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On Rings whose Simple Modules are Injective

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INTRODUCTION

By a well-known theorem due to Kaplansky a commutative ring R is von Neumann regular if and only if every simple R module is injective. In the noncommutative case neither the necessary nor the sufficient part of Kaplansky's theorem holds as has been shown by C. Faith [6] and J. Cozzens [4].

This paper is mainly concerned with the determination of the structure of a (not necessarily commutative) ring R (with identity) whose simple right R modules are injective. In Section 2 the main properties of such a ring R are stated; some of them were obtained by the second author many years ago (cf. [6]) without being published. In particular, it is shown that the center $Z(R)$ of R is von Neumann regular (Corollary 2.2 and Lemma 2.3), and that the class of rings R considered here is Morita invariant (Theorem 2.5). As an application of the results of Section 2 we obtain in Section 3 that the ring R is semisimple and artinian if and only if every cyclic semisimple R module is injective (Theorem 3.2). This answers an open question of Sandomierski and Cateforis [3]. In Section 4 it is shown that the rings R of right Krull-dimension at most one, whose simple right R modules are injective, are exactly the direct sums of finitely many simple rings S_i each of which is Morita equivalent to a right hereditary, right noetherian domain D_i whose torsion modules are injective and completely reducible (Theorem 4.2). If R is also left noetherian, then the simple left R modules are injective (Corollary 4.4).

The last two sections are concerned with the von Neumann regularity of rings R satisfying polynomial identities and having the property that the simple right R modules are injective. If R satisfies a homomorphic polynomial identity, then R is von Neumann regular if and only if every nonzero right ideal of every nonzero epimorphic image of R contains a nonzero idempotent (Theorem 5.4). Following C. Procesi the ring R is called affine, if each epimorphic image of R satisfies a polynomial identity with coefficients contained in its center, and if R is finitely generated as a ring over its center $Z(R)$. Using sheaves in Section 6 it is shown that the simple right R modules of the affine ring R are injective if and only if R is von Neumann regular, which again is equivalent to the biregularity of R (Theorem 6.3). If A is the group ring of the group G over the affine ring R, and if the simple right A modules are injective, then A is von Neumann regular (Corollary 6.8). Finally it is remarked that that Theorem 6.3 and Corollary 6.8 do not hold in general for rings R which are finitely generated as a ring over their centers $Z(R)$, as is easily seen by the examples of Cozzens [4].

Concerning our terminology and notations we refer to the books by Jacobson [8], Lambek [9] and Pierce [13].

2. Rings whose Simple Modules are Injective

In this section we give the main properties and characterizations of those rings R whose simple (right) R modules are injective. The R module M is semisimple, if the intersection of all maximal submodules of M is zero.

Theorem 2.1. *The following properties of the ring R are equivalent*:

(1) *Every simple R module is injective.*

(2) *Every R module is semisimple.*

(3) *Every cyclic R module is semisimple.*

(4) *Every right ideal of R is an intersection of maximal right ideals of R.*

Proof. Let M be a right R module, where R is a ring whose simple R modules are injective. If $0 \neq x \in M$, then by Zorn's lemma there is a submodule Y of M which is maximal among the submodules X of M with $x \notin X$. Let D denote the intersection of all submodules S of M with $S > Y$. Then $x \in D$, and $D/Y \neq 0$ is simple. Therefore $M/Y = D/Y \oplus K/Y$, where K is a submodule of M. Since x cannot be contained in K, it follows that Y is a maximal submodule of M. Hence M is semisimple, because $x \notin Y$.

Condition (2) trivially implies (3), and it is clear that (4) follows from (3). Now let S be a simple R module, and let X be a right ideal of R. If

$\alpha \in \hom_R(X, S)$, and if $Y = \ker \alpha$, then by (4) there is a maximal right ideal M of R such that $M \geqslant Y$, but $M \ngeqslant X$. Since X/Y is a simple R module, $M \cap X = Y$. Therefore

$$R/M = (M + X)/M \simeq X/M \cap X = X/Y \simeq S.$$

Thus α can be extended to an R module homomorphism $\hat{\alpha} \in \hom_R(R, S)$. Hence S is injective by [9, p. 88]. This completes the proof of Theorem 2.1.

COROLLARY 2.2. *If every simple R module is injective, then every right ideal X of the ring R is idempotent.*

Proof. Suppose that X^2 is different from X, then by Theorem 2.1 there is a maximal right ideal M of R such that $X^2 \leqslant M$, but $X \nleqslant M$. Hence $1 = x + m$ for some $x \in X$ and $m \in M$. Thus $x = x^2 + mx \in M$, and $1 \in M$, a contradiction!

The element $c \in R$ is right regular, if $cx = 0$ implies $x = 0$. A right and left regular element is called regular.

LEMMA 2.3. *If every right ideal of the ring R is idempotent, then R has the following properties*:

(a) *$R = RcR$ for every right regular element $c \in R$.*

(*b*) *The left singular ideal $T_l(R) = \{x \in R \mid x_l$ is essential$\} = 0$.*

(c) *R/K is a flat left R module for every two-sided ideal K.*

(d) *The center $Z(R)$ of R is von Neumann regular.*

Proof. (a) Since $cR = cRcR$, there are elements r_i, $s_i \in R$ such that $c = \sum_{i=1}^{n} cr_ics_i$. Hence $R = RcR$, because c is right regular.

(b) Let $x \in T_l(R)$. Then $xR = xRxR$ implies the existence of an element $z \in T_l(R)$ such that $x = xz$. Now z_l is an essential left ideal of R. Let $v \in Rx \cap z_l$. Then $v = rx$ for some $r \in R$, and $0 = vz = rxz = rx = v$. Hence $x = 0$.

(c) Let $0 \neq x \in K$. Then again there is a $z \in K$ such that $x = xz$. If $y = 1 - z$, then any $xy = 0$ and $ya = a$ for $a = 1 + K \in R/K$. Hence R/K is a flat left R module by Lemma 1 of [16].

(d) For each $a \in Z(R)$ there is an $x \in R$ such that $a = axa$, because $aR = aRa$. Clearly $e = xa = ax = e^2$, and $a = ae = ea$. For each $b \in R$ we have $ab - ba = 0$. Hence

$$a(be - eb) = abe - aeb = abe - bae = (ab - ba)e = 0.$$

Thus $be - eb \in a_r = e_r = (1 - e)R$, which implies $eb = ebe$. Similarly it follows that $be = ebe$. Therefore $e \in Z(R)$, and $aZ(R) = eZ(R)$. Hence $Z(R)$ is von Neumann regular by [9, p. 67].

Remark. Since every epimorphic image of a von Neumann regular ring is von Neumann regular, Kaplansky's theorem [15, Theorem 6] mentioned in the introduction follows at once from Theorem 2.1, Corollary 2.2 and Lemma 2.3(d).

For later use we now prove the Morita invariance of the equivalent properties (1)–(4) of Theorem 2.1. The rings R and S are Morita equivalent, if their categories $\mathfrak{M}_R$ and $\mathfrak{M}_S$ of right modules are equivalent. The ring-theoretical property $\mathfrak{e}$ is Morita invariant, if $\mathfrak{e}$ is inherited by Morita-equivalent rings.

LEMMA 2.4. *Let $e \neq 0$ be an idempotent of the ring R such that $R = ReR$. If the simple R modules are injective, then also the simple S modules are injective, where $S = eRe$.*

Proof. Suppose that Lemma 2.4 is false. Then by Theorem 2.1 there is a right ideal X of S which is not the intersection D of all maximal right ideals M of S with $M \geqslant X$. Hence there is an element $d \in D$ with $d \notin X$. Since $R = ReR$ it follows that $XR < DR \leqslant eR$. By Theorem 2.1 there is a maximal R submodule T of eR with $d \notin T$ and $XR \leqslant T$. Thus

$$S = eTe + dS.$$

Again using $R = ReR$ it is easy to see that eTe is a maximal right ideal of $S = eRe$. Since $XR \leqslant T$, it follows that $X \leqslant eTe$, and therefore $dS \leqslant eTe$, which implies $S = eTe + dS = eTe$, a contradiction!

THEOREM 2.5. *If the ring S is Morita equivalent to the ring R, and if the simple R modules are injective, then also the simple S modules are injective.*

Proof. Since S is Morita equivalent to R, there exists an integer n and an idempotent $e = e^2 \in R_n$ such that $S \simeq eR_ne$ and $R_n = R_neR_n$, where R_n denotes the ring of $n \times n$ matrices over R. Thus Lemma 2.4 implies Theorem 2.5.

3. RINGS WHOSE SEMI-SIMPLE MODULES ARE INJECTIVE

Using Theorem 2.1 in this section we give an affirmative answer to the following question by Cateforis and Sandomierski [3]: Is every ring R whose semisimple (right) R modules are injective a semisimple, artinian ring?

As is usual the ring R is called a right Goldie ring, if R satisfies the ascending chain conditions on annihilator right ideals and on direct sums of nonzero right ideals of R.

LEMMA 3.1. *The right Goldie ring R whose right ideals are idempotent is a direct sum of finitely many simple rings.*

Proof. Because of Theorem 2.1 the ring R is semisimple. Hence by Theorems 3.13 and 6.1 of [10] R is an irredundant subdirect sum of a finite number of prime rings $R_i (i = 1, 2, \ldots, k)$ which are also right Goldie rings. Each two-sided ideal $A \neq 0$ of R_i is an essential right ideal of R_i. By Theorem 3.9 of [7], there exists a regular element c of R in A. Hence $R = RcR \leqslant A \leqslant R$ by Lemma 2.3. Since R_i is simple for $i = 1, 2, \ldots, k$ it follows that

$$R = \sum_{i=1}^{k} \oplus R_i .$$

THEOREM 3.2. *The following conditions of the ring R are equivalent:*

(1) *R is semisimple and artinian.*

(2) *Every semisimple R module is injective.*

(3) *Every cyclic semisimple R module is injective.*

(4) *R is a right Goldie ring such that*

(a) *Every simple R module is injective, and*

(b) *indecomposable injective R modules with the same associated prime ideal of R are isomorphic.*

Proof. It is well known that (2) follows from (1), and the implication (2) $\to$ (3) is trivial. If (3) holds, then every simple R module S is injective, because S is cyclic. Therefore Theorem 2.1 implies that every R module is semisimple. Hence every cyclic R module is injective, which implies that R is semisimple and artinian by the theorem of Osofsky [12, p. 649]. Thus (4) follows from (3).

If (4) holds, then R is a direct sum of finitely many simple right Goldie rings R_i by Lemma 3.1. From (4b) now follows that R_i contains a minimal right ideal. Hence R_i is a simple ring with minimum condition, and (1) holds.

Remark. The examples of [4] show that in general condition (4b) of Theorem 3.2 cannot be omitted, though it holds for all commutative rings.

4. Structure of Rings with Krull Dimension one whose Simple Modules are Injective

The notion of right Krull dimension was introduced by Gabriel (cf. [11]). The ring R has *right Krull dimension* K-dim $\leqslant 1$, if each properly descending chain

$$A_1 > A_2 > \cdots > A_k > A_{k+1} > \cdots$$

of right ideals A_i of R, such that for each $i = 1, 2, \ldots$ the right R module A_i/A_{i+1} is not artinian, has only finitely many terms. If K-dim $R \leqslant 1$, but R is not right artinian, then R *has right Krull dimension one.*

Lemma 4.1. *If the simple right R modules of the ring R with right Krull dimension one are injective, then the right singular ideal*

$$T_r(R) = \{x \in R \mid x_r \text{ essential}\} = 0.$$

Proof. Assume that there is an element $0 \neq x \in T_r(R)$. Then by Corollary 2.2 there exists a $z \in T_r(R)$ such that $x = xz$. Clearly z is not nilpotent, and $(z^n)_r \leqslant x_r$ for all integers $n \geqslant 1$. Suppose that x_r/z_r is an artinian right R module, then it is a direct sum of finitely many simple right R modules, because it is semisimple by Theorem 2.1. Hence the ascending chain

$$z_r \leqslant (z^2)_r \leqslant \cdots \leqslant (z^k)_r \leqslant (z^{k+1})_r \leqslant \cdots$$

becomes stationary. Thus $(z^k)_r = (z^{2k})_r$ for some k. It follows that $z^kR \cap (z^k)_r = 0$, which is a contradiction. Hence x_r/z_r is not artinian.

By induction, we therefore obtain an infinite sequence of elements $0 \neq x_i \in T_r(R)$, such that $x_1 = x$, $x_2 = z$, $x_i = x_i x_{i+1}$, and $(x_{i+1})_r/(x_i)_r$ is is not an artinian right R module for all $i = 1, 2, 3, \ldots$. But then R has Krull dimension greater than one, a contradiction! Hence $T_r(R) = 0$.

Following A. W. Goldie the (right) R module M is called a torsion module, if $M = T(M)$, where $T(M) = \{m \in M \mid m_r = \{y \in R \mid my = 0\}$ is an essential right ideal of $R\}$.

Theorem 4.2. *The ring R is a ring with right Krull dimension at most one whose simple right R modules are injective if and only if R is a direct sum of finitely many simple rings S_i each of which is Morita equivalent to a right hereditary, right noetherian, simple domain D_i whose torsion modules are completely reducible and injective.*

Proof. If R is a direct sum $\sum_{i=1}^{k} \oplus S_i$ of simple rings S_i, each of which is Morita equivalent to a simple domain D_i having the properties stated in

Theorem 4.2, then R is a semiprime right noetherian right hereditary ring. In order to prove that $K\text{-dim } R \leqslant 1$ it suffices to show by Hilfsatz 2.1 of [11] that R/E is an artinian right R module for every essential right ideal E of R. Since R is right noetherian, we may assume that E is irreducible. Hence R/E is an indecomposable torsion R module. Thus R/E is simple, because it is completely reducible. Therefore $K\text{-dim } R \leqslant 1$.

If $D_i (i = 1, 2,..., k)$ is not a division ring, then each simple right D_i module is a torsion module. Hence it is injective. If D_i is a division ring, then any right D_i module is injective. Therefore Theorem 2.5 implies that every simple right R module is injective.

Conversely, if $K\text{-dim } R = 1$, and if the simple right R modules are injective, then the right singular ideal $T_r(R)$ of R is zero by Lemma 4.1. Furthermore, R is right finite dimensional in the sense of Goldie [7] by the proof of Hilfsatz 2.1 of [11]. Therefore R is a right Goldie ring by [9, pp. 106 and 108]. Hence Lemma 3.1 applies, and R is a direct sum of finitely many simple rings having the same properties as R. Thus we may assume that R is simple.

In order to prove that R is right noetherian, it suffices to show that each essential right ideal E of R is finitely generated, because by Zorn's lemma each right ideal X of R is a direct summand of an essential right ideal. Since R is a right Goldie prime ring, E contains a regular element c of R by Theorem 3.9 of [7]. By Hilfsatz 2.1 of [11] the right R module R/cR is artinian, because cR is an essential right ideal. Since R/cR is semisimple by Theorem 2.1, it follows that R/cR is a direct sum of finitely many simple right R modules. Hence $E/cR \leqslant R/cR$ is finitely generated, and R is right noetherian.

If the right R module $M \neq 0$ is torsion, then for every $0 \neq m \in M$ there is an essential right ideal E of R such that $mR \cong R/E$. Therefore mR is a direct sum of finitely many simple injective right R modules by Theorem 2.1 and [11, Hilfsatz 2.1]. Since R is right noetherian, it follows that M is completely reducible and injective. Thus each torsion right R module is injective.

The ring R is right hereditary if and only if each epimorphic image of an injective right R module Q is injective. Let $0 \neq U$ be a submodule of Q, and let V be the injective envelope of U in Q. Then Q/U is injective, if V/U is injective. Since $T_r(R) = 0$ by Lemma 4.1, the right R module V/U is torsion by [6, p. 15]. Thus V/U is injective, and R is right hereditary. As R is simple an easy application of the Morita theorems now shows that R is Morita equivalent to a simple right hereditary, right noetherian domain D with $K\text{-dim } D \leqslant 1$. Since the simple right D modules are injective by Theorem 2.5, it follows from the last paragraph that every torsion right D module is injective and completely reducible. This completes the proof of Theorem 4.2.

Remark 4.3. By a theorem due to Cateforis and Sandomierski [3] a *commutative* ring R whose torsion modules are injective is a semihereditary ring with finitely many essential ideals. But a noncommutative ring R whose torsion modules are injective in general has infinitely many essential right ideals as is easily seen by Theorem 4.2 and the examples given by Cozzens [4].

COROLLARY 4.4. *The following properties of the ring R are equivalent:*

(1) *R is a left noetherian ring with right Krull dimension at most one whose simple right R modules are injective.*

(2) *R is a right noetherian ring with left Krull dimension at most one whose simple left R modules are injective.*

(3) *R is a right and left noetherian hereditary ring whose one-sided simple R modules are injective.*

Proof. Since a right and left noetherian hereditary ring has right and left Krull dimension at most one by Theorem 4 of Webber [17] and Hilfsatz 2.1 of [11], it suffices to show that (1) implies (2). By Theorem 4.2 R is right hereditary. As R is also left noetherian a well-known theorem due to M. Auslander implies that R is also left hereditary. Thus R has left Krull dimension at most one. Therefore by Theorem 2.1 it remains to show that each left ideal X of R is an intersection of maximal left ideals of R. Because of Theorems 4.2 and 2.5 we may assume that R is a simple domain. Thus 0 is an intersection of maximal left ideals. If $X \neq 0$, then X is an essential left ideal. Let $X^* = \hom_R(X, R)$, and let Q be the quotient division ring of R. Then $X^* = \{y \in Q \mid Xy \leqslant R\}$, because Q is the injective hull of R. Hence X^*/R is a finitely generated, torsion right R module. Thus by Theorem 4.2 it is completely reducible, i.e., there are $k < \infty$ submodules $R < S_i < X^*$ of X^* such that

$$X^*/R = S_1/R \oplus S_2/R \oplus \cdots \oplus S_k/R.$$

where each S_i/R is simple and injective. Since X^* is finitely generated and projective, and since R is noetherian and hereditary, each right R module S_i is finitely generated and projective. Let $M_i = \hom_R(S_i, R)$ for all i. Then $M_i^* = S_i$, and $R > M_i > X$, because X and S_i are reflexive. Furthermore, M_i is a maximal left ideal of R, because S_i/R is simple. It is now easy to see that $X = \bigcap_{i=1}^k M_i$, because $X^* = \sum_{i=1}^k S_i$. This completes the proof of Corollary 4.4.

Remark 4.5. Corollary 4.4 suggests the following question: Is every simple left R module injective, if every simple right R module is injective?

Without any restrictions on R this question has a negative answer. One counterexample is the following von Neumann regular ring R (cf. also [6], p. 130): Let V be an infinite-dimensional right vector space over the field K. Let S be the socle of $E = \hom_K(V, V)$, and let R be the subring of E generated by S and the center K of E. Then V is a simple left R module. If $V^* = \hom_K(V, K)$ and if $V^{**} = \hom_K(V^*, K)$, then $V < V^{**}$, because V is infinite dimensional over K. Since the left R module V^{**} is an essential extension of V, the simple left R module V is not injective.

As R is von Neumann regular, the field K is a flat left R module. Hence $V^* = \hom_K(V, K)$ and K are injective right R modules by Cartan–Eilenberg [2, p. 123, Example 10]. Since it is well known that (up to isomorphisms) V^* and K are the only simple right R modules, it follows that every simple right R module is injective.

5. Rings with Polynomial Identities whose simple Modules are Injective

In this section we give necessary and sufficient conditions for the von Neumann regularity of rings with polynomial identities whose simple modules are injective.

Let Ω be a set of endomorphisms of the additive group R^+ of the ring R containing 1 and $-w$ whenever $w \in \Omega$ such that

$$w(xy) = (wx)y = x(wy) \qquad \text{for all} \quad w \in \Omega \quad \text{and} \quad x, y \in R.$$

Let $X = \{X_j \mid j \in I, I \text{ some index set}\}$ be a set of noncommutative indeterminates, and consider the polynomials

$$p[X] = p(X_1, X_2, \dots, X_n) = \sum w_{(i)} X_{i_1} X_{i_2} \dots X_{i_n}$$

with coefficients $w_{(i)} \in \Omega$, where all monomials $X_{i_1} X_{i_2} \dots X_{i_n}$ are different. Then by Amitsur [1, p. 470] *the ring R satisfies a nontrivial general polynomial identity*, if there exists a polynomial $p[X] \in \Omega[X]$ such that

$$p(r_1, r_2, \dots, r_n) = 0$$

for all $r_i \in R$, and

$$w_{(i)} R \neq 0$$

for some coefficient $w_{(i)}$ of $p[X]$.

It is clear that Ω induces linear mappings in each epimorphic image of R. If the polynomial $p[X]$ is nontrivially satisfied by all epimorphic images of R, then R is a ring *satisfying a homomorphic polynomial identity*. Clearly each standard identity is a homomorphic polynomial identity. If the simple right R modules are injective, then also the converse is true by

PROPOSITION 5.1. *If the simple right R modules are injective, then R satisfies a homomorphic polynomial identity if and only if R satisfies a standard identity.*

Proof. We may assume that R satisfies a homomorphic identity. Since the simple right R modules are injective, R is semiprimitive by Theorem 2.1. Hence R is a subring of a direct product of primitive rings R_α satisfying $p[X]$. By Kaplansky's theorem R_α is a central simple algebra of dimension $c \leqslant \frac{1}{4}d^2$ over its center C_α, where d is the degree of $p[X]$. By Proposition 1 of [8, p. 227] there is then an integer k such that each ring R_α satisfies the standard identity of degree k. Thus R satisfies a standard identity.

LEMMA 5.2. *If R is a prime ring satisfying a nontrivial polynomial identity of minimal degree d such that each right ideal of R is idempotent then R is a finite-dimensional, central simple algebra over its center C, and $[R : C] = \frac{1}{4}d^2$.*

Proof. By Theorem 7 of Amitsur [1] R has a classical ring of quotients Q which is a central simple algebra over its center C of dimension $\frac{1}{4}d^2$ over C. Let c be a nonzero divisor of R. Then cR is an essential right ideal of R. Hence by Theorem 9 of Amitsur [1] there is a nonzero two-sided ideal X contained in cR. Therefore

$$R = X \leqslant cR \leqslant R$$

by Lemma 3.1, and c is a unit in R. Thus $R = Q$.

Remark 5.3. The following results and notations from Pierce [13] will be used throughout the rest of the paper. Let $B(R)$ be the Boolean algebra of all central idempotents of R(cf. [13, p. 4]). For each maximal ideal M of $B(R)$ let $\overline{M} = MR = \{er \mid e \in M,\ r \in R\}$. Equipped with the hull-kernel topology the set $\mathfrak{X}(R)$ of all maximal ideals of $B(R)$ is a totally disconnected, compact, Hausdorff space. Let

$$\mathfrak{R}(R) = \bigcap_{M \in \mathfrak{X}(R)} R/\overline{M}.$$

Then in [13, p. 16] it is shown that $\mathfrak{R}(R)$ can be topologized such that $\mathfrak{R}(R)$ becomes a reduced sheaf of rings over the Boolean space $\mathfrak{X}(R)$. Furthermore, by Theorem 4.4 of Pierce [13], R is isomorphic to the ring $\Gamma(\mathfrak{X}(R), \mathfrak{R}(R))$ of all sections of $\mathfrak{R}(R)$ over $\mathfrak{X}(R)$. Therefore, by Proposition 3.4 of Pierce [13], the ring R is von Neumann regular, if each ring $R/\overline{M}$ with $M \in \mathfrak{X}(R)$ is von Neumann regular.

For the proof and the statement of the next result we repeat the following (well-known) definitions. The right ideal X of R is *potent*, if it is not a nilideal. By Jacobson [8, p. 210] the ring R is an *I ring*, if each potent right ideal of R contains an idempotent $x \neq 0$.

LEMMA 5.4. *Let R be a ring satisfying a homomorphic polynomial identity such that the simple R modules are injective. If each epimorphic image of R is an I ring, and if R does not contain nilpotent elements $x \neq 0$, then R is von Neumann regular.*

Proof. By Theorem 2.1 each epimorphic image of R is semisimple. Let $p[X]$ be a homomorphic polynomial identity satisfied by R, and let n be its degree. Let $M \in \mathfrak{X}(R)$, and let $\bar{R} = R/\bar{M}$. If P is a prime ideal of $\bar{R}$, then $\bar{R}/P$ is a central simple algebra over its center C of dimension $[\bar{R}/P\colon C] = t^2$ by Lemma 5.2, where t is a positive integer less or equal to n. Therefore $x^t = 0$ for all nilpotent elements $x \in \bar{R}/P$. Clearly $x^n = 0$. Since n is independent of the prime ideal P of $\bar{R}$, the ring $\bar{R}$ is an I ring having the property that $\bar{R}/P$ is of bounded index for every primitive ideal P of $\bar{R}$. By Theorem 2.1 the ring $\bar{R}$ is semiprimitive. Thus by Theorem 3 of [8, p. 239] every nonzero two-sided ideal B of $\bar{R}$ contains a central idempotent $e_M \neq 0$ of $\bar{R}$. Since $\mathfrak{R}(R)$ is a sheaf over the Boolean space $\mathfrak{X}(R)$, there is an open and closed neighborhood $\mathfrak{N}$ of M in $\mathfrak{X}(R)$ and a section $\tau \in \Gamma(\mathfrak{N}, \mathfrak{R}(R))$ such that $\tau(M) = e_M$ and $(\tau^2 - \tau)(Q) = 0_Q = 0 + QR \in R/Q$ for all $Q \in \mathfrak{N}$(cf. [13, Lemma 3.2, p. 11]). Define $\sigma \in \Gamma(\mathfrak{X}(R), \mathfrak{R}(R))$ by $\sigma(X) = \tau(X)$ for all $X \in \mathfrak{N}$ and $\sigma(X) = 0_X$ for all $X \in \mathfrak{X}(R) - \mathfrak{N}$. Then $\sigma = \sigma^2$. Since $\Gamma(\mathfrak{X}(R), \mathfrak{R}(R)) \cong R$, and since R does not contain nilpotent elements $x \neq 0$, it follows that σ is a central idempotent of $\Gamma(\mathfrak{X}(R), \mathfrak{R}(R))$ such that $\sigma(M) = e_M \neq 0$. As $\mathfrak{R}(R)$ is a reduced sheaf of rings over $\mathfrak{X}(R)$, it follows that $\sigma(M) = e_M$ is the identity of $\bar{R} = R/MR$ (cf. [13, p. 15]). Therefore $B = \bar{R}$ and $\bar{R}$ is simple. By Lemma 5.2 $\bar{R}$ is then a central simple algebra over its center. Thus $\bar{R}$ is von Neumann regular. This completes the proof of Lemma 5.4.

THEOREM 5.5. *Let R be a ring satisfying a homomorphic polynomial identity such that the simple R modules are injective.*

Then R is von Neumann regular if and only if every epimorphic image of R is an I ring.

Proof. Clearly each epimorphic image of a von Neumann regular ring is an I ring. Therefore it remains to prove the sufficiency of the condition.

Suppose that R is not von Neumann regular. An easy application of Zorn's lemma shows that the sum $T(R)$ of all two-sided ideals K of R which are von Neumann regular as a ring is a von Neumann regular ring, and that

$T(R/T(R)) = 0$. Clearly, $R \neq T(R)$. By Theorem 2.1 the simple right $R/T(R)$ modules are also injective. Hence we may assume that $T(R) = 0$.

By Theorem 2.1 and Lemma 4.1 the ring R is a semiprimitive I ring having the property that R/P is of bounded index for every primitive ideal P of R. Therefore by Theorem 3 of [8, p. 239] every nonzero ideal B of R contains a two-sided ideal $C \neq 0$ generated by a central idempotent $e \neq 0$ of R such that C is isomorphic to a ring of $n \times n$ matrices over a ring D without nilpotent elements. Since $R = C \oplus (1 - e)R$ each simple C module is injective, and each epimorphic image of C is an I ring. By Proposition 1 of [8, p. 40] each ideal of C is a ring of $n \times n$ matrices over an ideal of D. Therefore every epimorphic image of D is an I ring by [8, p. 211]. Since D is Morita equivalent to C, the simple D modules are also injective by Theorem 2.5. As $R = C \oplus (1 - e)R$, the ring C satisfies a homomorphic polynomial identity, say $p[X]$ with degree less or equal to d, where d is an even integer. Because of the semisimplicity of C it follows by application of Lemma 5.2 that C satisfies a standard identity of degree less or equal to d (cf. [1, p. 478]). Hence also D satisfies a homomorphic polynomial identity therefore D is von Neumann regular by Lemma 5.4. Thus the nonzero ideal C of R is von Neumann regular. This is a contradiction, because $T(R) = 0$. Hence Theorem 5.5 holds.

COROLLARY 5.6. *A π-regular ring R satisfying a homomorphic polynomial identity such that the simple R modules are injective is von Neumann regular.*

Proof. The proof follows at once from Theorem 4.3 and Proposition 1 of Jacobson [8, p. 210].

6. AFFINE RINGS WHOSE SIMPLE MODULES ARE INJECTIVE

The ring R is finitely generated as a ring over its center $Z(R)$, if R is an epimorphic image of a free (noncommutative) ring over $Z(R)$ generated by finitely many indeterminates $X_1, X_2, \ldots, X_n$ which only commute with the elements of $Z(R)$. Following C. Procesi the ring R is called an *affine ring*, if R is finitely generated over its center $Z(R)$, and if every epimorphic image of R satisfies a polynomial identity with coefficients contained in its center. It is well known that each ring R which is finitely generated as a module over its center is an affine ring. In this section Kaplansky's theorem mentioned in the introduction is generalized for affine rings.

With the notations of Remark 5.3 the following statement holds.

LEMMA 6.1. *Let R be an affine ring. Then $R/\overline{M}$ is an indecomposable ring for every maximal ideal M of $B(R)$.*

Proof. Suppose $R/\overline{M}$ is not indecomposable for some $M \in \mathfrak{X}(R)$. Then there is a nonzero central idempotent $e_M \in R/\overline{M}$. Let $e_M = a + \overline{M}$. If $f_1, f_2, \ldots, f_n$ are the generators of the ring R over its center $Z(R)$, then $a - a^2 \in M$, and $af_i - f_i a \in M$ for $i = 1, 2, \ldots, n$. Since $R \cong \Gamma = \Gamma(\mathfrak{X}(R), \mathfrak{R}(R))$, the ring Γ is finitely generated as a ring over its center Δ. Let φ_i, $i = 1, 2, \ldots, n$ be the corresponding generators. Since $\mathfrak{R}(R)$ is a sheaf over the Boolean space $\mathfrak{X}(R)$, by Lemma 3.2 of [13] there is an open neighborhood $\mathfrak{N}$ of M in $\mathfrak{X}(R)$ and a section $\tau \in \Gamma(\mathfrak{N}, \mathfrak{R}(R))$ such that for all $Q \in \mathfrak{N}$

(1) $(\tau - \tau^2)(Q) = 0_Q = 0 + \overline{Q} \in R/\overline{Q}$,

(2) $(\tau\varphi_i - \varphi_i\tau)(Q) = 0_Q = 0 + \overline{Q} \in R/\overline{Q}$.

Define $\sigma \in \Gamma(\mathfrak{X}(R), \mathfrak{R}(R))$ by $\sigma(X) = \tau(X)$ for all $X \in \mathfrak{N}$, and $\sigma(X) = 0_X$ for all $X \in \mathfrak{X}(R) - \mathfrak{N}$. Then $\sigma^2 = \sigma$, and $\varphi_i\sigma = \sigma\varphi_i$ for $i = 1, 2, \ldots, n$ by Proposition 3.4 of [13]. Thus $\sigma \in B(\Gamma(\mathfrak{X}(R), \mathfrak{R}(R))$. Since $\mathfrak{R}(R)$ is a reduced sheaf over $\mathfrak{X}(R)$, it follows from Pierce [13, p. 15] that

$$0 \neq e_M = \tau(M) = \sigma(M) = 1 + \overline{M},$$

a contradiction. Therefore Lemma 6.1 holds.

LEMMA 6.2. *Let R be an affine ring such that the right ideals of every epimorphic image of R are idempotent. Then R is biregular.*

Proof. Let M be a maximal ideal of the Boolean algebra $B(R)$ of central idempotents of R. Then using the notations of Remark 5.3 the ring $\bar{R} = R/\overline{M}$ is indecomposable by Lemma 6.1. Since each right ideal of $\bar{R}$ is idempotent, its center C is von Neumann regular by Lemma 2.3. Hence C is a field. As R is finitely generated as a ring over its center $Z(R)$, and as $(Z(R) + \overline{M})/\overline{M} \leqslant C$, the ring $\bar{R}$ is finitely generated over C. Let P be a prime ideal of $\bar{R}$. Then $R' = \bar{R}/P$ is a prime ring satisfying a nontrivial polynomial identity such that each right ideal of R' is idempotent. Thus by Lemma 5.2 R' is a simple ring with minimum condition on right and left ideals. Therefore Theorem 1 of Procesi [14] applies, and $\bar{R}$ is right artinian. Since $\bar{R}$ is semi-primitive and indecomposable, it follows that $\bar{R} = R/\overline{M}$ is simple for every maximal ideal M of $B(R)$. Hence $\mathfrak{R}(R)$ is a sheaf of simple rings over the Boolean space $\mathfrak{X}(R)$. Thus R is biregular by Remark 5.3 and the theorem of Dauns and Hofmann [5].

THEOREM 6.3. *The following properties of the affine ring R are equivalent:*

(1) *Every simple right R module is injective.*

(2) *R is von Neumann regular.*

(3) *R is biregular.*

(4) *R is isomorphic to the ring $\Gamma(\mathfrak{X}, \mathfrak{R})$ of all sections of a sheaf $\mathfrak{R}$ of central simple finite-dimensional algebras over a totally disconnected, compact, Hausdorff space.*

(5) *Every simple left R module is injective.*

Proof. Since (3) is symmetric in both sides, it remains to prove the equivalence of the first four statements.

If (4) holds, then R is von Neumann regular by Remark 5.3. In case R satisfies (2), then Lemma 6.2 applies, and R is biregular. Suppose that (3) holds. Then by the theorem of Dauns and Hofmann [5] R is isomorphic to the ring $\Gamma(\mathfrak{X}, \mathfrak{R})$ of all sections of a sheaf $\mathfrak{R}$ of simple rings R_x, $x \in \mathfrak{X}$, over a totally disconnected, compact Hausdorff space $\mathfrak{X}$. Since R is affine, the simple ring R_x satisfies a nontrivial polynomial identity. Hence R_x is a central simple finite-dimensional algebra by Theorems 7 and 9 of [1]. Therefore (2), (3) and (4) are equivalent.

If (1) holds, then every right ideal of every epimorphic image of R is idempotent by Theorem 2.1 and Corollary 2.2. Therefore R is biregular by Lemma 6.2. Thus it remains to show that (2) implies (1).

Let M be a simple right R module of the affine, von Neumann regular ring R, and let $P = \{x \in R \mid Mx = 0\}$. Then P is prime ideal of R, and R/P is a von Neumann regular prime ring satisfying a polynomial identity. Hence R/P is simple and right artinian by Theorems 7 and 9 of [1], and P is maximal. By Lemma 2.3 the center $Z(R)$ of R is a von Neumann regular ring. Therefore $Q = P \cap Z(R)$ is a maximal ideal of $Z(R)$. Thus $S = Z(R) - Q$ is a multiplicatively closed set contained in the center of R. Hence $R_S = \{rs^{-1} \mid r \in R, s \in S\}$ is a ring, and $V = \{x \in R \mid xs = 0 \text{ for some } s \in S\}$ is a two-sided ideal of R. By the equivalence of (2) and (3), the ring R is biregular. Thus for every $0 \neq p \in P$ there is an idempotent $0 \neq e \in Q$ such that $p = er$ for some $r \in R$. Since $s = 1 - e \in S$, and since $ps = res = re(1 - e) = 0$, it follows that $P \leqslant V$. As P is maximal, and as $V \neq R$, we obtain $P = V$. Furthermore, every $s \in S$ is a unit modulo V. Hence $R_S = R/P$ is simple and right artinian.

We now can employ an argument due to Rosenberg and Zelinsky [15]. Let E be the injective hull of the right R module M, and let $s \in S$. Then Es is a right R module contained in E. If $T = \{e \in E \mid es = 0\} \neq 0$, then $0 \neq T \cap M = M \leqslant T$, because T is a right R module, and E is an essential extension of the simple module M. Thus $s \in P$, a contradiction! Therefore $Es \simeq E$, and Es is injective. Since E is indecomposable, it follows that $E = Es$ for every $s \in S$. Therefore E is a right R_S module, so is M. As E is an essential extension of the right R module M, E is an essential extension of the right R_S module M. Since R_S is simple and right artinian, M is an injective right

R_S module. Hence $E = M$, and M is injective. This completes the proof of Theorem 6.3.

Remark. Theorem 6.3 does not hold for arbitrary rings which are finitely generated (as a ring) over their centers, because the examples constructed by Cozzens are finitely generated over their center.

COROLLARY 6.4. *If R is finitely generated as a module over its center, then the simple right R modules are injective if and only if R is von Neumann regular.*

Proof. Since R satisfies a standard identity, Corollary 6.4 follows at once from Theorem 6.3.

LEMMA 6.5. *Let A be the group ring RG of the group G over the ring R. Let ωG be the augmentation ideal of A. Then $R \cong A/\omega G$ is a flat left A module if and only if the following conditions hold*:

(a) *G is locally finite.*

(b) *The order n of every element $g \in G$ is a unit of R.*

Proof. If R and G satisfy the conditions (a) and (b), then R is a flat left A module by Villamayor [16, p. 949].

Conversely, if $R \cong A/\omega G$ is a flat left A module, then (a) and (b) hold by Lemmas 1 and 4 of Villamayor [16].

COROLLARY 6.6. *Let A be the group ring RG of the group G over the ring R. If every simple right A module is injective, then A is von Neumann regular if and only if R is von Neumann regular.*

Proof. By Corollary 2.2 and Lemma 2.3 the left A module $R \cong A/\omega G$ is flat. Hence Corollary 6.6 follows from Lemma 6.5 and Proposition 2 of [9, p. 155].

COROLLARY 6.7. *The simple right A modules of the group ring $A = RG$ of the finite group G over the ring R are injective if and only if the order n of G is a unit in R and every simple right R module is injective.*

Proof. If every simple right A module is injective, then every simple right R module is injective by Theorem 2.1. Furthermore, $n = | G |$ is a unit in R by Lemmas 6.5 and 2.3.

Conversely, let M be a simple right A module. Then M is a finitely generated right R module. By Zorn's lemma M contains a maximal R submodule V. Let $N = \bigcap_{g \in G} Vg$. Then N is a proper A submodule of M. Hence $N = 0$, and the right R module M is isomorphic to a direct summand of the completely reducible right R module $X = \sum_{g \in G} \oplus M/Vg$. Since every simple right R module is injective, X is an injective right R module. Let $0 \to P \to Q$ be an exact sequence of right A modules, and let $\varphi \in \hom_A(P, M)$. Then there is a $\psi \in \hom_R(Q, M)$ with $\psi(p) = \varphi(p)$ for every $p \in P$, because M_R is injective. If for every $q \in Q$

$$\hat{\varphi}(q) = \frac{1}{n} \sum_{g \in G} \psi(qg)g^{-1},$$

then $\hat{\varphi} \in \hom_A(Q, M)$, and $\hat{\varphi}(p) = \varphi(p)$ for every $p \in P$. Thus M is an injective right A module.

COROLLARY 6.8. *If the simple right A modules of the group ring $A = RG$ of the group G over the affine ring R are injective, then A is von Neumann regular.*

Proof. By Theorem 2.1 the simple right R modules are injective. Hence R is von Neumann regular by Theorem 6.3. Therefore A is von Neumann regular by Corollary 6.6.

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