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Hochschild (Co)Homology of Differential Operator Rings[1]

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We show that the Hochschild homology of a differential operator k-algebra $E = A\#_f U(g)$ is the homology of a deformation of the Chevalley–Eilenberg complex of $\mathfrak{g}$ with coefficients in $(M \otimes \overline{A}^*, b_*)$. Moreover, when A is smooth and k is a characteristic zero field, we obtain a type of Hochschild–Kostant–Rosenberg theorem for these algebras. When $A = k$ our complex reduces to the one obtained by C. Kassel (1988, *Invent. Math*. **91**, 221–251) for the homology of filtrated algebras whose associated graded algebras are symmetric algebras. In the last section we give similar results for the cohomology. © 2001 Academic Press

INTRODUCTION

Let k be a field and A an associative k-algebra with 1. An extension $E \supseteq A$ of A is a differential operator ring on A if there exists a k-Lie algebra $\mathfrak{g}$ and a vector space embedding $x \mapsto \bar{x}$, of $\mathfrak{g}$ into E, such that for all $x, y \in \mathfrak{g}$, $a \in A$:

(1) $\bar{x}a - a\bar{x} = a^x$, where $a \mapsto a^x$ is a derivation,

(2) $\overline{xy} - \overline{yx} = \overline{[x, y]_{\mathfrak{g}}} + f(x, y)$, where $[—, —]_{\mathfrak{g}}$ is the bracket of $\mathfrak{g}$ and $f : \mathfrak{g} \times \mathfrak{g} \to A$ is a bilinear map,

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(3) for a given basis $(x_i)_{i\in I}$ of $\mathfrak{g}$, E is a free left A-module with the standard monomials in the x_i's as a basis.

This general construction was introduced in [Ch, Mc-R]. Several particular cases of this type of extensions have been considered previously in the literature. For instance:

— when $\mathfrak{g}$ is one dimensional, f is trivial and E is the Ore extension $A[x, \delta]$, where $\delta(a) = a^x$,

— when $A = k$, one obtains the algebras studied by Sridharan in [S], which are the quasi-commutative algebras E, whose associated graded algebra is a symmetric algebra.

— in [Mc, Sect. 2] this type of extension was studied under the hypothesis that A is commutative and $(x, a) \mapsto a^x$ is an action and in [B-G-R, Theorem 4.2] the case in which the cocycle is trivial was considered.

In [B-C-M, D-T] the study of the crossed products $A\#_f H$ of an algebra A by a Hopf algebra H was begun and in [M] it was proved that the differential operator rings on A are the crossed products of A by enveloping algebras of Lie algebras.

In [G-G] we obtained a complex, simpler than the canonical one, giving the Hochschild homology of a general crossed product $E = A\#_f H$ with coefficients in an arbitrary E-bimodule M. In the present paper we show that, for differential operator rings, a complex simpler than the one obtained in [G-G] also works, and we give some applications of this result.

This paper is organized as follows: In Section 1 we recall the definition of differential operator rings following the Hopf algebra point of view of [B-C-M, D-T]. In Section 2 we recall a technical result, established in [G-G], that we need in order to carry out our computations. In Section 3 we get a resolution of a differential operator ring $E = A\#_f U(\mathfrak{g})$ as an E-bimodule. This resolution is a mixture of the canonical Hochschild normalized resolution of A and the Chevalley–Eilenberg resolution of $\mathfrak{g}$. In Section 4 we study the Hochschild homology of E with coefficients in an arbitrary E-bimodule M. The main result is Theorem 4.1, where the promised complex, which is a deformation of the Chevalley–Eilenberg complex of $\mathfrak{g}$ with coefficients in $(M \otimes \overline{A}^*, b_*)$, is obtained. Then we consider a natural filtration of this complex, and we derive from it the spectral sequence of [St] in a more explicit way than the original one. Then, we consider the case when A is a commutative smooth algebra. The result obtained by us under this condition is a common generalization of the Hochschild–Kostant–Rosenberg theorem and the computation given in [K] for the Hochschild homology of algebras whose associated graded algebras are symmetric algebras. Finally, in Section 5, we study the cohomology.

1. PRELIMINARIES

Let A be a k-algebra and H a Hopf algebra. A *weak action* of H on A is a bilinear map $(h, a) \mapsto a^h$ from $H \times A$ to A such that, for $h \in H$, $a, b \in A$,

(1) $(ab)^h = \sum_{(h)} a^{h^{(1)}} b^{h^{(2)}}$,

(2) $1^h = \epsilon(h)1$,

(3) $a^1 = a$.

By an *action* of H on A we mean a weak action such that

(4) $(a^1)^h = a^{hl}$ for all $h, l \in H$, $a \in A$.

Let A be a k-algebra and H a Hopf algebra with a weak action on A. Given a k-linear map $f : H \otimes H \to A$ we let $A\#_f H$ denote the k-algebra (in general non-associative and without 1) whose underlying vector space is $A \otimes H$ and whose multiplication is given by

$$(a \otimes h)(b \otimes l) = \sum_{(h)(l)} ab^{h^{(1)}} f\big(h^{(2)}, l^{(1)}\big) \otimes h^{(3)} l^{(2)},$$

for all $a, b \in A$, $h, l \in H$. The element $a \otimes h$ of $A\#_f H$ will usually be written $a\#h$ to remind us H is weakly acting on A. The algebra $A\#_f H$ is called a *crossed product* if it is associative with $1\#1$ as identity element. In [B-C-M] was proved that this happens if and only if f and the weak action satisfy the following conditions

(1) (normality of f) for all $h \in H$ we have $f(h,1) = f(1,h) = \epsilon(h)1_A$,

(2) (cocycle condition) for all $h, l, m \in H$ we have

$$\sum_{(h)(l)(m)} f\big(l^{(1)}, m^{(1)}\big)^{h^{(1)}} f\big(h^{(2)}, l^{(2)} m^{(2)}\big) = \sum_{(h)(l)} f\big(h^{(1)}, l^{(1)}\big) f\big(h^{(2)} l^{(2)}, m\big),$$

(3) (twisted module condition) for all $h, l \in H$, $a \in A$ we have

$$\sum_{(h)(l)} \big(a^{l^{(1)}}\big)^{h^{(1)}} f\big(h^{(2)}, l^{(2)}\big) = \sum_{(h)(l)} f\big(h^{(1)}, l^{(1)}\big) a^{h^{(2)} l^{(2)}}.$$

From now on, we assume that H is the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. In this case, item (1) of the definition of weak action implies that $(ab)^x = a^x b + ab^x$ for $x \in \mathfrak{g}$. So, a weak action determines a linear map $\delta : \mathfrak{g} \to \mathrm{Der}_k(A)$ by $\delta(x)(a) = a^x$. Moreover if $(h, a) \mapsto a^h$ is an action, then δ is a homomorphism of Lie algebras. Reciprocally given a linear map $\delta : \mathfrak{g} \mapsto \mathrm{Der}_k(A)$, there exists a (generality non-unique) weak action of $U(\mathfrak{g})$ on A such that $\delta(x)(a) = a^x$. When δ is a homomorphism of Lie algebras, there is a unique action of $U(\mathfrak{g})$ on A such that $\delta(x)(a) = a^x$. For a proof of these facts see [B-C-M].

Next we show that each normal cocycle $f : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to A$ is convolution invertible, giving a formula for f^{-1}.

Remark 1.1. Each normal cocycle $f : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to A$ is convolution invertible. Moreover, for each $h \in U(\mathfrak{g})$ and each family $x_1, \dots, x_r$ of elements of $\mathfrak{g}$, we have $f^{-1}(1, h) = f^{-1}(h, 1) = \epsilon(h)1_A$ and

$$\begin{aligned} &f^{-1}(x_1 \cdots x_r, h) \\ &\quad = \sum_{l=1}^{r} (-1)^l \sum_{\substack{1 \le p_1, \dots, p_l \\ p_1 + \cdots + p_l = r}} \sum_{\tau \in Sh_{p_1, \dots, p_l}} \sum_{(h)} f\big(x_{\tau(1)} \cdots x_{\tau(p_1)}, h^{(1)}\big) \\ &\qquad \times f\big(x_{\tau(p_1+1)} \cdots x_{\tau(p_1+p_2)}, h^{(2)}\big) \cdots f\big(x_{\tau(p_1+\cdots+p_{l-1}+1)} \cdots x_{\tau(r)}, h^{(1)}\big), \end{aligned}$$

where $Sh_{p_1,\dots,p_l}$ denotes the multishuffles associated to $p_1, \dots, p_l$. That is,

$$Sh_{p_1,\dots,p_l} = \left\{ \tau \in \mathfrak{S}_r : \tau\left(1 + \sum_{j=1}^{i} p_j\right) < \cdots < \tau\left(\sum_{j=1}^{i+1} p_j\right) \text{ for } 0 \le i < l \right\}.$$

This fact can be proved by a direct computation.

2. A METHOD FOR CONSTRUCTING RESOLUTIONS

Let k be a commutative ring with 1 and E a k-algebra. In this section we recall a result that we will use in Section 3. For the proof we remit to [G-G].

Let

$$\begin{tikzcd} \vdots \arrow[d, "\partial_3"] & & & \\ Y_2 \arrow[d, "\partial_2"] & X_{02} \arrow[l, "\mu_2"'] & X_{12} \arrow[l, "d^0_{12}"'] & \cdots \arrow[l, "d^0_{22}"'] \\ Y_1 \arrow[d, "\partial_1"] & X_{01} \arrow[l, "\mu_1"'] & X_{11} \arrow[l, "d^0_{11}"'] & \cdots \arrow[l, "d^0_{21}"'] \\ Y_0 & X_{00} \arrow[l, "\mu_0"'] & X_{10} \arrow[l, "d^0_{10}"'] & \cdots \arrow[l, "d^0_{20}"'] \end{tikzcd}$$

be a diagram of E-bimodules and morphisms of E-bimodules verifying:

(1) The column and the rows are chain complexes,

(2) Each X_{rs} is isomorphic to a free E-bimodule $E \otimes \overline{X}_{rs} \otimes E$,

(3) Each row is contractible as a complex of left E-modules, with a chain contracting homotopy $\sigma^0_{0s} : Y_s \to X_{0s}$ and $\sigma^0_{r+1,s} : X_{rs} \to X_{r+1,s}$ $(r \ge 0)$.

We define E-bimodule morphisms $d^l_{rs} : X_{rs} \to X_{r+l-1,\, s-l}$ ($r \geq 0$ and $1 \leq l \leq s$), recursively by

$$d^l_{rs}(\mathbf{x}) = \begin{cases} -\sigma^0_{0,\, s-1} \circ \partial_s \circ \mu_s(\mathbf{x}) & \text{if } r = 0 \text{ and } l = 1, \\ -\sum_{j=1}^{l-1} \sigma^0_{l-1,\, s-l} \circ d^{l-j}_{j-1,\, s-j} \circ d^j_{0s}(\mathbf{x}) & \text{if } r = 0 \text{ and } 1 < l \leq s, \\ -\sum_{j=0}^{l-i} \sigma^0_{r+l-1,\, s-l} \circ d^{l-j}_{r+j-1,\, s-j} \circ d^j_{rs}(\mathbf{x}) & \text{if } r > 0, \end{cases}$$

for $\mathbf{x} = 1 \otimes \bar{\mathbf{x}} \otimes 1$ with $\bar{\mathbf{x}} \in \overline{X}_{rs}$.

THEOREM 2.1. *Let $\tilde{\mu} : Y_0 \to E$ be a morphism of E-bimodules such that*

$$E \xleftarrow{\tilde{\mu}} Y_0 \xleftarrow{\partial_1} Y_1 \xleftarrow{\partial_2} Y_2 \xleftarrow{\partial_3} Y_3 \xleftarrow{\partial_4} Y_4 \xleftarrow{\partial_5} Y_5 \xleftarrow{\partial_6} Y_6 \xleftarrow{\partial_7} \dots,$$

is a complex that is contractible as a complex of left E-modules. Then

$$E \xleftarrow{\mu} X_0 \xleftarrow{d_1} X_1 \xleftarrow{d_2} X_2 \xleftarrow{d_3} X_3 \xleftarrow{d_4} X_4 \xleftarrow{d_5} X_5 \xleftarrow{d_6} X_6 \xleftarrow{d_7} \dots,$$

where

$$\mu = \tilde{\mu} \circ \mu_0, \qquad X_n = \bigoplus_{r+s=n} X_{rs}, \qquad \text{and} \qquad d_n = \sum_{\substack{r+s=n \\ r+l>0}} \sum_{l=0}^{s} d^l_{rs},$$

is a relative projective resolution of E as an E-bimodule.

3. A RESOLUTION FOR A DIFFERENTIAL OPERATOR RING

Let $E = A\#_f U(\mathfrak{g})$ be a crossed product. In this section we obtain an E-bimodule resolution (X_*, d_*) of E, that is simpler than the canonical of Hochschild. Then an explicit expression of the boundary maps of this resolution is given. To begin, we fix some notations:

(1) For each k-algebra B and each $r \in \mathbb{N}$, we write $\overline{B} = B/k$, $B^r = B \otimes \cdots \otimes B$ (r times) and $\overline{B}^r = \overline{B} \otimes \cdots \otimes \overline{B}$ (r times). Moreover, for $b \in B$, we also let b denote the class of b in $\overline{B}$.

(2) Given $a_0 \otimes \cdots \otimes a_r \in A^{r+1}$ and $0 \leq i < j \leq r$, we write $\mathbf{a}_{ij} = a_i \otimes \cdots \otimes a_j$.

(3) For each Lie k-algebra $\mathfrak{g}$ and each $s \in \mathbb{N}$, we write $\mathfrak{g}^{\wedge s} = \mathfrak{g} \wedge \cdots \wedge \mathfrak{g}$ (s times).

(4) Given $\mathbf{x} = x_1 \wedge \cdots \wedge x_s \in \mathfrak{g}^{\wedge s}$ and $1 \leq i \leq s$, we write $\mathbf{x}_{\hat{i}} = x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_s$.

(5) Given $\mathbf{x} = x_1 \wedge \cdots \wedge x_s \in \mathfrak{g}^{\wedge s}$ and $1 \leq i < j \leq s$, we write $\mathbf{x}_{\hat{i}\hat{j}} = x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_s$.

3.1. *The Complex* (Y'_*, ∂'_X)

Let $\tilde{\mathfrak{g}}$ be the direct sum of two copies $\{y_x : x \in \mathfrak{g}\}$ and $\{z_x : x \in \mathfrak{g}\}$ of $\mathfrak{g}$, endowed with the bracket given by $[y_x, y_{x'}]_{\tilde{\mathfrak{g}}} = y_{[x, x']_{\mathfrak{g}}}$ and $[y_x, z_{x'}]_{\tilde{\mathfrak{g}}} = [z_x, z_{x'}]_{\tilde{\mathfrak{g}}} = z_{[x, x']_{\mathfrak{g}}}$. Note that $\tilde{\mathfrak{g}}$ is the semi-direct sum arising from the adjoint action of $\mathbf{g}$ on itself. Let $\pi : U(\tilde{\mathfrak{g}}) \to U(\mathfrak{g})$ be the algebra map defined by $\pi(y_x) = \pi(z_x) = x$. Let $\Lambda(\mathfrak{g})$ be the exterior algebra generated by $\mathfrak{g}$. That is, the algebra generated by the elements $e_x (x \in \mathfrak{g})$ and the relations $e_{\lambda x + x'} = \lambda e_x + e_{x'}$ and $e_x^2 = 0$ $(\lambda \in k, x, x' \in \mathfrak{g})$. Let us consider the action of $U(\tilde{\mathfrak{g}})$ on $\Lambda(\mathfrak{g})$ determined by $e_{x'}^{y_x} = e_{[x, x']_{\mathfrak{g}}}$ and $e_{x'}^{z_x} = 0$. The enveloping algebra $U(\tilde{\mathfrak{g}})$ of $\tilde{\mathfrak{g}}$ acts weakly on $A \otimes \Lambda(\mathfrak{g})$ via $(a \otimes e)^u = a^{\pi(u)} \otimes e + a \otimes e^u (a \in A, e \in \Lambda(\mathfrak{g})$ and $u \in U(\tilde{\mathfrak{g}}))$. Moreover, the map $\tilde{f} : U(\tilde{\mathfrak{g}}) \times U(\tilde{\mathfrak{g}}) \to A \otimes \Lambda(\mathfrak{g})$, defined by $\tilde{f}(u, v) = f(\pi(u), \pi(v)) \otimes 1$, is a normal 2-cocycle which satisfies the twisted module condition.

THEOREM 3.1.1. *Let Y'_* be the graded algebra generated by A, the degree zero elements $y_x, z_x (x \in \mathfrak{g})$, the degree one elements $e_x (x \in \mathfrak{g})$, and the relations*

$$y_{\lambda x + x'} = \lambda_{y_x} + y_{x'}, \qquad y_x a = a^x + a y_x, \qquad e_{x'} y_x = y_x e_{x'} + e_{[x', x]_{\mathfrak{g}}},$$
$$z_{\lambda x + x'} = \lambda z_x + z_{x'}, \qquad z_x a = a^x + a z_x, \qquad e_{x'} z_x = z_x e_{x'},$$
$$e_{\lambda x + x'} = \lambda e_x + e_{x'}, \qquad e_x a = a e_x, \qquad e_x^2 = 0,$$
$$y_{x'} y_x = y_x y_{x'} + y_{[x', x]_{\mathfrak{g}}} + f(x', x) - f(x, x'),$$
$$z_{x'} y_x = y_x z_{x'} + z_{[x', x]_{\mathfrak{g}}} + f(x', x) - f(x, x'),$$
$$z_{x'} z_x = z_x z_{x'} + z_{[x', x]_{\mathfrak{g}}} + f(x', x) - f(x, x').$$

Let $(x_i)_{i \in I}$ be a basis of $\mathfrak{g}$ with indexes running on an ordered set I. For each $i \in I$ let us write $y_i = y_{x_i}, z_i = z_{x_i}$, and $e_i = e_{x_i}$. Then each Y'_s is a free A-module with basis

$$y_{i_1}^{m_1} e_{i_1}^{\delta_1} z_{i_1}^{n_1} \cdots y_{i_l}^{m_l} e_{i_l}^{\delta_l} z_{i_l}^{n_l} \qquad \begin{pmatrix} l \geq 0, i_1 < \cdots < i_l \in I, m_j, n_j \geq 0, \delta_j \in \{0, 1\} \\ m_j + \delta_j + n_j > 0, \delta_1 + \cdots + \delta_l = s \end{pmatrix}.$$

Proof. Let $\vartheta : Y'_* \to (A \otimes \Lambda(\mathfrak{g})) \#_{\tilde{f}} U(\tilde{\mathfrak{g}})$ be the homomorphism of algebras defined by $\vartheta(a) = (a \otimes 1)\#1$ for all $a \in A$ and $\vartheta(y_x) = (1 \otimes 1)\# y_x$, $\vartheta(z_x) = (1 \otimes 1)\# z_x$ and $\vartheta(e_x) = (1 \otimes e_x)\#1$ for all $x \in \mathfrak{g}$. Because of the Poincaré–Birkhoff–Witt theorem,

$$\vartheta\left(y_{i_1}^{m_1} e_{i_1}^{\delta_1} z_{i_1}^{n_1} \cdots y_{i_l}^{m_l} e_{i_l}^{\delta_l} z_{i_l}^{n_l}\right)$$
$$\left(l \geq 0, i_1 < \cdots < i_l \in I, m_j, n_j \geq 0 \text{ and } \delta_j \in \{0, 1\}\right),$$

is a basis of $(A \otimes \Lambda(\mathfrak{g})) \#_{\tilde{f}} U(\tilde{\mathfrak{g}})$ as an A-module. The theorem follows immediately from this fact. ∎

REMARK 3.1.2. Note that E is a subalgebra of Y'_* by embedding $a \in A$ to a and $x \in \mathfrak{g}$ to y_x. This gives rise to a structure of the left E-module on Y'_s. Similarly we consider Y'_* as a right E-module via the embedding of E in Y'_* that sends $a \in A$ to a and $x \in \mathfrak{g}$ to z_x.

THEOREM 3.1.3. *Let $\tilde{\mu}' : Y'_0 \to E$ be the algebra map defined by $\tilde{\mu}'(a) = a$ for $a \in A$ and $\tilde{\mu}'(y_i) = \tilde{\mu}'(z_i) = x_i$ for $i \in I$. There is a unique derivation $\partial'_* : Y'_* \to Y'_{*-1}$ such that $\partial'_1(e_i) = z_i - y_i$ for $i \in I$. Moreover, the chain complex of E-bimodules*

$$E \xleftarrow{\tilde{\mu}'} Y'_0 \xleftarrow{\partial'_1} Y'_1 \xleftarrow{\partial'_2} Y'_2 \xleftarrow{\partial'_3} Y'_3 \xleftarrow{\partial'_4} Y'_4 \xleftarrow{\partial'_5} Y'_5 \xleftarrow{\partial'_6} Y'_6 \xleftarrow{\partial'_7} \dots$$

is contractible as a complex of k-modules. A chain contracting homotopy is given by $\sigma_0(a\#x_{i_1}^{m_1}\cdots x_{i_l}^{m_l}) = az_{i_1}^{m_1}\cdots z_{i_l}^{m_l}$ and

$$\sigma_{s+1}\Big(ay_{i_1}^{m_1}e_{i_1}^{\delta_1}z_{i_1}^{n_1}\cdots y_{i_l}^{m_l}e_{i_l}^{\delta_l}z_{i_l}^{n_l}\Big) = -\sum_{\substack{j<\alpha\\ 0\le h<m_j}} az_{i_1}^{m_1+n_1}\cdots z_{i_{j-1}}^{m_{j-1}+n_{j-1}}y_{i_j}^{h}e_{i_j}$$
$$\times z_{i_j}^{m_j+n_j-h-1}y_{i_{j+1}}^{m_{j+1}}e_{i_{j+1}}^{\delta_{j+1}}z_{i_{j+1}}^{n_{j+1}}\cdots y_{i_l}^{m_l}e_{i_l}^{\delta_l}z_{i_l}^{n_l},$$

where $\alpha = \min\{k : \delta_k = 1\}$ (in particular $\delta_1 = \cdots = \delta_{\alpha-1} = 0$).

Proof. We must check that $\tilde{\mu}'\circ\sigma_0 = id$, $\sigma_0\circ\tilde{\mu}' + \partial'_1\circ\sigma_1 = id$ and $\partial'_{s+1}\circ\sigma_{s+1} + \sigma_s\circ\partial'_s = id$ for all $s > 0$. It is immediate that

$$\tilde{\mu}'\circ\sigma_0\big(a\#x_{i_1}^{m_1}\cdots x_{i_l}^{m_l}\big) = \tilde{\mu}'\big(az_{i_1}^{m_1}\cdots z_{i_l}^{m_l}\big) = a\#x_{i_1}^{m_1}\cdots x_{i_l}^{m_l}$$

and

$$\sigma_0\circ\tilde{\mu}'\big(ay_{i_1}^{m_1}z_{i_1}^{n_1}\cdots y_{i_l}^{m_l}z_{i_l}^{n_l}\big) = \sigma_0\big(a\#x_{i_1}^{m_1+n_1}\cdots x_{i_l}^{m_l+n_l}\big) = az_{i_1}^{m_1+n_1}\cdots z_{i_l}^{m_l+n_l}.$$

Let us compute $\partial'_{s+1}\circ\sigma_{s+1}$ for $s\ge 0$ and $\sigma_s\circ\partial'_s$ for $s>0$. To abbreviate we write

$$\mathbf{M}_{\mathbf{i}_{uv}}^{\mathbf{m}\delta\mathbf{n}} = y_{i_u}^{m_u}e_{i_u}^{\delta_u}z_{i_u}^{n_u}\cdots y_{i_v}^{m_v}e_{i_v}^{\delta_v}z_{i_v}^{n_v} \qquad \text{for } 1\le u<v\le l,$$
$$\mathbf{Z}_{\mathbf{i}_{uv}}^{\mathbf{m}+\mathbf{n}} = z_{i_u}^{m_u+n_u}\cdots z_{i_v}^{m_v+n_v} \qquad \text{for } 1\le u<v\le l,$$
$$|\delta|_h = \delta_1 + \cdots + \delta_h \qquad \text{for } 1\le h\le l.$$

We have

$$\partial'_{s+1}\circ\sigma_{s+1}(a\mathbf{M}_{\mathbf{i}_{1l}}^{\mathbf{m}\delta\mathbf{n}}) = \partial'_s\Bigg(-\sum_{\substack{u<\alpha\\ 0\le h<m_u}} a\mathbf{Z}_{\mathbf{i}_{1,u-1}}^{\mathbf{m}+\mathbf{n}}y_{i_u}^{h}e_{i_u}z_{i_u}^{m_u+n_u-h-1}\mathbf{M}_{\mathbf{i}_{u+1,l}}^{\mathbf{m}\delta\mathbf{n}}\Bigg)$$
$$= -\sum_{\substack{u<\alpha\\ 0\le h<m_u}} a\mathbf{Z}_{\mathbf{i}_{1,u-1}}^{\mathbf{m}+\mathbf{n}}\big(y_{i_u}^{h}z_{i_u}^{m_u+n_u-h} - y_{i_u}^{h+1}z_{i_u}^{m_u+n_u-h-1}\big)\mathbf{M}_{\mathbf{i}_{u+1,l}}^{\mathbf{m}\delta\mathbf{n}}$$
$$-\sum_{\substack{u<\alpha\\ 0\le h<m_u}}\sum_{v\ge\alpha}(-1)^{|\delta|_v}\delta_v a\mathbf{Z}_{\mathbf{i}_{1,u-1}}^{\mathbf{m}+\mathbf{n}}y_{i_u}^{h}e_{i_u}z_{i_u}^{m_u+n_u-h-1}$$

$$\times \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,\,v-1}} y_{i_v}^{m_v} z_{i_v}^{n_v+1} \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{v+1,\,l}}$$
$$+ \sum_{\substack{u<\alpha \\ 0\le h<m_u}} \sum_{v\ge\alpha} (-1)^{|\delta|_v} \delta_v a \mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,\,u-1}} y_{i_u}^{h} e_{i_u} z_{i_u}^{m_u+n_u-h-1}$$
$$\times \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,\,v-1}} y_{i_v}^{m_v+1} z_{i_v}^{n_v} \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{v+1,\,l}},$$

where $\alpha = \min\{k : \delta_k = 1\}$. Since

$$\sum_{\substack{u<\alpha \\ 0\le h<m_u}} a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,\,u-1}} \big(y_{i_u}^{h} z_{i_u}^{m_u+n_u-h} - y_{i_u}^{h+1} z_{i_u}^{m_u+n_u-h-1}\big) \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,\,l}}$$
$$= \sum_{u<\alpha} a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,\,u-1}} \big(z_{i_u}^{m_u+n_u} - y_{i_u}^{m_u} z_{i_u}^{n_u}\big) \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,\,l}}$$
$$= a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,\,\alpha-1}} \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{\alpha l}} - a\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{1l}},$$

we obtain

$$\partial'_{s+1} \circ \sigma_{s+1}(a\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{1l}}) = -a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,\,\alpha-1}} \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}\alpha l} + a\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{1l}}$$
$$- \sum_{\substack{u<\alpha \\ 0\le h<m_u}} \sum_{v\ge\alpha} (-1)^{|\delta|_v} \delta_v a \mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,\,u-1}} y_{i_u}^{h} e_{i_u} z_{i_u}^{m_u+n_u-h-1}$$
$$\times \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,\,v-1}} y_{i_v}^{m_v} z_{i_v}^{n_v+1} \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{v+1,\,l}}$$
$$+ \sum_{\substack{u<\alpha \\ 0\le h<m_u}} \sum_{v\ge\alpha} (-1)^{|\delta|_v} \delta_v a \mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,\,u-1}} y_{i_u}^{h} e_{i_u} z_{i_u}^{m_u+n_u-h-1}$$
$$\times \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,\,v-1}} y_{i_v}^{m_v+1} z_{i_v}^{n_v} \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{v+1,\,l}}.$$

On the other hand

$$\sigma_s \circ \partial'_s(a\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{1l}}) = \sigma_s\Bigg(\sum_{v\ge\alpha} (-1)^{|\delta|_v-1} \delta_v a \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{1,\,v-1}}$$
$$\times \big(y_{i_v}^{m_v} z_{i_v}^{n_v+1} - y_{i_v}^{m_v+1} z_{i_v}^{n_v}\big) \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{v+1,\,l}}\Bigg).$$

Since

$$\sigma_s\Big(a\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{1,\,\alpha-1}} \big(y_{i_\alpha}^{m_\alpha} z_{i_\alpha}^{n_\alpha+1} - y_{i_\alpha}^{m_\alpha+1} z_{i_\alpha}^{n_\alpha}\big) \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{\alpha+1,\,l}}\Big)$$
$$= a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,\,\alpha-1}} y_{i_\alpha}^{m_\alpha} e_{i_\alpha} z_{i_\alpha}^{n_\alpha} \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{\alpha+1,\,l}} - \sum_{\substack{u<\alpha \\ 0\le h<m_u}} a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,\,u-1}} y_{i_u}^{h} e_{i_u} z_{i_u}^{m_u+n_u-h-1}$$
$$\times \mathbf{M}^{\mathbf{m}\delta n}_{\mathbf{i}_{u+1,\,\alpha-1}} \big(y_{i_\alpha}^{m_\alpha} z_{i_\alpha}^{n_\alpha+1} - y_{i_\alpha}^{m_\alpha+1} z_{i_\alpha}^{n_\alpha}\big) \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{\alpha+1,\,l}}$$

and, for $v > \alpha$,

$$\sigma_s\Big(a\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{1,v-1}}\big(y_{i_v}^{m_v} z_{i_v}^{n_v+1} - y_{i_v}^{m_v+1} z_{i_v}^{n_v}\big)\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{v+1,l}}\Big)$$
$$= -\sum_{\substack{u<\alpha\\ 0\le h<m_u}} a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,u-1}} y_{i_u}^h e_{i_u} z_{i_u}^{m_u+n_u-h-1}$$
$$\times \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,v-1}}\big(y_{i_v}^{m_v} z_{i_v}^{n_v+1} - y_{i_v}^{m_v+1} z_{i_v}^{n_v}\big)\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{v+1,l}},$$

we obtain

$$\sigma_s \circ \partial'_s(a\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{1l}}) = a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,\alpha-1}}\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{\alpha l}}$$
$$+ \sum_{\substack{u<\alpha\\ 0\le h<m_u}} \sum_{v\ge\alpha} (-1)^{|\delta|_v}\delta_v a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,u-1}} y_{i_u}^h e_{i_u} z_{i_u}^{m_u+n_u-h-1}$$
$$\times \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,v-1}} y_{i_v}^{m_v} z_{i_v}^{n_v+1}\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{v+1,l}}$$
$$- \sum_{\substack{u<\alpha\\ 0\le h<m_u}} \sum_{v\ge\alpha} (-1)^{|\delta|_v}\delta_v a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,u-1}} y_{i_u}^h e_{i_u} z_{i_u}^{m_u+n_u-h-1}$$
$$\times \mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,v-1}} y_{i_v}^{m_v+1} z_{i_v}^{n_v}\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{v+1,l}}.$$

The result follows immediately from these facts. ∎

3.2. *The Resolution* (X_*, d_*)

Let $Y_s = E \otimes \mathfrak{g}^{\wedge s} \otimes U(\mathfrak{g})$ $(s \ge 0)$ and $X_{rs} = E \otimes \mathfrak{g}^{\wedge s} \otimes \bar{A}^r \otimes E$ $(r, s \ge 0)$. The groups X_{rs} are E-bimodules in an obvious way and the groups Y_s are E-bimodules via the left canonical action and the right action

$$(a_0 \otimes (v \otimes \mathbf{x} \otimes w))(a\#u) = \sum_{(u)(v)(w)} a_0\big(a^{w^{(1)}}\big)^{v^{(1)}} f\big(w^{(2)}, u^{(1)}\big)^{v^{(2)}}$$
$$\otimes\big(v^{(3)} \otimes \mathbf{x} \otimes w^{(3)}u^{(2)}\big),$$

where $\mathbf{x} = x_1 \wedge \cdots \wedge x_s$. Let us consider the diagram.

$$\begin{array}{ccccccc}
\vdots & & & & & & \\
\big\downarrow \partial_3 & & & & & & \\
Y_2 & \xleftarrow{\mu_2} & X_{02} & \xleftarrow{d^0_{12}} & X_{12} & \xleftarrow{d^0_{22}} & \cdots \\
\big\downarrow \partial_2 & & & & & & \\
Y_1 & \xleftarrow{\mu_1} & X_{01} & \xleftarrow{d^0_{11}} & X_{11} & \xleftarrow{d^0_{21}} & \cdots \\
\big\downarrow \partial_1 & & & & & & \\
Y_0 & \xleftarrow{\mu_0} & X_{00} & \xleftarrow{d^0_{10}} & X_{10} & \xleftarrow{d^0_{20}} & \cdots
\end{array},$$

where $\partial_* : Y_* \to Y_{*-1}$, $\mu_* : X_{0*} \to Y_*$ and $d^0_{**} : X_{**} \to X_{*-1,*}$, are defined by

$$\begin{aligned}\partial_s(a\#v\otimes\mathbf{x}\otimes w) = &\sum_{i=1}^{s}(-1)^i \sum_{(v)(x_i)} af\left(v^{(1)}, x_i^{(1)}\right)\#v^{(2)}x_i^{(2)}\otimes\mathbf{x}_{\hat{i}}\otimes w\\ &-\sum_{i=1}^{s}(-1)^i \sum_{(w)(x_i)} af\left(x_i^{(1)}, w^{(1)}\right)^{v^{(1)}}\#v^{(2)}\otimes\mathbf{x}_{\hat{i}}\otimes x_i^{(2)}w^{(2)}\\ &-\sum_{1\le i<j\le s}(-1)^{i+j}a\#v\otimes[x_i,x_j]\wedge\mathbf{x}_{\hat{i}\hat{j}}\otimes w,\end{aligned}$$

$$\mu_s(a_0\#v\otimes\mathbf{x}\otimes a_1\#w) = \sum_{(v)} a_0 a_1^{v^{(1)}}\#v^{(2)}\otimes\mathbf{x}\otimes w,$$

$$\begin{aligned}d^0_{rs}(a_0\#v\otimes\mathbf{x}\otimes\mathbf{a}_{1,r+1}\#w) = &\sum_{(v)} a_0 a_1^{v^{(1)}}\#v^{(2)}\otimes\mathbf{x}\otimes\mathbf{a}_{2,r+1}\#w\\ &+\sum_{i=1}^{r}(-1)^i a_0\#v\otimes\mathbf{x}\otimes\mathbf{a}_{1,i-1}\\ &\otimes a_i a_{i+1}\otimes\mathbf{a}_{i+1,r+1}\#w,\end{aligned}$$

where $\mathbf{a}_{1,r+1} = a_1\otimes\cdots\otimes a_{r+1}$ and $\mathbf{x} = x_1\wedge\cdots\wedge x_s$. It is immediate that the μ_s's and the d^0_{rs}'s are E-bimodule maps. In the proof of Theorem 3.2.1 we will see that the ∂_s's are also. Each horizontal complex X_{*s} is the tensor product $(E\otimes U(\mathfrak{g}))\otimes_A (A\otimes\overline{A}^*, b'_*)\otimes_A E$, where $E\otimes U(\mathfrak{g})$ is a right A-module via the canonical inclusion of A in E. Hence, the family $\sigma^0_{0s} : Y_s \to X_{0s}$, $\sigma^0_{r+1,s} : X_{rs} \to X_{r+1,s} (r \ge 0)$, of left E-module maps, defined by

$$\sigma^0_{r+1,s}(a_0\#v\otimes\mathbf{x}\otimes\mathbf{a}_{1,r+1}\#w) = (-1)^{r+1}a_0\#v\otimes\mathbf{x}\otimes\mathbf{a}_{1,r+1}\otimes 1\#w \qquad (r\ge -1),$$

is a contracting homotopy of

$$Y_s \xleftarrow{\mu_s} X_{0s} \xleftarrow{d^0_{1s}} X_{1s} \xleftarrow{d^0_{2s}} X_{2s} \xleftarrow{d^0_{2s}} X_{3s} \xleftarrow{d^0_{3s}} X_{4s} \xleftarrow{d^0_{4s}} \cdots.$$

Moreover each X_{rs} is a projective relative E-bimodule. We define E-bimodule maps

$$d^l_{rs} : X_{rs} \to X_{r+l-1,s-l} \qquad (r\ge 0 \text{ and } 1\le l\le s),$$

recursively by

$$d^l_{rs}(\mathbf{y}) = \begin{cases} -\sigma^0_{0,s-1}\circ\partial_s\circ\mu_s(\mathbf{y}) & \text{if } r=0 \text{ and } l=1,\\ -\sum_{j=1}^{l-1}\sigma^0_{l-1,s-l}\circ d^{l-j}_{j-1,s-j}\circ d^j_{0s}(\mathbf{y}) & \text{if } r=0 \text{ and } 1<l\le s,\\ -\sum_{j=0}^{l-1}\sigma^0_{r+l-1,s-l}\circ d^{l-j}_{r+j-1,s-j}\circ d^j_{rs}(\mathbf{y}) & \text{if } r>0,\end{cases}$$

where $\mathbf{y} = 1\#1\otimes x_1\wedge\cdots\wedge x_s\otimes\mathbf{a}_{1r}\otimes 1\#1 \in X_{rs}$.

THEOREM 3.2.1. *The complex*

$$E \xleftarrow{\mu} X_0 \xleftarrow{d_1} X_1 \xleftarrow{d_2} X_2 \xleftarrow{d_3} X_3 \xleftarrow{d_4} X_4 \xleftarrow{d_5} X_5 \xleftarrow{d_6} X_6 \xleftarrow{d_7} \dots,$$

where $\mu(a_0\#v \otimes a_1\#w) = \sum_{(v)(w)} a_0 a_1^{v^{(1)}} f\big(v^{(2)}, w^{(1)}\big)\#v^{(3)}w^{(2)}$,

$$X_n = \bigoplus_{r+s=n} X_{rs} \qquad \text{and} \qquad d_n = \sum_{\substack{r+s=n\\ r+l>0}} \sum_{l=0}^{s} d_{rs}^l$$

is a relative projective resolution of the E*-bimodule* E.

Proof. Let $\tilde{\mu}' : Y_0' \to E$ and (Y_*', ∂_*') be as in Theorem 3.1.1 and let $\tilde{\mu} : Y_0 \to E$ be the E-bimodule map defined by

$$\tilde{\mu}(a \otimes (v \otimes w)) = \sum_{(v)(w)} af\big(v^{(1)}, w^{(1)}\big)\#v^{(2)}w^{(2)}.$$

Let $\vartheta_* : (Y_*, \partial_*) \to (Y_*', \partial_*')$ be the isomorphism of E-bimodule complexes, determined by $\vartheta_s(x_1 \wedge \cdots \wedge x_s) = e_{x_1} \wedge \cdots \wedge e_{x_s}$. Since $\tilde{\mu} = \tilde{\mu}' \circ \vartheta_0$, we obtain from Theorem 3.1.1 that the complex of E-bimodules

$$E \xleftarrow{\tilde{\mu}} Y_0 \xleftarrow{\partial_1} Y_1 \xleftarrow{\partial_2} Y_2 \xleftarrow{\partial_3} Y_3 \xleftarrow{\partial_4} Y_4 \xleftarrow{\partial_5} Y_5 \xleftarrow{\partial_6} Y_6 \xleftarrow{\partial_7} \dots$$

is contractible as a complex of k-modules. Hence, the result follows immediately from Theorem 2.1. ∎

The boundary maps of the relative projective resolution of E that we just found are defined recursively. Next we compute these morphisms.

THEOREM 3.3. *For* $x_i, x_j \in \mathfrak{g}$, *we put* $\hat{f}_{ij} = f(x_i, x_j) - f(x_j, x_i)$. *We have*

$$\begin{aligned} d_{rs}^1(a_0\#1 \otimes \mathbf{x} \otimes \mathbf{a}_{1,r+1}\#1) = {} & \sum_{i=1}^{s} (-1)^{i+r+1} a_0\#x_i \otimes \mathbf{x}_{\hat{i}} \otimes \mathbf{a}_{1,r+1}\#1 \\ & + \sum_{i=1}^{s} (-1)^{i+r} a_0\#1 \otimes \mathbf{x}_{\hat{i}} \otimes \mathbf{a}_{1,r+1}\#x_i \\ & + \sum_{\substack{i=1\\ 1\le h\le r+1}}^{s} (-1)^{i+r} a_0\#1 \otimes \mathbf{x}_{\hat{i}} \otimes \mathbf{a}_{1,h-1} \otimes a_h^{x_i} \otimes \mathbf{a}_{h+1,r+1}\#1 \\ & + \sum_{1\le i<j\le s} (-1)^{i+j+r} a_0\#1 \otimes [x_i, x_j] \wedge \mathbf{x}_{\hat{i}\hat{j}} \otimes \mathbf{a}_{1,r+1}\#1, \end{aligned}$$

$$d_{rs}^2(a_0\#1 \otimes \mathbf{x} \otimes \mathbf{a}_{1,r+1}\#1) = \sum_{\substack{1\le i<j\le s\\ 0\le h\le r}} (-1)^{i+j+h} a_0\#1 \otimes \mathbf{x}_{\hat{i}\hat{j}} \otimes \mathbf{a}_{1h} \otimes \hat{f}_{ij} \otimes \mathbf{a}_{h,r+1}\#1$$

and $d_{rs}^l = 0$ *for all* $l \geq 3$, *where* $\mathbf{a}_{1,r+1} = a_1 \otimes \cdots \otimes a_{r+1}$ *and* $\mathbf{x} = x_1 \wedge \cdots \wedge x_s$.

Proof. To unify the expressions in the proof, we put $d^0_{0s} := \mu_s$, $d^1_{-1,s} = \partial_s$ and $d^2_{-1,s} = 0$. First, we compute the maps d^1_{rs}. For $r = -1$ the assertion is trivial. Suppose $r \geq 0$ and the result is valid for $d^1_{r's}$ with $-1 \leq r' < r$. Since, for all $r, s \geq 0$,

$$\sigma_{rs}(b_0\#v \otimes \mathbf{x} \otimes \mathbf{b}_{1,r-1} \otimes 1\#w) = 0, \tag{1}$$

then

$$\begin{aligned} d^1_{rs}(a_0\#1 \otimes \mathbf{x} \otimes \mathbf{a}_{1r} \otimes 1\#1) &= -\sigma^0_{r,s-1} \circ d^1_{r-1,s} \circ d^0_{rs}(a_0\#1 \otimes \mathbf{x} \otimes \mathbf{a}_{1r} \otimes 1\#1) \\ &= (-1)^{r+1}\sigma^0_{r,s-1} \circ d^1_{r-1,s}(a_0\#1 \otimes \mathbf{x} \otimes \mathbf{a}_{1r}\#1). \end{aligned}$$

Hence, the formula for $d^1_{rs}(a_0\#1 \otimes \mathbf{x} \otimes \mathbf{a}_{1r} \otimes 1\#1)$ follows immediately by induction on r. Now, let us compute d^2_{rs}. Suppose $r \geq 0$ and the result is valid for $d^2_{r's}$ with $0 \leq r' < r$. Using (1) twice, we get

$$\begin{aligned} &d^2_{rs}(a_0\#1 \otimes \mathbf{x} \otimes \mathbf{a}_{1r} \otimes 1\#1) \\ &\quad = -\sigma^0_{r+1,s-2} \circ \big(d^2_{r-1,s} \circ d^0_{rs} + d^1_{r,s-1} \circ d^1_{rs}\big)(a_0\#1 \otimes \mathbf{x} \otimes \mathbf{a}_{1r} \otimes 1\#1) \\ &\quad = \sigma^0_{r+1,s-2}\left(\sum_{1\leq i<j\leq s}\sum_{h=0}^{r-1}(-1)^{i+j+h+r+1}a_0\#1 \otimes \mathbf{x}_{\hat{\mathbf{i}}\hat{\mathbf{j}}} \otimes \mathbf{a}_{1h} \otimes \hat{f}_{ij} \otimes \mathbf{a}_{h+1,r}\#1\right) \\ &\qquad - \sigma^0_{r+1,s-2}\left(d^1_{r,s-1}\left(\sum_{j=1}^{s}(-1)^{j+r}a_0\#1 \otimes \mathbf{x}_{\hat{j}} \otimes \mathbf{a}_{1r} \otimes 1\#1\right)(1\#x_j)\right) \\ &\quad = \sum_{1\leq i<j\leq s}\sum_{h=0}^{r-1}(-1)^{i+j+h}a_0\#1 \otimes \mathbf{x}_{\hat{i}\hat{j}} \otimes \mathbf{a}_{1h} \otimes \hat{f}_{ij} \otimes \mathbf{a}_{h+1,r} \otimes 1\#1 \\ &\qquad - \sigma^0_{r+1,s-2}\left(\sum_{1\leq i<j\leq s}(-1)^{i+j}a_0\#1 \otimes \mathbf{x}_{\hat{i}\hat{j}} \otimes \mathbf{a}_{1r} \otimes \hat{f}_{ij}\#1\right) \\ &\quad = \sum_{1\leq i<j\leq s}\sum_{l=0}^{r}(-1)^{i+j+h}a_0\#1 \otimes \mathbf{x}_{\hat{i}\hat{j}} \otimes \mathbf{a}_{1h} \otimes \hat{f}_{ij} \otimes \mathbf{a}_{h+1,r} \otimes 1\#1. \end{aligned}$$

To prove that $d^l_{rs} = 0$ for all $l > 2$, it is sufficient to check that

$$\sigma^0_{r+2,s-4} \circ d^2_{r+1,s-2} \circ d^2_{rs}(a_0\#1 \otimes \mathbf{x} \otimes \mathbf{a}_{1r} \otimes 1\#1) = 0,$$

$$\sigma^0_{r+1,s-3} \circ d^1_{r+1,s-2} \circ d^2_{rs}(a_0\#1 \otimes \mathbf{x} \otimes \mathbf{a}_{1r} \otimes 1\#1) = 0,$$

$$\sigma^0_{r+1,s-3} \circ d^2_{r,s-1} \circ d^1_{rs}(a_0\#1 \otimes \mathbf{x} \otimes \mathbf{a}_{1r} \otimes 1\#1) = 0. \qquad \blacksquare$$

Next, we give an explicit formula for the comparison map between (X_*, d_*) and the canonical normalized Hochschild resolution $(E \otimes \overline{E}^* \otimes E, b'_*)$. Using this map it is easy to obtain explicit quasi-isomorphisms from the complex obtained in the following section for the Hochschild homology into the canonical one, and similarly for the cohomology.

Remark 3.4. There is a map of complexes $\theta_* : (X_*, d_*) \to (E \otimes \overline{E}^* \otimes E, b'_*)$, given by

$$\begin{aligned}\theta_{r+s}&(1_E \otimes x_1 \wedge \cdots \wedge x_s \otimes a_1 \otimes \cdots \otimes a_r \otimes 1_E)\\ &= \sum_{\tau \in \mathfrak{S}_s} \mathrm{sg}(\tau) 1_E \otimes ((1\# x_{\tau(1)}) \otimes \cdots \otimes (1\# x_{\tau(s)}))\\ &\quad * ((a_1\#1) \otimes \cdots \otimes (a_r\#1)) \otimes 1_E,\end{aligned}$$

where $*$ denotes the shuffle product defined by

$$(e_1 \otimes \cdots \otimes e_s) * (e_{s+1} \otimes \cdots \otimes e_n) = \sum_{\sigma \in \{(s, n-s)\text{-shuffles}\}} \mathrm{sg}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}.$$

4. THE HOCHSCHILD HOMOLOGY

Let $E = A\#_f U(\mathfrak{g})$ and M be an E-bimodule. We use Theorem 3.2.1 in order to construct a complex $\overline{X}_*(E, M)$, simpler than the canonical one, giving the Hochschild homology of E with coefficients in M.

The Complex $\overline{X}_*(E, M)$. Let $r, s, l \geq 0$ with $l \leq \min(2, s)$ and $r + l > 0$. We define the morphism $\bar{d}^l_{rs} : M \otimes \overline{A}^r \otimes \mathfrak{g}^{\wedge s} \to M \otimes \overline{A}^{r+l-1} \otimes \mathfrak{g}^{\wedge s-l}$, by

$$\begin{aligned}\bar{d}^0_{rs}(m \otimes \mathbf{a}_{1r} \otimes \mathbf{x}) &= ma_1 \otimes \mathbf{a}_{2r} \otimes \mathbf{x} + \sum_{i=1}^{r-1} (-1)^i m \otimes \mathbf{a}_{1,\, i-1} \otimes a_i a_{i+1}\\ &\quad \otimes \mathbf{a}_{i+2,\, r} \otimes \mathbf{x} + (-1)^r a_r m \otimes \mathbf{a}_{1,\, r-1} \otimes \mathbf{x},\\ \bar{d}^1_{rs}(m \otimes \mathbf{a}_{1r} \otimes \mathbf{x}) &= \sum_{i=1}^{s} (-1)^{i+r} ((1\#x_i)m - m(1\#x_i)) \otimes \mathbf{a}_{1r} \otimes \mathbf{x}_{\hat{i}}\\ &\quad + \sum_{\substack{i=1\\ 1\leq h\leq r}}^{s} (-1)^{i+r} m \otimes \mathbf{a}_{1,\, h-1} \otimes a_h^{x_i} \otimes \mathbf{a}_{h+1,\, r} \otimes \mathbf{x}_{\hat{i}}\\ &\quad + \sum_{1\leq i<j\leq s} (-1)^{i+j+r} m \otimes \mathbf{a}_{1r} \otimes [x_i, x_j] \wedge \mathbf{x}_{\hat{i}\hat{j}},\\ \bar{d}^2_{rs}(m \otimes \mathbf{a}_{1r} \otimes \mathbf{x}) &= \sum_{\substack{1\leq i<j\leq s\\ 0\leq h\leq r}} (-1)^{i+j+h} m \otimes \mathbf{a}_{1h} \otimes \hat{f}_{ij} \otimes \mathbf{a}_{h+1,\, r} \otimes \mathbf{x}_{\hat{i}\hat{j}},\end{aligned}$$

where $\mathbf{a}_{1r} = a_1 \otimes \cdots \otimes a_r$, $\mathbf{x} = x_1 \wedge \cdots \wedge x_s$, and $\hat{f}_{ij} = f(x_i, x_j) - f(x_j, x_i)$.

THEOREM 4.1. *The Hochschild homology* $\mathrm{H}_*(A, M)$, *of* E *with coefficients in* M, *is the homology of*

$$\overline{X}_*(E, M) = \overline{X}_0 \xleftarrow{\bar{d}_1} \overline{X}_1 \xleftarrow{\bar{d}_2} \overline{X}_2 \xleftarrow{\bar{d}_3} \overline{X}_3 \xleftarrow{\bar{d}_4} \overline{X}_4 \xleftarrow{\bar{d}_5} \overline{X}_5 \xleftarrow{\bar{d}_6} \overline{X}_6 \xleftarrow{\bar{d}_7} \ldots,$$

where

$$\overline{X}_n = \bigoplus_{r+s=n} M \otimes \overline{A}^r \otimes \mathfrak{g}^{\wedge s} \qquad \textit{and} \qquad \bar{d}_n = \sum_{\substack{r+s=n\\ r+l>0}} \sum_{l=0}^{\min(s,2)} \bar{d}^l_{rs}.$$

Proof. It follows from the fact that $\overline{X}_*(E, M)$ is the complex obtained by taking the tensor product $M \otimes_{E^e} (X_*, d_*)$, where (X_*, d_*) is the complex of Theorem 3.2.1, and using the identifications $\vartheta_{rs} : M \otimes \overline{A}^r \otimes \mathfrak{g}^{\wedge s} \to M \otimes_{E^e} E \otimes \mathfrak{g}^{\wedge s} \otimes \overline{A}^r \otimes E$, given by $\vartheta_{rs}(m \otimes \mathbf{a}_{1r} \otimes \mathbf{x}) = m \otimes (1\#1 \otimes \mathbf{x} \otimes a_{1r} \otimes 1\#1)$. ∎

Note that when f takes its values in k, then $\overline{X}_*(E, M)$ is the total complex of the double complex $(M \otimes \overline{A}^* \otimes \mathfrak{g}^{\wedge *}, \bar{d}^0_{**}, \bar{d}^1_{**})$.

4.2. *Stefan's Spectral Sequence*

Next we show that the complex $\overline{X}_*(E, M)$ has a natural filtration, which gives a more explicit version of the homology spectral sequence obtained in [St.]

For each $x \in \mathfrak{g}$, we have the morphism $\Theta^x_* : (M \otimes \overline{A}^*, b_*) \to (M \otimes \overline{A}^*, b_*)$, defined by $\Theta^x_r(m \otimes \mathbf{a}_{1r}) = ((1\#x)m - m(1\#x)) \otimes \mathbf{a}_{1r} + \sum_{1\le h\le r} m \otimes \mathbf{a}_{1,h-1} \otimes a^x_h \otimes \mathbf{a}_{h+1,r}$.

PROPOSITION 4.2.1. *For each $x, x' \in \mathfrak{g}$ the endomorphisms of $\mathrm{H}_*(A, M)$ induced by $\Theta^{x'}_* \circ \Theta^x_* - \Theta^x_* \circ \Theta^{x'}_*$ and by $\Theta^{[x,x']}_*$ coincide. Consequently $\mathrm{H}_*(A, M)$ is a right $U(\mathfrak{g})$-module.*

Proof. By a standard argument it is sufficient to prove it for $\mathrm{H}_0(A, M)$. In this case, the assertion can be easily checked, using that $(1\#x')(1\#x) = f(x', x)\#1 + 1\#x'x$ for all $x, x' \in \mathfrak{g}$. ∎

The chain complex $\overline{X}_*(E, M)$ has the filtration $F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$, where

$$F_i(\overline{X}_n) = \bigoplus_{\substack{r+s=n\\ r\ge 0,\, 0\le s\le i}} M \otimes \overline{A}^r \otimes \mathfrak{g}^{\wedge s}.$$

Using this fact and Proposition 4.2.1, we obtain the following:

COROLLARY 4.2.2. *There is a converging spectral sequence*

$$E^2_{rs} = \mathrm{H}_s(\mathfrak{g}, \mathrm{H}_r(A, M)) \Rightarrow \mathrm{H}_{r+s}(E, M).$$

Given an A-bimodule M we let $[A, M]$ denote the k-submodule of M generated by the commutators $am - ma$ ($a \in A$ and $m \in M$).

COROLLARY 4.2.3. *If A is separable, then* $\mathrm{H}_*(E, M) = \mathrm{H}_*\left(\mathfrak{g}, \frac{M}{[A,M]}\right)$.

4.3. *Smooth Algebras*

Let A be a commutative ring and let M be a symmetric A-bimodule. In the famous paper [H-K-R] was proved that if A is a commutative smooth k-algebra, then $H_n(A, M) = M \otimes_A \Omega^n_{A/k}$, where $\Omega^n_{A/k}$ denotes the A-module of differential n-forms of A. Next, we generalize this result by computing the Hochschild homology of a differential operator ring $E = A\#_f U(\mathfrak{g})$ with coefficients in an E-bimodule M which is symmetric as an A-bimodule, under the hypothesis that $\mathbb{Q} \subseteq k$ and A is a commutative smooth k-algebra.

Let us assume that $\mathbb{Q} \subseteq k$, A is a commutative ring, and M is symmetric as an A-bimodule. For each $r, s, l \geq 0$ with $1 \leq l \leq \min(2, s)$, we define the morphism $\tilde{d}^l_{rs} : M \otimes_A \Omega^r_{A/k} \otimes \mathfrak{g}^{\wedge s} \to M \otimes_A \Omega^{r+l-1}_{A/k} \otimes \mathfrak{g}^{\wedge s-l}$, by

$$\begin{aligned}
&\tilde{d}^1_{rs}(m \otimes_A da_1 \cdots da_r \otimes \mathbf{x}) \\
&\quad = \sum_{i=1}^{s} (-1)^{i+r}((1\#x_i)m - m(1\#x_i)) \otimes_A da_1 \cdots da_r \otimes \mathbf{x}_{\hat{i}} \\
&\qquad + \sum_{\substack{i=1 \\ 1\leq h\leq r}}^{s} (-1)^{i+r} m \otimes da_1 \cdots da_{h-1} da_h^{x_i} da_{h+1} \cdots da_r \otimes \mathbf{x}_{\hat{i}} \\
&\qquad + \sum_{1\leq i<j\leq s} (-1)^{i+j+r} m \otimes_A da_1 \cdots da_r \otimes [x_i, x_j] \wedge \mathbf{x}_{\hat{i}\hat{j}}, \\
&\tilde{d}^2_{rs}(m \otimes_A da_1 \cdots da_r \otimes \mathbf{x}) \\
&\quad = \sum_{1\leq i<j\leq s} (-1)^{i+j} m \otimes_A d\hat{f}_{ij} da_1 \cdots da_r \otimes \mathbf{x}_{\hat{i}\hat{j}},
\end{aligned}$$

where $\mathbf{x} = x_1 \wedge \cdots \wedge x_s$ and $\hat{f}_{ij} = f(x_i, x_j) - (x_j, x_i)$. Consider the complex

$$\widetilde{X}_*(E, M) = \widetilde{X}_0 \xleftarrow{\tilde{d}_1} \widetilde{X}_1 \xleftarrow{\tilde{d}_2} \widetilde{X}_2 \xleftarrow{\tilde{d}_3} \widetilde{X}_3 \xleftarrow{\tilde{d}_4} \widetilde{X}_4 \xleftarrow{\tilde{d}_5} \dots,$$

where

$$\widetilde{X}_n = \bigoplus_{r+s=n} M \otimes_A \Omega^r_{A/k} \otimes \mathfrak{g}^{\wedge s} \qquad \text{and} \qquad \tilde{d}_n = \sum_{\substack{r+s=n \\ r+l>0}} \sum_{l=1}^{\min(s,2)} \tilde{d}^l_{rs}.$$

Let $\vartheta_n : \overline{X}_n \to \widetilde{X}_n$ be the map $\vartheta_n(m \otimes \mathbf{a}_{1r} \otimes \mathbf{x}) = (1/r!) m \otimes_A da_1 \cdots da_r \otimes \mathbf{x}$. It is easy to check that $\vartheta_* : \overline{X}_*(E, M) \to \widetilde{X}_*(E, M)$ is a morphism of complexes, which is a quasi-isomorphism when A is smooth. Hence, in this case, the Hochschild homology of E with coefficients in M is the homology of $\widetilde{X}_*(E, M)$.

A filtrated algebra E is called quasi-commutative if its associated graded algebra is commutative. In [S] was proved that if E is a quasi-commutative algebra whose associated graded algebra is a polynomial ring, then E is isomorphic to a differential polynomial ring $k\#_f U(\mathfrak{g})$, with $\mathfrak{g}$ a finite dimensional Lie algebra. The Hochschild homology of this type of algebras was computed in [K, Theorem 3]. Next, we generalize this result.

Remark 4.3.1. Now, we assume that $\mathfrak{g}$ acts trivially on A. Consider the symmetric algebra $S = S_A(\mathfrak{g})$, endowed with the Poisson bracket defined by $\{a, x\} = \{a, b\} = 0$ and $\{x, y\} = f(x, y) - f(y, x) + [x, y]_{\mathfrak{g}}$ $(a \in A, x, y \in \mathfrak{g})$. It is easy to check that $\widetilde{X}_*(E, E)$ is isomorphic to the canonical complex $(\Omega^*_{S/k}, \delta_*)$ introduced by Brylinski and Koszul in [B, Ko], respectively. In fact an isomorphism $\Theta_* : (\Omega^*_{S/k}, \delta_*) \to \widetilde{X}_*(E, E)$ is given by $\Theta_{r+s}(P da_1 \cdots da_r dx_1 \cdots dx_s) = (-1)^s \eta(P) da_1 \cdots da_r x_1 \wedge \cdots \wedge x_s$, where $P \in S, a_1, \ldots, a_r \in A, x_1, \ldots, x_s \in \mathfrak{g}$ and $\eta : S \to E$ is the symmetrization $\eta(a y_1 \cdots y_n) = \frac{a}{n!} \sum_{\sigma \in \mathfrak{S}_n} y_{\sigma(1)} \cdots y_{\sigma(n)}$.

4.4. *Compatibility with the Canonical Decomposition*

Let us assume that $k \supseteq \mathbb{Q}$, A is a commutative ring, M is symmetric as an A-bimodule, and the cocycle f takes its values in k. In [G-S] was obtained a decomposition of the canonical Hochschild complex $(M \otimes \overline{A}^*, b_*)$. It is easy to check that the maps $\bar{d}_0$ and $\bar{d}_1$ are compatible with this decomposition. Since $\bar{d}_2$ is the zero map, we obtain a decomposition of $\overline{X}_*(E, M)$, and then a decomposition of $\mathrm{H}_*(E, M)$.

5. THE HOCHSCHILD COHOMOLOGY

Let $E = A\#_f U(\mathfrak{g})$ and M be an E-bimodule. Using again Theorem 3.5 we construct a complex $\overline{X}^*(E, M)$, simpler than the canonical one, giving the Hochschild cohomology of E with coefficients in M.

The Complex $\overline{X}^*(E, M)$. Let $r, s, l \geq 0$ with $l \leq \min(2, s)$ and $r + l > 0$. We define the morphism $\bar{d}^{rs}_l : \mathrm{Hom}_k(\overline{A}^{r+l-1} \otimes \mathfrak{g}^{\wedge s-l}, M) \to \mathrm{Hom}_k(\overline{A}^r \otimes \mathfrak{g}^{\wedge s}, M)$, by

$$\bar{d}^{rs}_0(\varphi)(\mathbf{a}_{1r} \otimes \mathbf{x}) = a_1 \varphi(\mathbf{a}_{2r} \otimes \mathbf{x}) + \sum_{i=1}^{r-1} (-1)^i \varphi(\mathbf{a}_{1,i-1} \otimes a_i a_{i+1} \otimes \mathbf{a}_{i+2,r} \otimes \mathbf{x}) + (-1)^r \varphi(\mathbf{a}_{1,r-1} \otimes \mathbf{x}) a_r,$$

$$\bar{d}^{rs}_1(\varphi)(\mathbf{a}_{1r} \otimes \mathbf{x}) = \sum_{i=1}^{s} (-1)^{i+r} [\varphi(\mathbf{a}_{1r} \otimes \mathbf{x}_{\hat{i}})(1\#x_i) - (1\#x_i)\varphi(\mathbf{a}_{1r} \otimes \mathbf{x}_{\hat{i}})]$$

$$+ \sum_{\substack{i=1 \\ 1\le h\le r}}^{s} (-1)^{i+r}\varphi(\mathbf{a}_{1,h-1}\otimes a_h^{x_i}\otimes \mathbf{a}_{h+1,r}\otimes \mathbf{x}_{\hat{i}})$$

$$+ \sum_{1\le i<j\le s} (-1)^{i+j+r}\varphi(\mathbf{a}_{1r}\otimes [x_i,x_j]\wedge \mathbf{x}_{\hat{i}\hat{j}}),$$

$$\bar{d}_2^{rs}(\varphi)(\mathbf{a}_{1r}\otimes \mathbf{x}) = \sum_{\substack{1\le i<j\le s \\ 0\le h\le r}} (-1)^{i+j+h}\varphi(\mathbf{a}_{1h}\otimes \hat{f}_{ij}\otimes \mathbf{a}_{h+1,r}\otimes \mathbf{x}_{\hat{i}\hat{j}}),$$

where $\mathbf{a}_{1r} = a_1\otimes\cdots\otimes a_r$, $\mathbf{x} = x_1\wedge\cdots\wedge x_s$, and $\hat{f}_{ij} = f(x_i,x_j) - f(x_j,x_i)$. Applying the functor $\mathrm{Hom}_{E^e}(-,M)$ to the complex (X_*,d_*) of Theorem 3.2.1, and using the identifications

$$\vartheta^{rs}:\ \mathrm{Hom}_k(\overline{A}^r\otimes \mathfrak{g}^{\wedge s},M)\to\ \mathrm{Hom}_{E^e}(E\otimes\mathfrak{g}^{\wedge s}\otimes\overline{A}^r\otimes E,M),$$

given by $\vartheta^{rs}(\varphi)(1\#1\otimes\mathbf{x}\otimes\mathbf{a}_{1r}\otimes 1\#1) = \varphi(\mathbf{a}_{1r}\otimes\mathbf{x})$, we obtain the complex

$$\overline{X}^*(E,M) = \overline{X}^0 \xrightarrow{\bar{d}^1} \overline{X}^1 \xrightarrow{\bar{d}^2} \overline{X}^2 \xrightarrow{\bar{d}^3} \overline{X}^3 \xrightarrow{\bar{d}^4} \overline{X}^4 \xrightarrow{\bar{d}^5} \dots,$$

where

$$\overline{X}^n = \bigoplus_{r+s=n} \mathrm{Hom}_k(\overline{A}^r\otimes\mathfrak{g}^{\wedge s},M) \qquad \text{and} \qquad \bar{d}^n = \sum_{\substack{r+s=n \\ r+l>0}} \sum_{l=0}^{\min(s,2)} \bar{d}_l^{rs}.$$

Note that when f takes its values in k, then $\overline{X}^*(E,M)$ is the total complex of the double complex $(\mathrm{Hom}_k(\overline{A}^*\otimes\mathfrak{g}^{\wedge *},M),\bar{d}_0^{**},\bar{d}_1^{**})$.

THEOREM 5.1. *The Hochschild cohomology $H^*(E,M)$, of E with coefficients in M, is the homology of $\overline{X}^*(E,M)$.*

Proof. It is an immediate consequence of the above discussion. ∎

5.2. *Stefan's Spectral Sequence*

Next we show that the complex $\overline{X}^*(E,M)$ has a natural filtration, which gives a more explicit version of the cohomology spectral sequence obtained in [St].

For each $x\in\mathfrak{g}$, we have the map

$$\Theta_x^*\colon (\mathrm{Hom}_k(\overline{A}^*,M),b^*)\to(\mathrm{Hom}_k(\overline{A}^*,M),b^*),$$

defined by $\Theta_x^r(\varphi)(\mathbf{a}_{1r}) = (1\#x)\varphi(\mathbf{a}_{1r}) - \varphi(\mathbf{a}_{1r})(1\#x) - \sum_{h=1}^r \varphi(\mathbf{a}_{1,h-1}\otimes a_h^x\otimes\mathbf{a}_{h+1,r})$.

PROPOSITION 5.2.1. *For each $x,x'\in\mathfrak{g}$ the endomorphisms of $\mathrm{H}^*(A,M)$ induced by $\Theta_{x'}^*\circ\Theta_x^* - \Theta_x^*\circ\Theta_{x'}^*$ and by $\Theta_{[x',x]}^*$ coincide. Consequently $\mathrm{H}^*(A,M)$ is a left $U(\mathfrak{g})$-module.*

Proof. It is similar to the proof of Proposition 4.2.1. ∎

The cochain complex $\overline{X}^*(E, M)$ has the filtration $F^0 \supseteq F^1 \supseteq \dots$, where

$$F^i(\overline{X}^n) = \bigoplus_{\substack{r+s=n \\ r\geq 0,\, s\geq i}} \mathrm{Hom}_k(\overline{A}^r \otimes \mathfrak{g}^{\wedge s}, M).$$

From this fact and Proposition 5.2.1, we obtain the following:

COROLLARY 5.2.2. *There is a converging spectral sequence*

$$E_2^{rs} = \mathrm{H}^s(\mathfrak{g}, \mathrm{H}^r(A, M)) \Rightarrow \mathrm{H}^{r+s}(E, M).$$

Given an A-bimodule M we let M^A denote the k-submodule of M consisting of the elements m verifying $am = ma$ for all $a \in A$.

COROLLARY 5.2.3. *If A is separable, then* $\mathrm{H}^*(E, M) = \mathrm{H}^*(\mathfrak{g}, M^A)$.

5.3. *Smooth Algebras*

Let us assume that $k \supseteq \mathbb{Q}$, A is a commutative ring, and M is symmetric as an A-bimodule. For each $r, s, l \geq 0$ with $1 \leq l \leq \min(2, s)$, we define the morphism $\tilde{d}_l^{rs} : \mathrm{Hom}_A(\Omega_{A/k}^{r+l-1} \otimes \mathfrak{g}^{\wedge s-l}, M) \to \mathrm{Hom}_A(\Omega_{A/k}^r \otimes \mathfrak{g}^{\wedge s}, M)$, by

$$\begin{aligned}
\tilde{d}_1^{rs}(\varphi)(da_1 \cdots da_r \otimes \mathbf{x}) &= \sum_{i=1}^{s} (-1)^{i+r} [\varphi(da_1 \cdots da_r \otimes \mathbf{x}_{\hat{i}}), (1\# x_i)] \\
&\quad + \sum_{\substack{i=1 \\ 1\leq h\leq r}}^{s} (-1)^{i+r} \varphi(da_1 \cdots d_{h-1} da_h^{x_i} d_{h+1} \cdots d_r \otimes \mathbf{x}_{\hat{i}}) \\
&\quad + \sum_{1\leq i<j\leq s} (-1)^{i+j+r} \varphi(da_1 \cdots da_r \otimes [x_i, x_j] \wedge \mathbf{x}_{\hat{i}\hat{j}}), \\
\tilde{d}_2^{rs}(\varphi)(da_1 \cdots da_r \otimes \mathbf{x}) &= \sum_{1\leq i<j\leq s} (-1)^{i+j} \varphi(d\hat{f}_{ij} da_1 \cdots da_r \otimes \mathbf{x}_{\hat{i}\hat{j}}),
\end{aligned}$$

where $\mathbf{x} = x_1 \wedge \cdots \wedge x_s$, $\hat{f}_{ij} = f(x_i, x_j) - f(x_j, x_i)$ and $[\varphi(da_1 \cdots da_r \otimes \mathbf{x}_{\hat{i}}), (1\# x_i)] = \varphi(da_1 \cdots da_r \otimes \mathbf{x}_{\hat{i}})(1\# x_i) - (1\# x_i)\varphi(da_1 \cdots da_r \otimes \mathbf{x}_{\hat{i}})$. Consider the complex

$$\widetilde{X}^*(E, M) = \widetilde{X}^0 \xrightarrow{\tilde{d}^1} \widetilde{X}^1 \xrightarrow{\tilde{d}^2} \widetilde{X}^2 \xrightarrow{\tilde{d}^3} \widetilde{X}^3 \xrightarrow{\tilde{d}^4} \widetilde{X}^4 \xrightarrow{\tilde{d}_5} \dots,$$

where

$$\widetilde{X}^n = \bigoplus_{r+s=n} \mathrm{Hom}_k(\Omega_{A/k}^r \otimes \mathfrak{g}^{\wedge s}, M) \qquad \text{and} \qquad \tilde{d}^n = \sum_{\substack{r+s=n \\ r+l>0}} \sum_{l=1}^{\min(s,2)} \tilde{d}_l^{rs}.$$

Let $\vartheta : \widetilde{X}^n \to \overline{X}^n$ be the map $\vartheta^n(\varphi)(\mathbf{a}_{1r} \otimes \mathbf{x}) = \frac{1}{r!}\varphi(da_1 \cdots da_r \otimes \mathbf{x})$. It is easy to check that $\vartheta^* : \widetilde{X}^*(E, M) \to \overline{X}^*(E, M)$ is a morphism of complexes, which is a quasi-isomorphism when A is smooth. Hence, in this case, the Hochshild cohomology of E with coefficients in M is the cohomology of $\widetilde{X}^*(E, M)$.

5.4. *Compatibility with the Canonical Decomposition*

Let us assume that $k \supseteq \mathbb{Q}$, A is a commutative ring, M is symmetric as an A-bimodule, and the cocycle f takes its values in k. Then the Hochschild cohomology $\mathrm{H}^*(E, M)$ has a decomposition similar to the one obtained in paragraph 4.4 for the Hochschild homology.

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