mo, p. 91]:

Lemma 10.1. *Let R be an algebra, C a coalgebra, and let $\mathrm{End}_R^C(R \otimes C)$ be the k-algebra of all left R-linear and right C-colinear endomorphisms of $R \otimes C$. The map $T_R^C : \mathrm{Hom}_k(C, R) \to \mathrm{End}_R^C(R \otimes C)$, given by $T_R^C(g)(r \otimes c) = rg(c_{(1)}) \otimes c_{(2)}$, is an anti-isomorphism of algebras (here $\mathrm{Hom}_k(C, R)$ is considered as an algebra via the convolution product and $\mathrm{End}_R^C(R \otimes C)$ is considered as an algebra via the composition of endomorphisms). The inverse map of T_R^C is given by $(T_R^C)^{-1}(g)(c) = (R \otimes \epsilon)g(1 \otimes c)$.*

Propositions 10.3 and 10.4 are direct generalizations of Propositions 4.7 and 4.8 of [GG]. The proofs given there work in our setting.

Definition 10.2. Let H be a braided Hopf algebra, (B, s) a right H-comodule algebra, and $i : A \hookrightarrow B$ an algebra inclusion. We say that $(i : A \hookrightarrow B, s)$ is an H-extension of A if $i(A) = B^{\mathrm{coH}}$. Let $(i' : A \hookrightarrow B', s')$ be another H-extension of A. We say that $(i : A \hookrightarrow B, s)$ and $(i' : A \hookrightarrow B', s')$ are equivalent if there is an H-comodule algebra isomorphism $f : (B, s) \to (B', s')$ which is also a left A-module homomorphism.

Proposition 10.3. *Let s be a transposition of H on A, $\rho : H \otimes A \to A$ a weak s-action, and $f : H^2 \to A$ a normal cocycle compatible with s verifying the twisted module condition. Let $A\,\#_f\, H$ be the crossed product associated with (s, ρ, f). The map $\hat{s} := (A \otimes c)(s \otimes H)$ is a transposition of H on $A\,\#_f\, H$.*

Proposition 10.4. *$(A \#_f H, \hat{s})$ is an H-comodule algebra via $\nu := A \otimes \Delta$. Moreover, $(A \hookrightarrow A \#_f H, \hat{s})$ is an H-extension and the map $\gamma : (H, c) \to (A \#_f H, \hat{s})$, defined by $\gamma(h) = 1 \# h$, is an H-comodule homomorphism.*

Definition 10.5. An H-extension $(i : A \hookrightarrow B, s)$ of A is a cleft if there is a convolution invertible H-comodule homomorphism $\gamma : (H, c) \to (B, s)$; it is H-Galois if the map $\beta_B : B \otimes_A B \to B \otimes H$, defined by $\beta(b \otimes b') = (b \otimes 1)\nu(b')$ where ν denotes the coaction of B, is bijective; and it has the normal basis property if there exists a left A-linear right H-comodule isomorphism $\phi : (A \otimes H, \hat{s}) \to (B, s)$, where the coaction of $A \otimes H$ is $A \otimes \Delta$ and $\hat{s} = (A \otimes c)(s \otimes H)$.

Let $\gamma : (H, c) \to (B, s)$ be a cleft map. For $h \in H$, write $s(h \otimes \gamma^{-1}(1)) = \sum_i \gamma_i^{-1} \otimes h_i$. Since $s(h \otimes \gamma(1)) = \gamma(1) \otimes h$, we have that $\gamma(1)\gamma^{-1}(1) \otimes h = 1 \otimes h = s(h \otimes 1) = s(h \otimes \gamma(1)\gamma^{-1}(1)) = \sum_i \gamma(1)\gamma_i^{-1} \otimes h_i$. Hence, $s(h \otimes \gamma^{-1}(1)) = \gamma^{-1}(1) \otimes h$. Using this fact we obtain that $\gamma' := \gamma(1)^{-1}\gamma$ is a cleft map verifying $\gamma'(1) = 1$.

If $(A \hookrightarrow B, s)$ is an H-extension with a normal basis $\phi : (A \otimes H, \hat{s}) \to (B, s)$ satisfying $\phi(1 \otimes 1) = 1$, then B is isomorphic via ϕ to the crossed product $A \#_f H$ constructed from the transposition $s_A : H \otimes A \to A \otimes H$ induced by s (see Remark 5.1), the weak s_A-action $\rho(h \otimes a) = (A \otimes \epsilon)\phi^{-1}(\phi(1 \otimes h)\phi(a \otimes 1))$, and the cocycle $f(h \otimes l) = (A \otimes \epsilon)\phi^{-1}(\phi(1 \otimes h)\phi(1 \otimes l))$. In fact, arguing as in [S, Section 3], we see that the multiplication map $\mu_{A\#_f H}$ of $A \otimes H$, obtained by transporting through ϕ the multiplication map of B, has the form

$$\mu_{A\#_f H} = (\mu \otimes H)(\mu \otimes \mathcal{F})(A \otimes \chi \otimes A),$$

where $\chi = (\rho \otimes H)(H \otimes s_A)(\Delta \otimes A)$ and $\mathcal{F} = (f \otimes \mu)\Delta_{H \otimes^c H}$. By Theorems 2.3 and 6.3, we know that ρ satisfies the first three conditions of Remark 6.2. Using that $\widehat{s_A} : H \otimes A \#_f H \to A \#_f H \otimes H$ is compatible with $\mu_{A\#_f H}$, it is easy to check that it also satisfies Remark 6.2(4), and that the cocycle f is compatible with s_A. Finally, from Theorems 2.3 and 9.3 it follows that f is a normal cocycle satisfying the twisted module condition. Conversely, it is clear that each crossed product is an H-extension, which has the normal basis property in an obvious way.

Theorem 10.6. *Let H be a braided Hopf algebra and $(A \hookrightarrow B, s)$ an H-extension. The following assertions are equivalent*:

1. *$(A \hookrightarrow B, s)$ is cleft.*
2. *$(A \hookrightarrow B, s)$ is H-Galois with a normal basis.*
3. *There is an isomorphism $(B, s) \to (A \#_f H, \hat{s})$, where $A \#_f H$ is a crossed product whose cocycle $f : H \otimes^c H \to A$ is convolution invertible.*

Proof. (1) $\Leftrightarrow$ (2). Let $\gamma : (H, c) \to (B, s)$ be a cleft map. Then, $\nu\gamma = (\gamma \otimes H)\Delta$ is a convolution invertible map, since $\nu : B \to B \otimes_s H$ is an algebra map. Moreover, for $b \in B$,

$$T^H_{B\otimes_s H}(\nu\gamma)\big(\nu\big(b_{(0)}\gamma^{-1}(b_{(1)})\big)\otimes b_{(2)}\big) = T^H_{B\otimes_s H}(\nu\gamma)\big(T^H_{B\otimes_s H}\big(\nu\gamma^{-1}\big)\big(\nu(b_{(0)})\otimes b_{(1)}\big)\big)$$
$$= b_{(0)}\gamma^{-1}(b_{(1)})\gamma(b_{(2)})\otimes b_{(3)}\otimes b_{(4)}$$
$$= T^H_{B\otimes_s H}\big((\gamma\otimes H)\Delta\big)\big(b_{(0)}\gamma^{-1}(b_{(1)})\otimes 1\otimes b_{(2)}\big).$$

Applying $(B\otimes H\otimes\epsilon)(T^H_{B\otimes_s H})^{-1}(\nu\gamma)$ to this equality we obtain $\nu(b_{(0)}\gamma^{-1}(b_{(1)})) = b_{(0)}\gamma^{-1}(b_{(1)})\otimes 1$. Thus, the map $\phi:(A\otimes H,\hat{s})\to(B,s)$, given by $\phi(a\otimes h)=a\gamma(h)$ is a normal basis, since $b\mapsto b_{(0)}\gamma^{-1}(b_{(1)})\otimes b_{(2)}$ is a well-defined map from B to $A\otimes H$, which is the composition inverse of ϕ. Let $\alpha: B\otimes H\to B\otimes_A B$ be the map $\alpha := B\otimes_A\gamma$. For $b\in B$ and $h\in H$, we have

$$\beta_B\alpha(b\otimes h) = \beta_B\big(b\otimes\gamma(h)\big) = b\gamma(h_{(1)})\otimes h_{(2)} = T^H_B(\gamma)(b\otimes h). \tag{6}$$

Hence, $\beta_B\alpha = T^H_B(\gamma)$. Then β_B is an isomorphism, since $T^H_B(\gamma)$ and α are. Now, assume that $(A\hookrightarrow B,s)$ is Galois with a normal basis. Let $\phi:(A\otimes H,\hat{s})\to(B,s)$ be a normal basis. Let $\gamma:(H,c)\to(B,s)$ be the map defined by $\gamma(h)=\phi(1\otimes h)$ and let $\alpha: B\otimes H\to B\otimes_A B$ be as above. Then the equality (6) holds. Since $\beta_B\alpha$ is a bijective map and T^H_B is an algebra isomorphism, this shows that γ is convolution invertible.

(2) ⇒ (3). Let $\phi:(A\otimes H,\hat{s})\to(B,s)$ be a normal basis of $(A\hookrightarrow B,s)$. From the equivalence between items (1) and (2), and the discussion following Definition 10.5, we can assume that $\phi(1\otimes 1)=1$ (by simply taking $\phi(a\otimes h)=a\gamma(h)$, where $\gamma:(H,c)\to(B,s)$ is a cleft map satisfying $\gamma(1)=1$), that (B,s) is a crossed product $(A\#_f H,\hat{s})$, and that the cleft map γ is the inclusion $h\mapsto 1\# h$. For $a\# h\in A\#_f H$, we have

$$\beta_B(B\otimes\gamma)(a\otimes h\otimes l) = (a\# h)(1\# l_{(1)})\otimes l_{(2)} = (A\otimes\beta_H)T_A^{H\otimes^c H}(f)(a\otimes h\otimes l).$$

Since $\beta_B(B\otimes\gamma)$ and β_H are bijective maps and $T_A^{H\otimes^c H}$ is an algebra isomorphism, f is convolution invertible.

(3) ⇒ (1). We can assume that $(B,s)=(A\#_f H,\hat{s})$ for some crossed product $A\#_f H$ with convolution invertible cocycle. Let γ be the inclusion $h\mapsto 1\# h$. For $a\# h\in A\#_f H$, we have

$$T^H_{A\#_f H}(\gamma)(a\otimes h\otimes l) = (a\# h)(1\# l_{(1)})\otimes l_{(2)} = (A\otimes\beta_H)T_A^{H\otimes^c H}(f)(a\otimes h\otimes l).$$

Since $(A\otimes\beta_H)T_A^{H\otimes^c H}(f)$ is bijective and $T^H_{A\#_f H}$ is an algebra isomorphism, γ is convolution invertible. □

Let $A\#_f H$ be a crossed product with f a convolution invertible normal cocycle. Let $\gamma: H\to A\#_f H$ be the cleft map $h\mapsto 1\# h$. The map $\gamma\mu$ is convolution invertible, since γ is and $\mu: H\otimes^c H\to H$ is a coalgebra map. Moreover, $(\gamma\mu)^{-1}=\gamma^{-1}\mu$. On the other hand, it is immediate that $\mu_{A\#_f H}(\gamma\otimes\gamma) = (f\otimes 1_H)*(\gamma\mu)$. Hence,

$$f\otimes 1_H = \big(\mu_{A\#_f H}(\gamma\otimes\gamma)\big)*\big(\gamma^{-1}\mu\big). \tag{7}$$

This gives a formula for f in terms of γ.

Next, we obtain a formula for the convolution inverse of γ. From the proof of Theorem 10.6 it follows that

$$\gamma^{-1} = \big(T^{H}_{A\#_f H}\big)^{-1}\big(T^{H\otimes^c H}_{A}\big(f^{-1}\big)\big(A\otimes\beta_H^{-1}\big)\big),$$

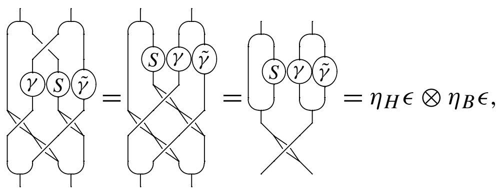
where f^{-1} denotes the convolution inverse of $f : H\otimes^c H\to A$. Making the calculations we obtain

$$\gamma^{-1}(h) = \big(f^{-1}\otimes H\big)(S\otimes H\otimes S)(H\otimes c)(c\otimes H)(\Delta\otimes H)\Delta(h).$$

Lemma 10.7. *Let $(A\hookrightarrow B, s)$ be an H-extension. Assume that s is bijective. If $\gamma : (H, c)\to(B, s)$ is a cleft map, then*

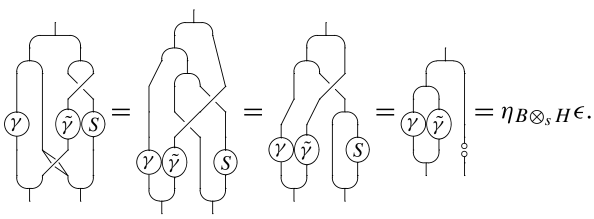
(1) $(H\otimes\gamma^{-1})c^{-1} = s^{-1}(\gamma^{-1}\otimes H)$.
(2) $\nu\gamma^{-1} = (\gamma^{-1}\otimes S)\,c\Delta$, *where ν is the coaction of B.*

Proof. (1) It suffices to check that $s(H\otimes\gamma^{-1}) = (\gamma^{-1}\otimes H)c$. By the fact that $(\gamma\otimes S)s = s(S\otimes\gamma)$ and the compatibility of s with μ_H and μ_B, we have



where $\tilde\gamma$ is the convolution inverse of γ. Hence $s(S\otimes\gamma^{-1})$ is a convolution right inverse of $s(H\otimes\gamma) : H\otimes^c H\to B\otimes_s H$. In a similar way, we can check that $(\gamma^{-1}\otimes S)c$ is a convolution left inverse of $(\gamma\otimes H)c : H\otimes^c H\to B\otimes_s H$. Since, $s(H\otimes\gamma) = (\gamma\otimes H)c$, we get that $s\,(S\otimes\gamma^{-1}) = (\gamma^{-1}\otimes S)c$. The assertion follows easily from this fact.

(2) First note that since $\nu : B\to B\otimes_s H$ is an algebra map, $\nu\,\gamma^{-1}$ is the inverse of $\nu\gamma = (\gamma\otimes H)\Delta$. On the other hand, by item (1), the coassociativity of Δ, and the compatibility of c with Δ and ϵ, we have



Thus, $(\gamma^{-1}\otimes S)c\Delta$ is a right inverse of $\nu\gamma$, and so $(\gamma^{-1}\otimes S)c\Delta = \nu\gamma^{-1}$. □

Next, we generalize Proposition 6.9. Our proof is very close to that given in [Mo, Corollary 7.2.11].

Proposition 10.8. *Let H be a braided Hopf algebra, s a transposition of H on A, $\rho: H\otimes A\to A$ a weak s-action, and $f: H^2\to A$ a normal cocycle compatible with s satisfying the twisted module condition. Let $B=A\#_f H$ be the crossed product built from (s,ρ,f). If s and the antipode S of H are bijective maps and f is convolution invertible, then $B\simeq H\otimes A$ as right A-modules.*

Proof. Let $s_B: H\otimes B\to B\otimes H$ be the transposition $s_B=(A\otimes\tau)(s\otimes H)$ and let $\tau_B: H\otimes B\to B\otimes H$ be the flip. By Proposition 5.5, we know that $(B^{\mathrm{op}},\widetilde{s_B}^{-1})$ is an $\widetilde{H_c^{\mathrm{cop}}}$-comodule algebra with coaction $\tilde\nu=\tau_B\widetilde{s_B}^{-1}\nu$, where $\nu=A\otimes\Delta$. We assert that $(B^{\mathrm{op}})^{\mathrm{coH}}=A^{\mathrm{op}}$. It is clear that $A^{\mathrm{op}}\subseteq(B^{\mathrm{op}})^{\mathrm{coH}}$. Since $B=(A\otimes k)\oplus(A\otimes\ker\epsilon)$, to prove the converse inclusion it suffices to see that $(B^{\mathrm{op}})^{\mathrm{coH}}\cap(A\otimes\ker\epsilon)=0$. Now, using that $\tau s^{-1}=(A\otimes\epsilon\otimes H)\tilde\nu$, it follows easily that τs^{-1} and $A\otimes\eta_H\epsilon$ coincide on $(B^{\mathrm{op}})^{\mathrm{coH}}$. Hence, $\tau s^{-1}=0$ on $(B^{\mathrm{op}})^{\mathrm{coH}}\cap(A\otimes\ker\epsilon)$. Since τs^{-1} is an injective map, we have that $(B^{\mathrm{op}})^{\mathrm{coH}}\cap(A\otimes\ker\epsilon)=0$. So, $(A^{\mathrm{op}}\hookrightarrow B^{\mathrm{op}},\widetilde{s_B}^{-1})$ is an $\widetilde{H_c^{\mathrm{cop}}}$-extension of A^{op}. Let $\gamma:(H,c)\to(B,s_B)$ be the map defined by $\gamma(h)=1\# h$. From Theorem 10.6 we know that γ is a convolution invertible map of right H-comodules. Let $\overline{S}$ be the composition inverse of S. The equalities

$$\mu_{B^{\mathrm{op}}}\big(\gamma^{-1}\overline{S}\otimes\gamma\overline{S}\big)\Delta_{\widetilde{H_c^{\mathrm{cop}}}}=\mu_B\big(\gamma\overline{S}\otimes\gamma^{-1}\overline{S}\big)c^{-1}\Delta=\mu_B\big(\gamma\otimes\gamma^{-1}\big)\Delta\overline{S}=\eta_B\epsilon$$

show that $\gamma^{-1}\overline{S}$ is a convolution left inverse of $\gamma\overline{S}:\widetilde{H_c^{\mathrm{cop}}}\to B^{\mathrm{op}}$. Similarly, we can show that $\gamma^{-1}\overline{S}$ is also a convolution right inverse of $\gamma\overline{S}$. Moreover, by Lemma 10.7(1) and the fact that $(\overline{S}\otimes H)\tau c^{-1}\tau=\tau c^{-1}\tau(H\otimes\overline{S})$ and $(H\otimes\gamma)\tau c^{-1}\tau=\tau_B s_B^{-1}\tau_B(\gamma\otimes H)$, we have

$$\begin{aligned}\big(\gamma\overline{S}\otimes H\big)\widetilde{c^{-1}}&=\big(\gamma\overline{S}\otimes H\big)\tau c^{-1}\tau=\tau_B\big(H\otimes\gamma\overline{S}\big)c^{-1}\tau\\&=\tau_B(H\otimes\gamma)c^{-1}\tau\big(H\otimes\overline{S}\big)=\tau_B s_B^{-1}(H\otimes\gamma)\tau\big(H\otimes\overline{S}\big)\\&=\tau_B s_B^{-1}\tau_B\big(H\otimes\gamma\overline{S}\big)=\widetilde{s_B}^{-1}\big(H\otimes\gamma\overline{S}\big).\end{aligned}$$

Finally, by Lemma 10.7 and the fact that $\tau c^{-1}(\overline{S}\otimes H)=(\overline{S}\otimes H)\tau c^{-1}$, we have

$$\begin{aligned}\tilde\nu\gamma^{-1}\overline{S}&=\tau_B\widetilde{s_B}^{-1}\nu\gamma^{-1}\overline{S}=\tau_B\widetilde{s_B}^{-1}\big(\gamma^{-1}\otimes S\big)c\Delta\overline{S}\\&=\tau_B\widetilde{s_B}^{-1}\big(\gamma^{-1}\otimes S\big)\big(\overline{S}\otimes\overline{S}\big)\Delta=\tau_B\widetilde{s_B}^{-1}\big(\gamma^{-1}\overline{S}\otimes H\big)\Delta\\&=\big(\gamma^{-1}\otimes H\big)\tau c^{-1}\big(\overline{S}\otimes H\big)\Delta=\big(\gamma^{-1}\overline{S}\otimes H\big)\tau c^{-1}\Delta\\&=\big(\gamma^{-1}\overline{S}\otimes H\big)\Delta_{\widetilde{H_c^{\mathrm{cop}}}}.\end{aligned}$$

Therefore, $(A^{\mathrm{op}}\hookrightarrow B^{\mathrm{op}},\widetilde{s_B}^{-1})$ is a cleft via $\gamma^{-1}\overline{S}:(\widetilde{H_c^{\mathrm{cop}}},\widetilde{c^{-1}})\to(B^{\mathrm{op}},\widetilde{s_B}^{-1})$. Consequently, by Theorem 10.6, the map $\phi: A^{\mathrm{op}}\otimes\widetilde{H_c^{\mathrm{cop}}}\to B^{\mathrm{op}}$, given by $\phi(a\otimes h)=$

$a(\gamma^{-1}\overline{S}(h))$, is an isomorphism of left $A^{\rm op}$-modules. This implies that $H \otimes A \simeq B$ as right A-modules via $h \otimes a \mapsto \gamma^{-1}\overline{S}(h)a$. □

11. Maschke's Theorem

In [LS] the classical theorem of Maschke about the semisimplicity of the group algebras was extended to Hopf algebras: A finite-dimensional Hopf k-algebra is semisimple iff $\epsilon(x) \neq 0$ for a left integral x of H. The proof was obtained by a similar argument to that used in the classical proof of Maschke. Let $A \# H$ be a smash product. Using an extension of this argument [CF, Theorem 4], it was proved that if A and H are Artinian semisimple, then $A \# H$ also is. In [BM, Theorem 2.6] this theorem was generalized to Hopf crossed products (see [BCM]) with invertible cocycle. Now, let H be a braided Hopf algebra and let $s : H \otimes A \to A \otimes H$ be a bijective transposition. In this section we show that Maschke's Theorem remains valid for crossed products $A \#_f H$, constructed from a weak s-action $h \otimes a \mapsto h \cdot a$ and a convolution invertible cocycle compatible with s. This generalizes Theorem 5.1 of [GG], where the case when H is a standard Hopf algebra was considered.

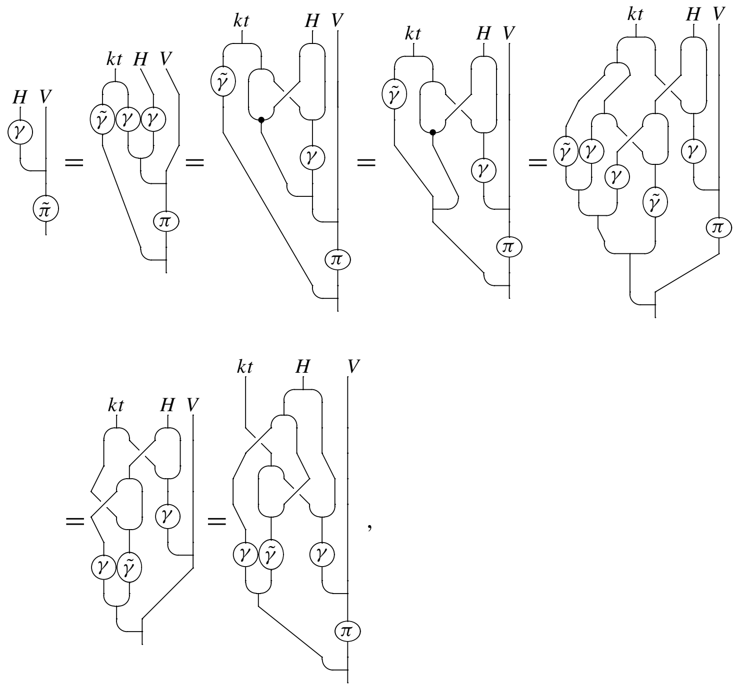
Theorem 11.1. *Let H be a semisimple braided Hopf algebra, s a bijective transposition of H on A, let $h \otimes a \mapsto h \cdot a$ be a weak s-action of H on A, and $f : H^2 \to A$ a normal cocycle compatible with s, satisfying the twisted module condition. Let $A \#_f H$ be the crossed product constructed from these data. Assume that f is convolution invertible. The following assertions hold*:

(1) *If V is a left $A \#_f H$-module and $W \subseteq V$ is a submodule which has a complement in the category of A-modules, then W has a complement in the category of $A \#_f H$-modules.*
(2) *If A is Artinian semisimple, then so is $A \#_f H$.*

Proof. Clearly, (1) implies (2). In order to prove (1), consider an A-linear projection $\pi : V \to W$ and choose $t \in \int_H^r$ with $\epsilon(t) = 1$. As in [BM], let $\tilde{\pi} : V \to W$ be the map defined by

$$\tilde{\pi}(v) = \sum_{(t)} \gamma^{-1}(t_{(1)})\pi\big(\gamma(t_{(2)})v\big) \quad \text{for all } v \in V,$$

where $\gamma : H \to A \#_f H$ is the map defined by $\gamma(h) = 1 \# h$. It is easy to check that $\tilde{\pi}$ is a projection of V onto W. We must see that $\tilde{\pi}$ is A-linear and H-linear. The former assertion can be proved as in [GG, Theorem 5.1]. Let us see the latter one. Let $\tilde{\gamma}$ denote the convolution inverse γ^{-1} of γ. We have



where the first equality follows from the definition of $\tilde{\pi}$ and the fact that V is a left $A\,\#_f\,H$-module, the second one follows from the definition of the multiplication of E, the third from using the fact that π is A-linear, the fourth follows from (7) and the fact that V is a left $A\,\#_f\,H$-module, the fifth from the coassociativity and the fact that $\tilde{\gamma}$ is the convolution inverse of γ, and the sixth follows from the fact that c is compatible with the comultiplication. Since, by Corollary 3.16 and the discussion following Theorem 3.20,

$$\begin{aligned}&(H\otimes\mu\otimes\mu)(H\otimes\Delta_{H\otimes^c H})(c\otimes H)(H\otimes\Delta)(t\otimes h)\\&\quad= h_{(1)}\otimes\Delta(th_{(2)}) = h_{(1)}\otimes\Delta\big(t\epsilon(h_{(2)})\big) = h\otimes\Delta(t)\end{aligned}$$

for all $h\in H$, we obtain that $\tilde{\pi}(\gamma(h)v) = \gamma(h)\gamma^{-1}(t_{(1)})\pi(\gamma(t_{(2)})v) = \gamma(h)\tilde{\pi}(v)$ for all $h\in H$, $v\in V$, as desired. □

12. Equivalence of crossed products

The purpose of this section is to give necessary and sufficient conditions for two crossed products to be equivalent. As a consequence we obtain that a crossed product $A\,\#_f\,H$ is equivalent to one of the form $A^s_f[H]$ iff the action is inner (see Definition 12.5). It will be convenient to work in the more general context allowing $A\,\#_f\,H$ to be not necessarily an associative algebra.

Fix an associative unitary k-algebra A and a braided bialgebra H. Let s be a transposition of H on A, $\rho : H \otimes A \to A$ a map satisfying conditions of Remark 6.2(2)–(4), and $f : H^2 \to A$ a normal map compatible with s. Let $A \#_f H$ be the (not necessarily associative) unitary k-algebra, constructed as in Definition 9.6, from the triple (s, ρ, f). It is easy to check that:

(1) the map $\hat{s} = (A \otimes \tau)(s \otimes H) : H \otimes A \#_f H \to A \#_f H \otimes H$ satisfies the properties required by Definition 4.1;
(2) $A \#_f H$ is a right H-comodule via $A \otimes \Delta : A \#_f H \to A \#_f H \otimes H$ and $(A \#_f H)^{\mathrm{coH}} = A$;
(3) $A \otimes \Delta : A \#_f H \to A \#_f H \otimes_{\hat{s}} H$ is a homomorphism of unitary algebras, where $A \#_f H \otimes_{\hat{s}} H$ is $(A \#_f H) \otimes H$ with the multiplication twisted by $\hat{s}$;
(4) $A \otimes \Delta : (A \#_f H, \hat{s}) \to (A \#_f H, \hat{s}) \otimes (H, c)$ is a map of $\mathcal{B}_H$.

In fact, it is easy to see that in the proof of Propositions 10.3 and 10.4 the associativity of $A \#_f H$ is not used. So, it has sense to say that $(A \#_f H, \hat{s})$ is a not necessarily associative H-comodule algebra, and that $(A \subseteq A \#_f H, \hat{s})$ is an H-extension of A. Finally, the notion of morphism of general H-comodule unitary algebras is identical to that in the associative setting.

Definition 12.1. Fix a Hopf algebra H and an algebra A. Let (s, ρ, f) and (s', ρ', f') be two triples, as above. Let $(A \subseteq A \#_f H, \hat{s})$ and $(A \subseteq A \#_{f'} H, \widehat{s'})$ be the H-extensions associated with (s, ρ, f) and (s', ρ', f'), respectively. We say that $(A \subseteq A \#_f H, \hat{s})$ and $(A \subseteq A \#_{f'} H, \widehat{s'})$ are equivalent if there is an isomorphism of H-comodule algebras $g : (A \#_f H, \hat{s}) \to (A \#_{f'} H, \widehat{s'})$, which is also an A-linear map.

Remark 12.2. If there is an A-linear H-comodule algebra map $g : (A \#_f H, \hat{s}) \to (A \#_{f'} H, \widehat{s'})$, then $s = s'$. In fact, write $s(h \otimes a) = \sum_i a_i \otimes h_i$ and $s'(h \otimes a) = \sum_{i'} a_{i'} \otimes h_{i'}$. We have,

$$\sum_{i'} a_{i'} \# 1 \otimes h_{i'} = \widehat{s'}(H \otimes g)(h \otimes a \# 1) = (g \otimes H)\hat{s}(h \otimes a \# 1) = \sum_i a_i \# 1 \otimes h_i,$$

where the first and the third equality follow from the fact that g is an A-linear map and $g(1 \# 1) = 1 \# 1$. In spite of this remark, we will follow using the notations s and s' to emphasize the difference between $\hat{s}$ and $\widehat{s'}$, which are not equal as transpositions, since the products of $A \#_f H$ and $A \#_{f'} H$ are different.

Let $g : A \otimes H \to A \otimes H$ be an A-linear and H-colinear map. By the definition of the coaction ν of $A \otimes H$, we have:

$$\begin{aligned} g(1 \otimes h) &= (A \otimes \epsilon \otimes H)\nu g(1 \otimes h) = (A \otimes \epsilon \otimes H)(g \otimes H)\nu(1 \otimes h) \\ &= (A \otimes \epsilon)g(1 \otimes h_{(1)}) \otimes h_{(2)}. \end{aligned}$$

Hence, $g(a \otimes h) = ag(1 \otimes h) = au(h_{(1)}) \otimes h_{(2)}$ for all $a \in A$ and $h \in H$, where $u(h) = (A \otimes \epsilon)g(1 \otimes h)$. This gives a bijective correspondence between the maps from H to A and the A-linear and H-colinear maps from $A \otimes H$ to $A \otimes H$. We assert that g is bijective iff u is convolution invertible. Moreover, in this case, $g^{-1}(a \otimes h) = au^{-1}(h_{(1)}) \otimes h_{(2)}$ for all $a \in A$ and $h \in H$. In fact, suppose g is invertible. Let $v(h) = (A \otimes \epsilon)g^{-1}(1 \otimes h)$. We have

$$1 \otimes h = g^{-1} g(1 \otimes h) = g^{-1}\big(u(h_{(1)}) \otimes h_{(2)}\big) = u(h_{(1)})v(h_{(2)}) \otimes h_{(3)}.$$

Applying $A \otimes \epsilon$ to both sides of this equality, we obtain $u(h_{(1)})v(h_{(2)}) = \epsilon(h)1$. A similar argument shows that $v(h_{(1)})u(h_{(2)}) = \epsilon(h)1$. So v is the convolution inverse of u. The converse is immediate.

In the next theorem we obtain necessary and sufficient conditions on u in order for g to be a morphism of H-comodule algebras.

This result, for the case when H is a standard Hopf algebra and the transposition is the flip, was independently obtained by Doi [D2] and R.J. Blattner (unpublished).

Theorem 12.3. *Let g and u be as in the previous discussion. Then g is a morphism of H-comodule algebras from $(A \#_f H, \hat{s})$ to $(A \#_{f'} H, \widehat{s'})$ iff*

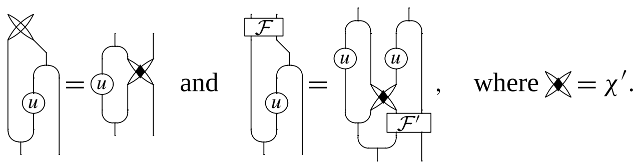
(1) $u(1) = 1$,
(2) $(u \otimes H)c = s'(H \otimes u)$,
(3) $\mu(A \otimes u)\,\chi = \mu(u \otimes \rho')(\Delta \otimes A)$,
(4) $\mu(A \otimes u)\mathcal{F} = \mu^2(\rho \otimes A^2)(H \otimes u \otimes u \otimes f')(H \otimes c \otimes H^2)(\Delta \otimes H^2)\Delta_{H\otimes^c H}$,

where $\chi = (\rho \otimes H)(H \otimes s)(\Delta \otimes A)$, $\mathcal{F} = (f \otimes \mu)\Delta_{H\otimes^c H}$, and $\mu: A^2 \to A$ is the multiplication map. Moreover, g is an isomorphism iff u is convolution invertible. In this case, ρ is a weak s-action and f is a cocycle satisfying the twisted module condition iff ρ' and f' respectively are.

Proof. Assume that g is a H-comodule algebra map. From $g(1 \# 1) = 1 \# 1$ we obtain that $u(1) = 1$, and applying $A \otimes \epsilon \otimes H$ to the both sides of the equality

$$(g \otimes H)\hat{s}(h \otimes 1 \# l) = \widehat{s'}(H \otimes g)(h \otimes 1 \# l) \quad \text{for all } h, l \in H,$$

we obtain $(u \otimes H)c = s'(H \otimes u)$. Moreover, since $g((1 \# h)(a \# 1)) = g(1 \# h)g(a \# 1)$ and $g((1 \# h)(1 \# l)) = g(1 \# h)g(1 \# l)$ for all $a \in A$ and $h, l \in H$, we have



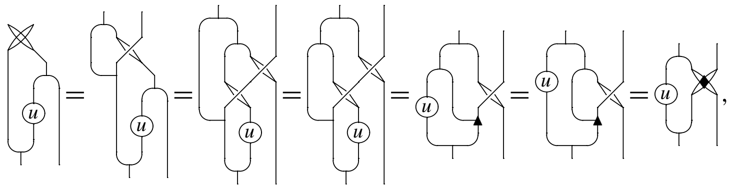
Applying $A \otimes \epsilon$ to these equalities, we obtain condition (3) and the following equality:

$$\mu(A \otimes u)\mathcal{F} = \mu(\mu \otimes f')(A \otimes \chi' \otimes H)(u \otimes H \otimes u \otimes H)(\Delta \otimes \Delta). \tag{4'}$$

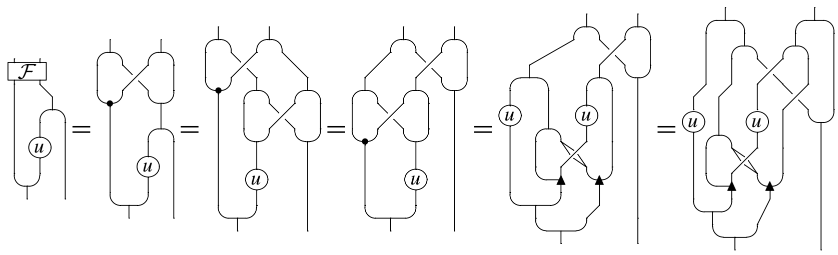
Conversely, suppose conditions (1)–(3) and $(4')$ are satisfied. From condition (1) we have $g(1 \# 1) = 1 \# 1$ and from condition (2) it follows easily that $(g \otimes H)\hat{s} = \widehat{s'}(H \otimes g)$. Let

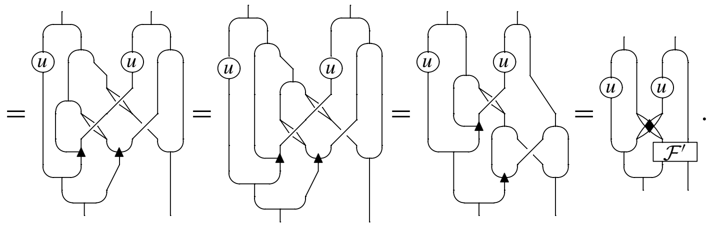
 $= \rho'$ and  $= f'$.

By the compatibility of s with the comultiplication, the coassociativity, and condition (3), we get

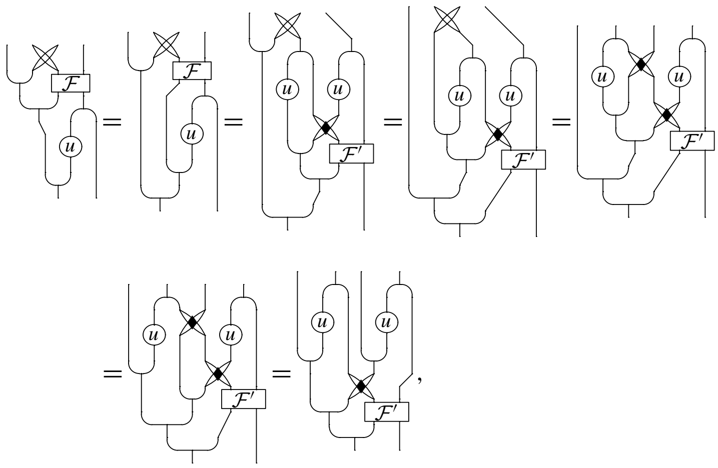


and using the coassociativity, the compatibility of c and s with the comultiplication, conditions (2) and $(4')$, and the fact that $\Delta\mu = (\mu \otimes \mu)(H \otimes c \otimes H)(\Delta \otimes \Delta)$,

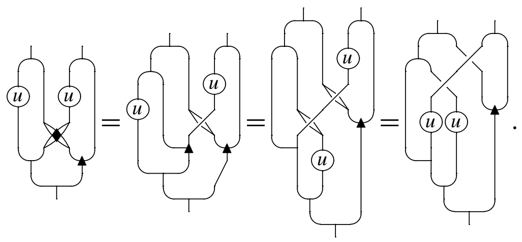




Using these facts, the associativity of μ_A, and the compatibility of χ' with μ_A, we obtain



which proves that g preserves the multiplication. Since the left sides of the equalities (4) and (4′) coincide, to finish the proof of the first assertion it suffices to check that if conditions (1)–(3) hold, then the right sides of (4) and (4′) also coincide. But, by the coassociativity of Δ and conditions (2) and (3),

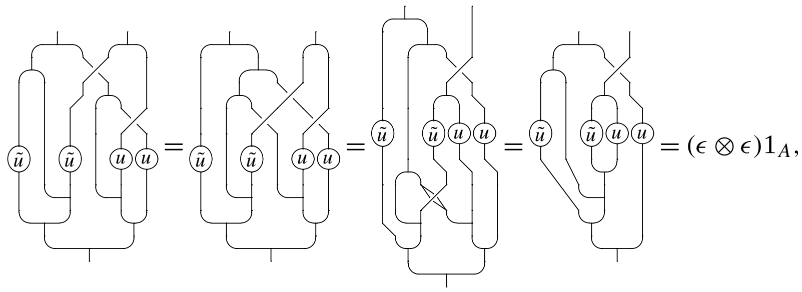


The discussion above shows that g is an isomorphism iff u is convolution invertible. The last assertion follows from Theorems 2.3, 6.3, and 9.3 and the fact that $A\,\#_f\,H$ is associative iff $A\,\#_{f'}\,H$ is. □

Corollary 12.4. *Let ρ, ρ', f, f', χ, $\mathcal{F}$, g, and u be as in Theorem* 12.3. *Assume that u is convolution invertible, $u(1) = 1$, $(u \otimes H)c = s'(H \otimes u)$, and ρ is a weak s-action. Then g is an equivalence between $(A\,\#_f\,H, \hat{s})$ and $(A\,\#_{f'}\,H, \widehat{s'})$ iff*

(1) $\rho' = \mu(\mu \otimes u)(u^{-1} \otimes \chi)(\Delta \otimes A)$,
(2) $f' = \mu^2(A \otimes \rho \otimes \mu)(u^{-1} \otimes H \otimes u^{-1} \otimes A \otimes u)(\Delta \otimes H \otimes \mathcal{F})\Delta_{H \otimes^c H}$.

Proof. It is immediate that u being convolution invertible, items (3) of Theorem 12.3 and (1) of the present corollary are equivalent. Hence, we only need to check that item (4) of the mentioned theorem and item (2) of the present corollary also are. Since the right side of the equality in item (4) of the previous theorem is the convolution product of $\mu(\rho \otimes A)(H \otimes u \otimes u)(H \otimes c)(\Delta \otimes H)$ and f' in $\mathrm{Hom}_k(H \otimes^c H, A)$, to do this will be sufficient to prove that the first map is left convolution invertible in $\mathrm{Hom}_k(H \otimes^c H, A)$ with inverse $\mu(A \otimes \rho)(u^{-1} \otimes H \otimes u^{-1})(\Delta \otimes H)$. But, by the coassociativity of Δ, the compatibility of Δ with c, the fact that $(u \otimes H)c = s'(H \otimes u)$, the coassociativity of μ_A, and by Remark 6.2(1)–(2),



where $\tilde{u}$ denotes the convolution inverse of u. □

Definition 12.5. A weak s-action $\rho : H \otimes A \to A$ is inner if there is a convolution invertible element $u \in \mathrm{Hom}_k(H, A)$ satisfying $(u \otimes H)c = s(H \otimes u)$ such that

$$\rho = \mu^2\bigl(u \otimes A \otimes u^{-1}\bigr)(H \otimes s)(\Delta \otimes A), \tag{9}$$

where u^{-1} denotes the inverse of u.

Let $u \in \mathrm{Hom}_k(H, A)$ be a convolution invertible map such that $(u \otimes H)c = s(H \otimes u)$. Let $g : A \otimes H \to A \otimes H$ be the A-linear and H-colinear map, defined by $g(a \otimes h) = au(h_{(1)}) \otimes h_{(2)}$. By the discussion preceeding Theorem 12.3 and its proof,

$$\begin{aligned}(u \otimes H)c = s(H \otimes u) \quad &\iff \quad (g \otimes H)\hat{s} = \widehat{s'}(H \otimes g)\\ &\iff \quad \bigl(g^{-1} \otimes H\bigr)\widehat{s'} = \hat{s}\bigl(H \otimes g^{-1}\bigr)\\ &\iff \quad \bigl(u^{-1} \otimes H\bigr)c = s\bigl(H \otimes u^{-1}\bigr).\end{aligned}$$

Therefore, u^{-1} satisfies the equality $(u^{-1} \otimes H)\,c = s(H \otimes u^{-1})$. Using this fact it is easy to check that such a map u defines a weak s-action by formula (8) iff $u(1)$ belongs to the center $\mathrm{Z}(A)$ of A. We leave this and the proof of Proposition 12.7 to the reader.

Definition 12.6. Let $u, v \in \mathrm{Hom}_k(H, A)$ be convolution invertible maps such that $(u \otimes H)c = s(H \otimes u)$, $(v \otimes H)c = s(H \otimes v)$, $u(1) \in \mathrm{Z}(A)$, and $v(1) \in \mathrm{Z}(A)$. We say that u is equivalent to v and we write $u \simeq v$, if u and v induce the same weak s-action of H on A.

Proposition 12.7. *Let u, v as in Definition* 12.5. *We have*

(1) $u \simeq w$, *where* $w \in \operatorname{Hom}_k(H, A)$ *is the map* $h \mapsto u(h)u(1)^{-1}$,
(2) $u \simeq v$ *iff* $\mu_A^2(v^{-1} \otimes u \otimes A)(\Delta \otimes A) = \mu_A^2(A \otimes v^{-1} \otimes u)(A \otimes \Delta)s$.

Proposition 12.8. *Let s be a transposition of H on A, $\rho : H \otimes A \to A$ a weak s-action, and $f : H^2 \to A$ a map. Suppose ρ is inner via $u \in \operatorname{Hom}_k(H, A)$. Assume that $u(1) = 1$. Let $f' \in \operatorname{Hom}_k(H^2, A)$ be the map given by*

$$f' := \mu^2(A \otimes \rho \otimes \mu)\big(u^{-1} \otimes H \otimes u^{-1} \otimes A \otimes u\big)(\Delta \otimes H \otimes \mathcal{F})\,\Delta_{H \otimes^c H},$$

where $\mathcal{F} = (f \otimes \mu)\Delta_{H\otimes^c H}$. Then f' is a normal cocycle compatible with s, satisfying the twisted module condition in respect to the trivial action of H on A, iff f is a normal cocycle compatible with s, satisfying the twisted module condition in respect to ρ. Moreover, in this case $(A \#_f H, \hat{s}) \simeq (A_{s'}^{f}[H], \widehat{s'})$, where $s' = s$. Conversely, if $(A \#_f H, \hat{s}) \simeq (A_{s'}^{f}[H], \widehat{s'})$, then ρ is inner.

Proof. It is immediate that f' is normal iff f is, and a direct computation shows that f' is compatible with s iff f is. The other assertions follow easily from Corollary 12.4. □

Let s be a transposition of H on A. Suppose an inner weak s-action of H on A is implemented by $u \in \operatorname{Hom}_k(H, A)$ with $u(1) = 1$. Then the map $f : H^2 \to A$ defined by

$$f := \big(u \otimes u \otimes u^{-1}\mu\big)\Delta_{H\otimes^c H}$$

is a normal cocycle compatible with s, satisfying the twisted module condition. We call f the inner cocycle defined by u. By Proposition 12.8, we obtain that $A \#_f H \simeq A \otimes_s H$.

Acknowledgments

We thank Nicolas Andruskiewitsch and Constanza di Luigi for helpful comments and the referee for a substantial improvement in the presentation of Sections 5 and 6.

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