JOURNAL OF ALGEBRA **208,** 575–603 (1998)
ARTICLE NO. JA987490

Localization on (Necessarily) Topological Coalgebras and Cohomology

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Communicated by Wilberd van der Kallen

Received September 17, 1997

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1. INTRODUCTION

Given a k-coalgebra C, two cohomology theories have been associated to it by Doi [3]. One of them, denoted *Hoch**, plays the role of Hochschild homology of algebras, and the other one corresponds to Hochschild cohomology.

Their resemblance to Hochschild (co)homology includes the fact that they are invariant with respect to Morita–Takeuchi equivalence. We proved this result in a previous paper [4], where we also considered the problem of invariance of *Hoch** under Azumaya extensions, which we only

* Research partially supported by UBACYT EX230, CONICET PIA 6213/96 and Fundación Antorchas.
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0021-8693/98 $25.00

solved in a partial way due to the lack of a good concept of localization for coalgebras.

So, this paper is dedicated to the study of localization in the coalgebra context.

Given a coalgebra C, after choosing the "localizing set" S and defining $C_{[S]}$, it is clear that $C_{[S]}$ is not a coalgebra. It becomes necessary then to work in the category of topological coalgebras, as it has been noticed by Takeuchi [10]. The main difference is given by the fact that $C \tilde{\otimes} C$ changes, according to the topology chosen, for which $C \tilde{\otimes} C$ is completed.

All C-comodules will be topological k-vector spaces. However, Takeuchi's work concerns only linear topologies (i.e., topologies where the basis of neighbourhoods of zero is given by a family of subspaces). This is not a satisfactory frame if we consider $k = \mathbb{C}$ and locally convex $\mathbb{C}$-vector spaces where many other interesting topologies may be chosen.

In this work we study localization of topological coalgebras C with respect to multiplicatively closed subsets of $Z(C')$ (the center of the topological dual algebra). It includes both the linear and the locally convex case, with special emphasis on the last one.

Our main interest was at first the behaviour of *Hoch** with respect to localization. Although $C_{[S]}$ needs not be a coflat C-comodule, we establish that coflatness holds under certain conditions.

In this case, we prove that *Hoch** commutes with localization.

We give examples of coalgebras and multiplicatively closed sets satisfying these conditions.

In Section 2, after giving some definitions (recalled from [10] in the linear case), we list some examples that will appear afterwards.

Section 3 is devoted to the study of the category ${}^{C}\mathcal{M}$ of complete topological left C-comodules, for a topological coalgebra C. Concepts such as injectivity, freeness, and resolutions appearing here are always relative to k-split morphisms. We also define topological versions of cohomology theories *Hoch** and H^*, proving some of their properties.

In the first part of Section 4, we study carefully the localization theory for topological algebras and prove that some local-global results (for A commutative) still hold in this context. We give there the definition of the tensor product of two bimodules in the category of locally convex complete Hausdorff A-modules (A is a topological $\mathbb{C}$-algebra), which is in fact, a generalization [1] (afterwards we establish a long exact sequence for relative $Tor_*^{A^e|k}$ in this frame). The main step of Takeuchi's linear localization theory for coalgebras is the fact that topological tensor products commute with projective limits. In the locally convex case, considering complete Hausdorff comodules and the completion of the projective topology on the tensor product, this fact still holds, allowing then the construction of localization which is carried out in Section 4, where its universal property is studied.

However, the (co)flatness problem in this situation differs a lot from the algebraic case; even flatness over the field k is not guaranteed and is only verified under an additional hypothesis (cf. Proposition 5.1). So we devote Section 5 to establishing conditions assuring coflatness of a localized coalgebra $C_{[S]}$ as a C-comodule.

In Section 6, we prove under the hypothesis of the previous section our motivating result, that is:

THEOREM 6.2. *Let C be a Fréchet nuclear coalgebra, S a multiplicative subset of $Z(C')$, and M a Fréchet nuclear $C_{[S]}$-bicomodule* (*hence a C-bicomodule*). *If $C_{[S]}$ is C-coflat and Fréchet, then there is a natural isomorphism*

$$Hoch^*\big({}_S M_S, C_{[S]}\big) \overset{f^M{}_*}{\to} {}_S Hoch^*(M, C)_S.$$

The last section consists of the study of localization of the examples of Section 1. For all of them, localization is coflat and Fréchet. The hypothesis of $C_{[S]}$ being Fréchet is superfluous when S is numerable (and also C is Fréchet itself), however, $C_{[S]}$ can be a Fréchet space without S being numerable.

All vector spaces considered are k-symmetric and algebras are either unital or H-unital.

We want to thank Enrico Boasso for our discussions.

2. BASIC DEFINITIONS

We will consider two types of topological vector spaces:

(1) When k is an arbitrary field (with the discrete topology), a topology in a k-vector space M will mean a linear topology, i.e., a topology given by a family of neighbourhoods of zero consisting of subspaces of M.

We reproduce from [10] the basic definitions of linear topologies and topological co-representations that will be used here.

If M is a topological k-vector space and $\{V_\alpha\}_{\alpha \in \mathscr{A}}$ is a family of subspaces defining a base of neighbourhoods of zero, the completion of M is defined as $\hat{M} = \lim_{\overrightarrow{\mathscr{A}}} M/V_\alpha$. If $(M, \{V_\alpha\})$ and $(N, \{W_\beta\})$ are two such spaces, we give $M \otimes_k N$ the topology defined by the family of subspaces $\{V_\alpha \otimes_k N + M \otimes_k W_\beta\}_{\alpha, \beta}$. In this context, $M \tilde{\otimes} N$ will mean $M \hat{\otimes}_k N$, i.e., the completion of the tensor product with respect to the above topology.

(2) When $k = \mathbb{C}$ we will work only with locally convex vector spaces. Then $M \tilde{\otimes} N$ may have different meanings, in general $M \tilde{\otimes} N$ will mean

$M \hat{\otimes}_\pi N$, i.e., the completion of the algebraic tensor product with respect to the projective topology (see [13]). This will also be the case when k is another field which is provided with a topology inherent to its structure.

Notice that (in both cases) $M \tilde{\otimes} k \cong k \tilde{\otimes} M \cong \hat{M}$. For that reason (and for some others that will appear later) we will restrict our attention to the category of complete topological vector spaces.

The basic definitions of topological coalgebras and co-representations can be found in [10] for linear topologies. In Section 3 we shall give definitions that apply to locally convex spaces.

Let $\mathscr{C}$ be the category of complete topological vector spaces over k. It is a monoidal category with $\otimes = \hat{\otimes}_k$. As we shall see in the next section, different tensor products satisfying certain conditions may be defined in the category of topological vector spaces [6, 13] providing it with a structure of a monoidal category. Here we must suppose we have chosen one of them. A complete topological coalgebra C is simply a coalgebra in the monoidal category $(\mathscr{C}, \otimes)$.

EXAMPLES. (1) Let G be a (finite dimensional) Lie group. Then $C^\infty(G)$ is a nuclear Fréchet space which has the structure of a topological coalgebra with coproduct $C^\infty(G) \to C^\infty(G \times G) = C^\infty(G) \hat{\otimes}_\pi C^\infty(G)$ induced by multiplication $G \times G \to G$, and counit induced by the inclusion $\{1_G\} \to G$. If X is a smooth manifold provided with a differentiable action of G, then $C^\infty(X)$ is a $C^\infty(G)$-topological comodule by means of the map $C^\infty(X) \to C^\infty(G \times X) = C^\infty(G) \hat{\otimes}_\pi C^\infty(X)$ induced by the action $G \times X \to X$. The same holds for a Hausdorff locally compact group G and a Hausdorff locally compact topological space X with continuous action $G \times X \to X$. Here the coalgebra is the set of continuous functions $C(G)$ and the comodule is $C(X)$.

(2) Let A be a discrete k-algebra. Then its dual A^* is not a coalgebra (unless A is finite dimensional), however, it is a topological colgebra as $A^* \hat{\otimes} A^* = (A \otimes A)^*$ (the topology in A^* is induced by $A^* \subset k^A$ which has the product topology). Also, a discrete A-module M gives rise to a topological A^*-comodule M^* (see [10, Theorem 1.19]).

(3) The continuous dual of a topological k-algebra is a topological coalgebra, and duals of continuous representations of the algebra are comodules. Examples of this kind are (regular) compact-supported measures over a locally compact Hausdorff space X, or distributions over a smooth manifold.

3. THE CATEGORY ${}^{C}\mathscr{M}$

We now look more carefully at the category ${}^{C}\mathscr{M}$ of complete topological left C-comodules. Objects $M \in {}^{C}\mathscr{M}$ are complete topological k-vector spaces provided with a coassociative continuous structural map $\rho_M: M \to C \tilde{\otimes} M$, where $\tilde{\otimes}$ means the completion with respect to a fixed arbitrary topology on the tensor product, for which C is a coalgebra, satisfying the following.

If V, V_1, V_2, W are topological k-vector spaces, then:

(1) The subspace spanned by $\operatorname{Im}(V \times W \to V \tilde{\otimes} W)$ is dense in $V \tilde{\otimes} W$.

(2) For each continuous linear map $f: V_1 \to V_2$, there are continuous linear maps $f \times 1: V_1 \tilde{\otimes} W \to V_2 \tilde{\otimes} W$ and $1 \otimes f: W \tilde{\otimes} V_1 \to W \tilde{\otimes} V_2$ such that $(f \otimes 1)(v \otimes w) = f(v) \otimes w$ and $(1 \otimes f)(w \otimes v) = w \otimes f(v)$ for all $v \in V_1$, $w \in W$.

(3) $V \tilde{\otimes}(W \tilde{\otimes} V_1) \cong (V \tilde{\otimes} W) \tilde{\otimes} V_1 (v \otimes (w \otimes v') \mapsto (v \otimes w) \otimes v')$ is a topological isomorphism.

(4) The twisting $V \tilde{\otimes} W \cong W \tilde{\otimes} V$ $(v \otimes w \mapsto w \otimes v)$ is a topological isomorphism.

One also asks structural comodule maps to satisfy $(\epsilon \otimes \mathrm{id}_M) \circ \rho_M = \mathrm{id}_M$ where ϵ is the counit of the topological coalgebra and we identify $k \tilde{\otimes} M = \hat{M} = M$.

For locally convex Hausdorff spaces over $\mathbb{C}$, the most usual topologies on the tensor product are the projective one, the injective one, or an intermediate one. If C is a $\mathbb{C}$-topological coalgebra which is nuclear, then they all agree.

The *morphisms* in ${}^{C}\mathscr{M}$ are continuous C-colinear morphisms. A *C-subcomodule* N of M will be a closed linear subspace of M (endowed with the induced topology) such that $\rho_M(N) \subseteq C \tilde{\otimes} N$.

For *quotients*, take a C-subcomodule N of M and consider the k-vector space $(M/N)\hat{}$ and the composition of canonical projection and inclusion $j \circ \pi: M \to (M/N)\hat{}$. By property (2) of $\tilde{\otimes}$, we have a continuous map $(1 \otimes \pi)\rho_M: M \to C \tilde{\otimes} M/N$ that annihilates the subcomodule N. Then it induces a structure map on the quotient which can be extended to its completion, as indicated in the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\pi} & M/N & \hookrightarrow & (M/N)\hat{} \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle \rho_{M/N}} & & \downarrow{\scriptstyle \hat{\rho}_{M/N}} \\ C \tilde{\otimes} M & \xrightarrow{1\otimes\pi} & C \tilde{\otimes} M/N & = & C \tilde{\otimes}(M/N)\hat{} \end{array}$$

Let $M = M_1 \oplus M_2$ be a direct sum of two closed C-subcomodules. It is clear that if N is a topological k-vector space then $M \tilde{\otimes} N = M_1 \tilde{\otimes} N \oplus M_2 \tilde{\otimes} N$ in ${}^C\mathscr{M}$.

If $X, Y, Z \in {}^C\mathscr{M}$, we will call

$$0 \to X \to^f Y \to^g Z \to 0$$

a *short exact sequence* of topological comodules if f is continuous colinear and injective, g is continuous colinear and surjective, $\operatorname{Im}(f) = \operatorname{Ker}(g)$, and the topology on X (resp. Z) coincides with the initial topology (resp. final topology) given by f (resp. g).

A *k-split* short exact sequence will be a short exact sequence which splits continuously (as k-vector spaces).

We adapt from [3] the following definition of free comodules,

DEFINITION 3.1. A left C-comodule M is called *free* if there is a k-space V such that $M \cong C \tilde{\otimes} V$ as left C-comodules by a continuous isomorphism.

Remark. As $C \tilde{\otimes} V = C \tilde{\otimes} \hat{V}$, then the space V of the definition above can always be assumed to be complete.

PROPOSITION 3.2. *Let N be a left C-comodule and V a k-space. Then* $\operatorname{Com}_C(N, C \tilde{\otimes} V)$ *is (algebraically) isomorphic to* $\operatorname{Hom}_k(N, \hat{V})$ *(where* $\operatorname{Com}_C(-,-)$ *denotes the morphisms in the topological comodule category and* $\operatorname{Hom}_k(-,-)$ *are the morphisms as topological k-spaces). The isomorphism is given by*

$$\phi\colon \operatorname{Com}_C(N, C \tilde{\otimes} V) \to \operatorname{Hom}_k(N, \hat{V})$$
$$f \mapsto (\epsilon \otimes \mathrm{id}) f$$

with inverse

$$\psi\colon \operatorname{Hom}_k(N, \hat{V}) \to \operatorname{Com}_C(N, C \tilde{\otimes} V)$$
$$g \mapsto (\mathrm{id} \otimes g)\rho_N.$$

The proof follows the arguments of [3, Proposition 3], as the existence of the continuous map $(\mathrm{id} \otimes g)$ is guaranteed by property (2) of $\tilde{\otimes}$.

DEFINITION 3.3. A C-comodule M is *relatively injective* if and only if whenever $f\colon X_1 \to X_2$ is a morphism of topological C-comodules such that $0 \to X_1 \xrightarrow{f} X_2 \to \operatorname{Coker}(f) \to 0$ is k-split, then, given a morphism $h\colon X_1 \to M$ in the category, there exists $\tilde{h}\colon X_2 \to M$ extending h, i.e., such that $\tilde{h} \circ f = h$.

EXAMPLE. Every free C-comodule is relatively injective, because given a free comodule $C \tilde{\otimes} V$, if we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & X_1 \arrow[r, "f"] \arrow[d, "h"] & X_2 \arrow[r] & \mathrm{Coker}(f) \arrow[r] & 0 \\ & C \tilde{\otimes} V & & & \end{tikzcd}$$

in the category ${}^C\mathscr{M}$, such that f is k-split with splitting g: $X_2 \to X_1$, we want to find $\tilde{h}$: $X_2 \to C \tilde{\otimes} V$ C-colinear such that $\tilde{h} \circ f = h$. But $\mathrm{Com}_C(X_2, C \tilde{\otimes} V) \cong \mathrm{Hom}_k(X_2, \hat{V})$. Taking then $\tilde{h} := \psi((\epsilon \otimes \mathrm{id}_V) \circ h \circ g)$, it is clear that $\tilde{h} \circ f = h$.

COROLLARY 3.4. *Every complete topological comodule M can be embedded in an injective comodule, and so the category ${}^C\mathscr{M}$ has enough* (*relative*) *injectives*.

Proof. As in the algebraic case, the embedding can be chosen as the structure morphism ρ_M: $M \to C \tilde{\otimes} M$. This is an embedding because it is k-split, with splitting $(\epsilon \otimes \mathrm{id})$: $C \tilde{\otimes} M \to k \tilde{\otimes} M \cong \hat{M} = M$ (notice that the completeness of M is essential).

Remark. Not every topological injective C-comodule is free, however, it is injective if and only if it is a direct summand of a free comodule. When we consider the projective topology on the tensor product, the definition of injective comodule coincides with [5, Definition 1.1.2]. One of Getzler's examples is the following: let G be a Lie group, and consider the (differential graded) coalgebra $\mathscr{A}^*(G)$ of differential forms on G and let M be a manifold provided of a smooth left action. Then the space $\mathscr{A}^*(M)$ of differential forms on M is a left topological $\mathscr{A}^*(G)$-(differential graded)comodule, and it is injective if the action of G on M is locally trivial.

From now on, injective comodule will mean (k-split) relative injective. In the algebraic context , given C-comodules M and N, C' is a k-algebra and $\mathrm{Com}_C(M, N) = \mathrm{Hom}_{C'}(M, N)$, where M and N are considered as right C'-modules by means of the evaluation map. But this fact is no longer true for topological C-comodules, as, depending on the topology of C', the evaluation map may not be continuous. Nevertheless, the evaluation map is always separately continuous, so every C-comodule M is a C'-module with respect to an appropriate tensor product (for example, the injective tensor product for locally convex spaces). If the coalgebra C is a nuclear (and Fréchet) space, it is reflexive and the correspondence between C-comodules and C'-modules is 1-1.

Considering linear topologies, the one needed in C' in order to have an isomorphism is the Ω-topology of [10]. In this case, if V is an arbitrary topological vector space, $\mathrm{Hom}_{cont}(C' \otimes_{\Omega} V, \hat{V}) \cong \mathrm{Hom}_{cont}(V, V \hat{\otimes} C)$, where $\mathrm{Hom}_{cont}(X, Y)$ denotes the continuous k-linear maps from X to Y.

DEFINITION 3.5. Let M be a topological right C-comodule and N a topological left C-comodule. Consider the continuous map

$$0 \to \mathrm{Ker} \to M \tilde{\otimes} N \xrightarrow{\rho_M^+ \otimes I - I \otimes \rho_N^-} M \tilde{\otimes} C \tilde{\otimes} N.$$

The kernel is a closed subspace of a complete space, hence complete. It is called the *topological cotensor product* of M and N, and is denoted by $M \square_C N$.

Remark. The notation $M \square_C N$ does not reflect the topology; a better notation would be $M \tilde{\square}_C N$ because it refers to the tensor product topology $\tilde{\otimes}$. However, we shall use the first one, as the topology will be clear from the context.

PROPOSITION 3.6. *The cotensor product* $- \square_C -$ *is a bifunctor satisfying*:

(1) (*Functoriality*) *Let* $f\colon M \to M'$ *be a morphism of topological right* C-*comodules and* $g\colon N \to N'$ *be a morphism of topological left* C-*comodules. Then they induce* (*in a functorial way*) *a morphism* $f \square g\colon M \square_C N \to M' \square_C N'$.

(2) (*Associativity*) *Let* C *and* D *be two topological coalgebras,* M *a right* C-*comodule,* T *a left* D-*comodule, and* N *a left* C-*comodule and right* D-*comodule such that both structure morphisms commute. Then* $(M \square_C N) \square_D T = M \square_C (N \square_D T)$.

(3) (*Left-exactness*) *Let* $0 \to X \to Y \to Z \to 0$ *be a* k-*split exact sequence of left* C-*comodules and* M *a right* C-*comodule. Then*

$$0 \to M \square_C X \to M \square_C Y \to M \square_C Z$$

is an exact sequence of topological k-*spaces.*

(4) *For every right* C-*comodule* M, $M \square_C C$ *is canonically isomorphic to* M.

(5) *If* $D \to C$ *is a morphism of topological coalgebras* (*so* D *is a left and right* C-*comodule*), *then* $D^e \square_{C^e} C \cong D \square_C C \square_C D \cong D \square_C D$.

Proof. These are well-known facts in the algebraic context; the only thing to do is to verify that these constructions behave well with respect to topologies. The only properties that may be worth a proof are (1), (3), and (4).

(1) Property (3) of $\tilde{\otimes}$ assures that we have morphisms $f \otimes g \colon M \tilde{\otimes} N \to M' \tilde{\otimes} N'$ and $f \otimes \mathrm{id}_C \otimes g \colon M \tilde{\otimes} C \tilde{\otimes} N \to M' \tilde{\otimes} C \tilde{\otimes} N'$. The colinearity of f and g imply that we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' \square_C N' & \longrightarrow & M' \tilde{\otimes} N' & \xrightarrow{\rho^+_{M'} \otimes I - I \otimes \rho^-_{N'}} & M' \tilde{\otimes} C \tilde{\otimes} N' \\ & & & & {\scriptstyle f \otimes g}\uparrow & & \uparrow{\scriptstyle f \otimes \mathrm{id}_C \otimes g} \\ 0 & \longrightarrow & M \square_C N & \longrightarrow & M \tilde{\otimes} N & \xrightarrow{\rho^+_{M} \otimes I - I \otimes \rho^-_{N}} & M \tilde{\otimes} C \tilde{\otimes} N \end{array}$$

By composition we have a (continuous) map $M \square_C N \to M' \tilde{\otimes} N'$. As the square is commutative, it follows that the image of this map is contained in $M' \square_C N'$, then it induces a map $M \square_C N \to M' \square_C N'$ which is continuous because $M' \square_C N'$ has the induced topology of $M' \tilde{\otimes} N'$.

(3) Given a C-comodule M and a k-split exact sequence $0 \to X \to Y \to Z \to 0$ then $0 \to M \tilde{\otimes} X \to M \tilde{\otimes} Y \to M \tilde{\otimes} Z \to 0$ and $0 \to M \tilde{\otimes} C \tilde{\otimes} X \to M \tilde{\otimes} C \tilde{\otimes} Y \to M \tilde{\otimes} C \tilde{\otimes} Z \to 0$ also split, and hence the are exact. The rest of the proof is similar to the algebraic case (use the definition of $\square_C$ and chasing).

(4) The sequence

$$0 \to M \xrightarrow{\rho_M} M \tilde{\otimes} C \xrightarrow{\rho_M \otimes I - I \otimes \Delta} M \tilde{\otimes} C \tilde{\otimes} C$$

is exact and k-split (with splitting $\mathrm{id} \otimes \epsilon$). Then M is the kernel of the right map. Remark that completeness of M is essential, and M has the induced topology because the map $\rho_M \colon M \to M \tilde{\otimes} C$ splits.

The relation between coflatness and injectivity is not the same as in the algebraic case, where both notions are coincident. This situation is due to the lack of a finiteness theorem stating that every (topological) comodule is the union of its finite dimensional subcomodules. However, (relative) injective left C-comodules are (relative) C-coflat comodules.

Definition 3.7. A topological C-bicomodule M is a left and right C-comodule, with structure maps $\rho^+ \colon M \to M \tilde{\otimes} C$ and $\rho^- \colon M \to C \tilde{\otimes} M$, such that $(\rho^- \otimes \mathrm{id}_C)\rho^+ = (\mathrm{id}_C \otimes \rho^+)\rho^-$.

The same reasons of the algebraic context (property (3) and essentially property (4) of $\tilde{\otimes}$) assure that the category of topological C-bicomodules is isomorphic to the category of (for example, left) $C \tilde{\otimes} C^{op}$-comodules. The coalgebra $C \tilde{\otimes} C^{op}$ will be denoted by C^e.

We recall from [3] that given a coalgebra C and a bicomodule M, there are two cohomology theories $Hoch^*(M, C)$ and $H^*(M, C)$, which play the role of Hochschild homology and cohomology, respectively, and have consequently a description in terms of relative cohomological functors.

These thories measure the failure of a bicomodule being relative injective (H^*) or relative coflat (*Hoch**), in the algbebraic case these two notion (coflatness and injectivity) agree, so the topological context justifies in a deeper way the existence of two different cohomology theories for coalgebras.

We are now able to define these theories in the context of topological coalgebras.

Let M be a C-bicomodule. As the category has enough relative injectives, we take a relative injective resolution of C as C-bicomodule

$$0 \to C \to X_0 \to X_1 \to \cdots .$$

Then

DEFINITION 3.8.

$$Hoch^n(M,C) := H^n(M \,\square_{C^e} X_*) = Cotor^n_{C^e|k}(M,C)$$

$$H^n(M,C) := H^n\big(\mathrm{Com}_{C^e}(M, X_*)\big) = \mathrm{Ext}^n_{C^e|k}(M,C).$$

More precisely, we can always take the standard resolution of C as C^e-comodule, i.e., $X_n = C^{\tilde{\otimes} n+2} \cong C^e \tilde{\otimes} C^{\tilde{n}}$ with the same maps of [3], which are continuous. Each X_n is relative injective because it is free. This resolution is k-split by means of continuous maps $s_n\colon X_n \to X_{n-1}$, $s_n(c_0 \otimes \cdots \otimes c_{n+1}) = \epsilon(c_0)c_1 \otimes \cdots \otimes c_{n+1}$ (in degree zero, $s_0\colon C^e \to k \tilde{\otimes} C \cong \hat{C} = C$ because C is complete).

We finish this section with a basic property of topological *Hoch**:

PROPOSITION 3.9. *Let M and P be C-bicomodules. Then*

(1) $Hoch^0(M,C) = M \,\square_{C^e} C$.

(2) *If P is C^e-(relatively)coflat, $Hoch^n(P,C) = 0$ if $n > 0$.*

(3) *If $Q \in \mathscr{M}^C$, $Cotor^1_C(Q,N) = 0$ $\forall N \in {}^C\mathscr{M}$ if and only Q is relative C-coflat.*

The proof of this proposition is similar to the algebraic case.

4. THE LOCALIZATION FUNCTOR

The procedure of localization of topological coalgebras in [10] begins by considering that the localization A_S^{alg} of an algebra A (where the superscript *alg* denotes localization in the usual sense) with respect to a multiplicative subset $S \subseteq Z(A)$ is the direct limit of the system $(A^{(q)} \xrightarrow{.r} A^{(p)})_{p,q \in S,\ p=q.r}$ (here we look at the elements of S as endomorphisms of A).

We begin this section by analyzing localization of topological algebras and modules. As we shall see, this procedure is a combination of the algebraic one and topological considerations.

Given a topological $\mathbb{C}$-algebra A, a multiplicatively closed subset $S \subseteq Z(A)$, and M a topological A-module, the localized algebra A_S^{alg} and the localized A_S^{alg}-module M_S^{alg} may be topologized in two different ways, which in fact are coincident. The first one is the inductive limit topology as $A_S^{alg} = \lim_{\substack{\to \\ p\in S}} A^{(p)}$ and $M_S^{alg} = \lim_{\substack{\to \\ p\in S}} M^{(p)}$; the second is the final topology given by the canonical map $A \to A_S^{alg}$ (resp. $M \to M_S^{alg}$). Nevertheless, even if M and A are complete and Hausdorff, A_S^{alg} or M_S^{alg} need not be. As a consequence, the above will not be the definition of localization for topological algebras and modules.

Let A be a locally convex Hausdorff complete $\mathbb{C}$-algebra, M an (locally convex Hausdorff complete) A-module, and $S \subseteq Z(A)$. Then

$$A_S := \lim_{\substack{\to \\ \rho\in S}} A^{(p)}.$$

The difference between this definition and the algebraic one is that we are taking limits in the above category. The way to recover this localization from the algebraic one is simply

$$A_S = \left(A_S^{alg}/\overline{0}\right)^{\wedge},$$

where the topology on A_S^{alg} is (for example) the final topology of the canonical map $A \to A_S^{alg}$. In an analogous way, $M_S := \lim_{\substack{\to \\ p\in S}} M^{(p)} = (M_S^{alg}/\overline{0})^{\wedge}$.

Example. Let X be a smoothy manifold, $U \subseteq X$ an open subset, $A = C^\infty(X)$, and $S_U = \{f \in A/f(x) \neq 0 \ \forall x \in U\}$. S_U is a multiplicatively closed subset of A. In this case, the topological and algebraic localization coincide, moreover $A_{S_U} = A_{S_U}^{alg} = C^\infty(U)$ [8]. Given a closed submanifold $Y \subset X$, $C^\infty(Y)$ is an A-module, and $C^\infty(Y)_{S_U} = C^\infty(Y \cap U)$.

We adopt the following definition of tensor product over a topological algebra:

Definition 4.1. Let M be a topological right A-module and N a topological left A-module (with respect to a fixed $\tilde{\otimes}$). Then

$$M \bigotimes_A N := \left(M \mathbin{\tilde{\otimes}} N/\overline{\langle m.a \otimes n - m \otimes a.n\rangle_{m,a,n}}\right)^{\wedge},$$

i.e., the completion of the quotient of $M \mathbin{\tilde{\otimes}} N$ by the closure of the subspace spanned by the elements of type $m.a \otimes n - m \otimes a.n$ (where $M \mathbin{\tilde{\otimes}} N/\overline{\langle ma \otimes n - m \otimes an\rangle}$ is endowed with the quotient topology).

We remark that this tensor product is the (categorical) cokernel of the map $\rho_M \otimes \mathrm{Id}_N - \mathrm{Id}_M \otimes \rho_N$: $M \tilde{\otimes} A \tilde{\otimes} N \to M \tilde{\otimes} N$ in the category of Hausdorff complete locally convex vector spaces. A particular case of this definition is for example the Haagerup tensor product of operator spaces over a C^*-algebra (see for instance [1]).

An alternative definition of topological tensor product could be done as

$$M \bigotimes_A N := \left(M \otimes N / \overline{\langle m.a \otimes n - m \otimes a.n \rangle}^{M \otimes N}\right)\hat{} = \left(M \bigotimes_A^{alg} N)/\overline{0}\right),$$

namely, we first perform the algebraic tensor product, next we "make it Hausdorff," and finally we complete it. These definitions turn out to be equivalent, as both spaces are canonically homeomorphic ($M \otimes_A^{alg} N$ has the quotient topology from $M \otimes N$).

This alternative definition of topological tensor product may be useful in order to study properties of topological localization.

PROPOSITION 4.2. *Let A and S be as above, M a right A-module. Then*

$$M_S \cong M \bigotimes_A A_S.$$

Proof. $M \otimes_A A_S = (M \otimes_A^{al} A_S)/\overline{0})\hat{} \cong (M_S^{alg}/\overline{0})^{\wedge} = M_S$.

The second version of the definition of topological tensor product also suggests that we can analyze flatness of topological localization in three steps. The first one (i.e., algebraic tensor product) is purely algebraic and then preserves exactness; we can then isolate the topological pathologies looking at the other two steps. So, let

$$0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$$

be a short exact sequence of topological vector spaces. Then

$$0 \to X/\overline{0_X} \xrightarrow{\bar{i}} Y/\overline{0_Y} \xrightarrow{\bar{\pi}} Z/\overline{0_Z} \to 0$$

is a sequence of Hausdorff spaces, and

$$0 \to \left(X/\overline{0_X}\right)\hat{} \xrightarrow{\tilde{i}} \left(Y/\overline{0_Y}\right)\hat{} \xrightarrow{\tilde{\pi}} \left(Z/\overline{0_Z}\right)\hat{} \to 0$$

is a sequence of complete Hausdorff spaces.

- *Hausdorification.* It preserves set-injections and surjections, quotient topologies, and $\overline{\mathrm{Im}(\bar{i})} = \mathrm{Ke}(\bar{\pi})$.
- *Completion.* It preserves categorical epimorphisms, and if $(X/\overline{0_X})$ has the topology induced by $(Y/\overline{0_Y})$, it preserves monomorphisms. Also

when $(X/\overline{0_X})$ has the topology induced by $(Y/\overline{0_Y})$ we can easily verify that $\mathrm{Ker}(\tilde{\pi}) = \overline{\mathrm{Ker}(\overline{\pi})} = \overline{\mathrm{Im}(\bar{i})} = \mathrm{Im}(\tilde{i})$.

As a consequence, localization woul be flat if $(X \otimes_A^{alg} A_S)/\overline{0_X}$ has the topology induced by $(Y \otimes_A^{alg} A_S/\overline{0_Y})$, in fact, as we will see later, it is enough to look if $X \otimes_A^{alg} A_S$ has the topology induced by $Y \otimes_A^{alg} A_S$. Although there is no reason to suppose that this condition always holds and so A_S need not be A-flat, we do not have any counterexample.

It is interesting to notice that as the topological tensor product is associative, then given a right A-module N_A and an $A-B$-bimodule ${}_AM_B$, whenever N is A-flat and M is B-flat, then $N \otimes_A M$ is also B-flat. As a particular case, given a continuous k-algebra morphism $\phi : A \to B$, taking ${}_AM_B =_\phi B$, the extension of a flat right A-module is B-flat.

One important thing concerning topological tensor products is the fact that given M_A and ${}_AN$ Hausdorff A-modules, the topology in the algebraic tensor product which is used for $M \otimes_A N$, is seldom Hausdorff. However, if, for example, $N = A^n$ (and M complete) then $M \otimes_A N = M \otimes_A^{alg} N = M^n$ which is obviously Hausdorff. The next proposition shows that another situation in which one can assure that $M \otimes_A^{alg} N$ is Hausdorff is when $N = A_S$.

PROPOSITION 4.3. *Let M be a topological Hausdorff A-module. Then M_S is Hausdorff for any multiplicatively closed subset $S \subset Z(A)$.*

Proof. As we want to show that $\bar{0} = 0$, we take an element $x/s \neq 0$ in M_S^{alg} and we will find a functional $\phi \in (M_S^{alg})'$ such that $\phi(x/s) \neq 0$.

By definition $x/s \neq 0$ implies $t.x \neq 0$ (in M) for all $t \in S$, in particular $s.x \neq 0$. As M is Hausdorff there exists a functional $\phi \in M'$ such that $\phi(s.x) \neq 0$. As

$$\left(M_S^{alg}\right)' = (M_S)' = \left(\varinjlim_{S} M\right)' = \varprojlim_{S} M'$$

(the last equality is in general just as vector spaces) in order to find a functional in $(M_S^{alg})'$, it is enough to find a family $\{\phi_t\}_{t \in S}$ of linear functionals in M' satisfying the compatibility conditions $\phi_t(q.m) = \phi_{t.q}(m)$ $(t, q \in S,\ m \in M)$.

Let us take then the family defind by $\phi_t(m) := \phi(t.m)$. Then $\{\phi_t\}_{t \in S}(x/s) = \phi_s(x) = \phi(s.x) \neq 0$.

There are several properties concerning (algebraic) A-modules for A commutative which can be studied locally. One of them states that, given an A-module M, $M = 0$ if and only if $M_{\mathscr{M}} = 0$ for all maximal ideals $\mathscr{M} \subset A$. A similar property in the topological context should say that a topological (Hausdorff) A-module $M = 0$ if and only if $M_{\mathscr{M}} = 0$ for all maximal closed ideals $\mathcal{M} \subset A$. Even if we are not able to prove this statement, we can show that

PROPOSITION 4.4. *Let M be a Hausdorff topological A-module, then $M = 0$ if and only if $M_{\mathcal{M}} = 0$ for all maximal ideals $\mathcal{M} \subset A$.*

Proof. $M = 0 \Leftrightarrow M_{\mathcal{M}}^{alg} = 0$ $\forall \mathcal{M} \subset A$ maximal ideal. $M_{\mathcal{M}}$ being Hausdorff, also

$$M_{\mathcal{M}} = \overline{M_{\mathcal{M}}^{alg}} = 0 \Leftrightarrow M_{\mathcal{M}}^{alg} = 0.$$

Then this proposition is a direct consequence of the algebraic result.

Remark. There are some algebras where every maximal ideal is closed, such as Banach algeras or more generally Q-algebras (see for instance [9]), but in general this is not true even in the locally convex Fréchet case (take $A = C^{\infty}(\mathbb{R})$ and a maximal ideal containing the ideal of compactly supported smooth functions).

COROLLARY 4.5. *With the same notations as in the above proposition, given $f: M \to N$ a morphism of (topological) A-modules, then*

- *f is injective if and only if $f_{\mathcal{M}}: M_{\mathcal{M}} \to N_{\mathcal{M}}$ is injective for all maximal ideal $\mathcal{M} \subset A$.*
- *f has dense image if and only if $f_{\mathcal{M}}: M_{\mathcal{M}} \to N_{\mathcal{M}}$ does, for all maximal ideals $\mathcal{M} \subset A$.*

Proof. f is injective if and only if $\mathrm{Ker}(f) = 0$, which, by the proposition, is equivalent to $\mathrm{Ker}(f)_{\mathcal{M}} = 0$ for all maximal ideals $\mathcal{M} \subset A$.

Given $\mathcal{M}$, there is a morphism $\mathrm{Ker}(f)_{\mathcal{M}} \to \mathrm{Ker}(f_{\mathcal{M}})$ whose image is dense, and the restriction of this morphism to the (dense) subspace $\mathrm{Ker}(f)_{\mathcal{M}}^{alg}$ is injective. Then $\mathrm{Ker}(f)_{\mathcal{M}} = 0 \Leftrightarrow \mathrm{Ker}(f_{\mathcal{M}}) = 0$.

For the second part of the corollary, as $- \otimes_A A_{\mathcal{M}}$ preserves categorical epimorphisms, $f_{\mathcal{M}}$ has dense image whenever f is a categorical epimorphism.

For the converse, the argument is similar to the above: there is a morphism $N_{\mathcal{M}}/\overline{\mathrm{Im}(f_{\mathcal{M}})} \to (N/\overline{\mathrm{Im}(f)})_{\mathcal{M}}$ with dense image (containing the elements of type $[x]/t$, $x \in N$, and $t \in A - \mathcal{M}$). Then $\overline{\mathrm{Im}(f_{\mathcal{M}})} = N_{\mathcal{M}}$ for all maximal ideals $\mathcal{M}$ implies $(N/\overline{\mathrm{Im}(f)})_{\mathcal{M}} = 0$ $\forall \mathcal{M}$ and so $N/\overline{\mathrm{Im}(f)} = 0$, i.e., $\overline{\mathrm{Im}(f)} = N$.

Now, we return to localization of coalgebras.

If C is a coalgebra, and S is a multiplicative subset of the center of the dual algebra, then each element of S defines an endomorphism of C (that we want to make invertible). So, Takeuchi [10] defines the localization $C_{[S]}$ of C as the inverse limit of the system $(C^{(p)} \to^{\cdot r} C^{(q)})_{\{p,q \in S,\ \ p=q.r\}}$.

Although he considers only cocommutative coalgebras, the same construction may be done if $S \subseteq Z(C')$ (notice that the only hypothesis used in the proof that $C_{[S]}$ is a topological coalgebra [10, Lemma 2.5] is that the elements $f \in S$ satisfy $(f \otimes 1) \circ \Delta = (1 \otimes f) \circ \Delta$, i.e., $f \in Z(C')$).

However, for arbitrary C, this limit is not a coalgebra, as inverse limits do not commute with $\otimes_k$. It is necessary then to consider complete topological coalgebras and completed tensor products. So, even if the (topological) localization theory for coalgebras appears as a dualization of the corresponding theory for algebras, it is actually more natural, noticing that the topological considerations are essential for the construction of the object $C_{[S]}$. We do not have (as in the algebra context) different steps, first the algebraic one and then the topological one.

The topologies considered by Takeuchi are linear topologies. He proved [10, Theorem 2.3] that in this context $\lim_{\leftarrow}$ and $\tilde{\otimes}_k$ commute and so $C_{[S]}$ is a coalgebra.

When considering locally convex topological coalgebras, we recall [13, Part III] that if the topological spaces are Hausdorff, and the topology in the tensor product space is the projective topology (denoted by $\otimes_\pi$), then:

- The direct product commutes with completion.
- $\hat{\otimes}_\pi$ commutes with Kernels.
- $\hat{\otimes}_\pi$ commutes with direct products.

So, $\hat{\otimes}_\pi$ commutes with projective limits, and $C_{[S]}$ will also have in this context a topological coalgebra structure, given by $\Delta_S : C_{[S]} \to C_{[S]} \hat{\otimes}_\pi C_{[S]}$, where Δ_S is the unique continuous map such that the following diagram commutes

$$\begin{tikzcd}
C^{(pq)} \arrow[r, "\Delta_{p,q}"] & C^{(p)} \hat{\otimes}_\pi C^{(q)} \\
C_{[S]} \arrow[u, "proj_{pq}"] \arrow[r, "\Delta_S"] & C_{[S]} \hat{\otimes}_\pi C_{[S]} \arrow[u, "proj_{(p,q)}"']
\end{tikzcd}$$

Now consider $M \in {}^C\mathscr{M}$. Clearly, if S is a multiplicative subset of $Z(C')$, then each element s of S acts on M by an endomorphism:

$$M \xrightarrow{\rho_M} C \hat{\otimes} M \xrightarrow{s \otimes \mathrm{id}_M} k \hat{\otimes} M = M.$$

We define now the localization of M with respect to S:

DEFINITION 4.6. Let M be a C-comodule. We define M_S as the projective limit of the system $\{r : M^{(q)} \to M^{(p)}: \ p, q \in S, \text{ and } q = p.r\}$, where $M^{(p)} = M \ \forall p \in S$ and r: $M \to M$ is the multiplication by r given by the C'-module structure of M.

We have a canonical map $\pi_M\colon M_S \to M$ given by $.p : M^{(p)} \to M^{(1)} = M$ ($.p$ = multiplication by $p \in S$).

PROPOSITION 4.7. *The pair* $(M_S, \pi_M\colon M_S \to M)$ *has the following universal property*: $.p : M_S \to M_S$ (*multiplication by* $p \in S$) *is an ismorphism for all* $p \in S$, *and for every* C-*comodule* N *such that* $.p\colon N \to N$ *is an isomorphism for all* $p \in S$, *every morphism* $f : N \to M$, *induced a* C-*comodule morphism* $\tilde{f}\colon N \to M_S$ *such that* $f = \pi_M \circ \tilde{f}$.

Proof. Let $s \in S$ and consider the morphisms $\xi_p : M^{(p.s)} \to M^{(p)}$ given by $\xi_p = \mathrm{id}_M$, where p is any element of S. Composing these maps with $\mathrm{proj}_{p.s} : M_S \to M^{(p.s)}$ we have a system of morphisms defining a map $\xi : M_S \to M_S$ which can be easily checked to be the inverse of multiplication by s in M_S.

For the second part of the property let N be a left C-comodule such that for every $p \in S$, $.p : N \to N$ is invertible and let $f : N \to M$ be a C-comodule morphism. Then consider the family of morphisms $\{f_p\}_{p \in S}$ defined by the composition

$$N \xrightarrow{(.p)^{-1}} N \xrightarrow{f} M = M^{(p)}.$$

It is clear that if $p = q.r$, then $f_q = (.r)f_p$, so, we have $\tilde{f}\colon N \to \lim_{\leftarrow} M^{(p)}$ such that $\mathrm{proj}_1 \circ \tilde{f} = f$.

Remark. If M is a C-comodule such that $.p : M \to M$ (multiplication by p) is an isomorphism for all $p \in S$, then $M_S = M$. To see this, we may check that $\pi_M = \mathrm{id}_M\colon M = M_S$ obviously verifies the universal property.

LEMMA 4.8. *Let* C *be topological coalgebra with respect to a fixed* $\tilde{\otimes}$, M *a right* C-*comodule such that* $\tilde{\otimes}$ *commutes with projective limits* (*for example*, $\tilde{\otimes} = \hat{\otimes}_\pi$ *for locally convex spaces*). *Then* M_S *is naturally isomorphic to* $M \,\square_C\, C_{[S]}$ *as* $C_{[S]}$-*comodules*.

Proof. As $\tilde{\otimes}$ commutes with projective limits we have the commutative diagram

$$\begin{array}{ccccc}
M \,\square_C\, C_{[S]} & \hookrightarrow & M \,\tilde{\otimes}\, C_{[S]} & \longrightarrow & M \,\tilde{\otimes}\, C \,\tilde{\otimes}\, C_{[S]} \\
 & & \| & & \| \\
\mathrm{Ker} & \hookrightarrow & \lim\limits_{\leftarrow} M \,\tilde{\otimes}\, C^{(p)} & \xrightarrow{\lim_{\leftarrow} \tau^p} & \lim\limits_{\leftarrow} M \,\tilde{\otimes}\, C \,\tilde{\otimes}\, C^{(p)},
\end{array}$$

where $\tau^P = \rho_M \otimes \mathrm{Id}_{C^{(p)}} - \mathrm{Id}_M \otimes \Delta$.

As projective limits also commute with kernels, then $\mathrm{Ker} = \lim_{\leftarrow} \mathrm{Ker}(\tau^p)$. This last one equals $\lim_{\leftarrow} M \,\square_C\, C^{(p)}$. It follows from the commutative diagram

$$\begin{array}{ccc} M^{(p)} & \xrightarrow{.r} & M^{(q)} \\ \| & & \| \\ M \,\square_C C^{(p)} & \xrightarrow{.r} & M \,\square_C C^{(q)} \end{array}$$

that $\lim_{\leftarrow} M \,\square_C C^{(p)} = \lim_{\leftarrow} M^{(p)} = M_S$. ∎

As a direct corollary of the above two lemmas we have that if M is a C-comodule where multiplication by elements of S are isomorphisms (for example, $M = C_{[S]}$), then $M \,\square_C C_{[S]} \cong M_{[S]} = M$.

When $Z(C')'$ and C'' are topological coalgebras we have canonical coalgebra morphisms $C \to C'' \to Z(C')'$. This will be, for example, the case in Section 5 where C is a nuclear Fréchet space. Let $S \subset Z(C') \subseteq (Z(C')')'$. If M is a C-comodule then it is also a $Z(C')'$-comodule; therefore it makes sense to localize it either as a C-comodule or as a $Z(C')'$-comodule. The next lemma shows that both notions agree.

LEMMA 4.9. *Let M, C, and S be as in the above paragraph. Then $M \,\square_{Z(C')'} Z(C')'_{[S]}$ is a right C-comodule and*

$$M \,\square_{Z(C')'} Z(C')'_{[S]} \cong M \,\square_C C_{[S]}$$

as C-comodules.

Proof. We shall denote $Z := Z(C')'$ and $Z_S := Z(C')'_{[S]}$. As $M \,\square_Z Z_S$ is the localization with respect to the set S in the full subcategory of right Z-comodules, $M \,\square_Z Z_S = \lim_{\substack{\leftarrow \\ p \in S}} M^{(p)}$ which is naturally a right C-comodule, and also computes the localizatin of M with respect to S in the category of right C-comodules, i.e., computes $M \,\square_C C_{[S]}$ (note that limits in both categories preserve the underlying objects).

It would be desirable to have results that allow us to study some properties locally in the cocommutative case. In order to obtain them, we are going to consider from now on nuclear Fréchet (or DF) coalgebras (hence reflexive). These kinds of coalgebras will be the objects of the next sections.

PROPOSITION 4.10. *Let C be a cocommutative coalgebra, and let M be a Hausdorff topological C-comodule. Then $M = 0$ if and only if $M_{[\mathscr{M}]} = 0$ for all maximal ideals $M \subset C'$.*

Proof. Given M, it is clear that $M = 0 \Rightarrow M_{[\mathscr{M}]} = 0$ for all maximal ideals $\mathscr{M} \subset C'$. Also, $M = 0 \Leftrightarrow M' = 0$. M' begin a C'-module, $M' = 0 \Leftrightarrow M'_{\mathscr{M}} = 0\ \forall \mathscr{M}$.

Notice that the hypothesis of C being nuclear is necesary in order to provide M' of a C'-module structure by means of the map $C' \hat{\otimes} M' \cong (M \hat{\otimes} C)' \to^{\rho'_M} M'$.

But the natural inclusion $M \to M''$ (and hence $\lim_{\substack{\leftarrow \\ (C'-\mathscr{M})}} M \to \lim_{\substack{\leftarrow \\ (C'-\mathscr{M})}} M''$) gives a bilinear map

$$M'_{\mathscr{M}} \times M_{[\mathscr{M}]} \to \mathbb{C}$$
$$\left(\left[\frac{\phi}{s} \right], \{x_t\}_{t \notin \mathscr{M}} \right) \mapsto \phi(x_s)$$

defining a separating duality. Then $M'_{\mathscr{M}} = 0 \Leftrightarrow M_{[\mathscr{M}]} = 0$ and this completes the proof.

In general, for an A-module M and a maximal ideal $\mathscr{M}$, the canonical map $M \to M_{\mathscr{M}}$ is neither injective nor surjective. Nevertheless, Proposition 4.4 states that the induced map

$$M \to \prod_{\mathscr{M} \text{ max}} M_{\mathscr{M}}$$

is injective. The above proposition states that

$$\bigoplus_{\mathscr{M} \text{ max}} M_{[\mathscr{M}]} \to M$$

has dense image, as we can also deduce from the duality given by

$$\bigoplus_{\mathscr{M}} M_{[\mathscr{M}]} \otimes \prod_{\mathscr{M}} (M')_{\mathscr{M}} \to \mathbb{C}$$
$$\left(\sum_{\mathscr{M}} \{x_t\}_{t \notin \mathscr{M}}, \left\{ \left[\frac{\phi_u}{s_u} \right] \right\}_{u \in \mathscr{M}} \right) \mapsto \sum_{u \in \mathscr{M}} (\phi_u(x_u)).$$

As a corollary one gets a statement analogous to the one which holds in the case of modules, concerning morphism of comodules:

COROLLARY 4.11. *With the same notations as in the above proposition, given f: $M \to N$ a morphism of* (*topological*) *C-comodules, then f is injective if and only if $f_{[\mathscr{M}]}$: $M_{[\mathscr{M}]} \to N_{[\mathscr{M}]}$ is injective for all maximal ideals $\mathscr{M} \subset C'$.*

Proof. The proof of this assertion is easier than in the algebra context because $Ker(f_{[\mathscr{M}]})$ is exactly $(Ker(f))_{[\mathscr{M}]}$, and then it follows straightforward from the above proposition.

One natural question is the fact that "$f: M \to N$ is a categorical epimorphism (dense image)" is equivalent to "$f_{[\mathscr{M}]}$ is a categorical epimorphism for all maximal ideals $\mathscr{M}$." As far as we know, if both M and N are reflexive, then f with dense image implies $f_{[\mathscr{M}]}$ with dense image for all $\mathscr{M}$. If in addition $M_{[\mathscr{M}]}$ and $N_{[\mathscr{M}]}$ are also reflexive, then, dualizing the result for modules, the equivalence holds.

There are some other results concerning localization of commutative rings which generalize to the context of topological algebras and coalgebras. These facts will not be treated here but in a subsequent paper.

5. LOCALIZATION OF TOPOLOGICAL (CO)MODULES AND (CO)FLATNESS

We are interested in the behaviour of *Hoch** with respect to localization.

First we state:

LEMMA 5.1. *Let C be a coalgebra and M a C-bicomodule. For every $n \in \mathbb{N}_0$ consider $M \tilde{\otimes} C^{\tilde{\otimes} n}$ with the C-bicomodule structure induced by M. Then the Hochschild coboundary*

$$b\colon M \tilde{\otimes} C^{\tilde{\otimes} n} \to M \tilde{\otimes} C^{\tilde{\otimes} n+1}$$

is a morphism of $Z(C)$-bicomodules.

As a corollary, the cohomology groups $Hoch^*(M, C)$ are $Z(C)$-bicomodules so it makes sense to localize them with respect to $S \subseteq Z(C')$, where two-sided localization of a C-bicomodule M is defined as the $C_{[S]}$-bicomodule

$${}_S M_S := C_{[S]} \square_C M \square_C C_{[S]}.$$

Let C be a topological k-coalgebra. If $0 \to X \to Y \to Z \to 0$ is an exact sequence of complete topological k-spaces, which is (continuously) k-split, then

$$0 \to X \tilde{\otimes} C \to Y \tilde{\otimes} C \to Z \tilde{\otimes} C \to 0 \tag{$*$}$$

is exact and splits.

However, if the short exact sequence doesn't split, it is no longer true, in general, that $(*)$ is exact, as for example, the topology of $Z \tilde{\otimes} C$ may not be the quotient topology. This fact, that may be disturbing as k-split sequences are not so frequent, can be avoided if the spaces considered verify some conditions, for example:

PROPOSITION 5.2 [11, Proposition 4.2].

$$0 \to X \overset{\alpha}{\to} Y \overset{\beta}{\to} Z \to 0$$

be an exact sequence of topological spaces and morphisms. Then

$$0 \to X \hat{\otimes}_{\pi} F \to Y \hat{\otimes}_{\pi} F \to Z \hat{\otimes}_{\pi} F \to 0$$

is also an exact sequence in the category, provided that one of the following conditions holds:

(1) *F and Y are both Fréchet spaces and F or X is nuclear.*

(2) *F and Y are both DF-spaces and F or Y is nuclear.*

Although some of the results we shall obtain do not require an additional hypothesis, as the short exact sequences considered will be k-split, we are especially interested in coalgebras and comodules which are Fréchet $\mathbb{C}$-spaces, and coalgebras which are also nuclear spaces.

Also, from now on (taking into account Proposition 5.2) the algebra A will be a nuclear DF space, and the topological A-modules considered will also be DF spaces. $S \subset Z(A)$ will be a countable multiplicatively closed subset of A.

Noticing that A_S is a direct limit of spaces $A^{(p)} = A$ $(p \in S)$, if S is countable, then A_S is DF and nuclear, as it is a countable direct limit of such spaces, so, it is also $\mathbb{C}$-flat (see Proposition 5.2).

Remark. In the case of algebras, the countability of S is necessary to assure that A_S is DF and also that it is nuclear, in contrast with the case of coalgebras, where we only need it to obtain that $C_{[S]}$ is Fréchet, because arbitrary projective limits of nuclear spaces are nuclear.

Considering $S \subset Z(C')$, S countable and multiplicatively closed, then $C_{[S]}$ is Fréchet (as it is a projective limit of a countable system of Fréchet spaces) and nuclear, so, it is also $\mathbb{C}$-flat in the category considered.

LEMMA 5.3. *If*

$$0 \to M \to N \to T \to 0$$

is a k-split exact sequence of C^e-comodules, then $Hoch^*_{C^e}(-, C)$ gives a long exact sequence

$$0 \to M \,\square_{C^e} C \to N \,\square_{C^e} C \to T \,\square_{C^e} C \to Hoch^1(M, C) \to Hoch^1(N, C)$$
$$\to Hoch^1(T, C) \to \cdots .$$

Proof. Considering relative injective C^e-resolutions I_* and J_* of M and T, respectively, we have a commutative diagram with exact rows and first and third columns

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & I_1 \arrow[u] \arrow[r] & I_1 \oplus J_1 \arrow[u, dashed, "f_0"] \arrow[r] & J_1 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & I_0 \arrow[u, "\epsilon_0"] \arrow[r] & I_0 \oplus J_0 \arrow[u, dashed] \arrow[r] & J_0 \arrow[u, "\eta_0"'] \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[u] \arrow[r] & N \arrow[u, dashed] \arrow[r] & T \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u, dashed] & 0 \arrow[u] &
\end{tikzcd}$$

which may be completed (continuously) in the middle. We do not reproduce the details of the step by step algebraic construction, where each map of this construction turns out to be continuous, even when one modifies the splittings of the initial short exact sequence.

We also remark that, denoting by $\{s_i\}_{i \geq 0}$ and $\{t_i\}_{i \geq 0}$ the contracting homotopies of the resolutions of M and T, respectively, as $s_n \epsilon_n + \epsilon_{n+1} s_{n+1} = \mathrm{Id}_{I_n}$ and $t_n v_n + \eta_{n+1} t_{n+1} = \mathrm{Id}_{J_n}$, it follows (by chasing) that we can construct a family $\{g_i\}_{i \geq 0}$ satisfying that $g_n f_n + f_{n+1} g_{n+1} = \mathrm{Id}_{I_n \oplus J_n}$.

Remark. As the category of topological vector spaces is additive, it follows that for any pair of objects E and F their direct sum $E \oplus F$ and the product $E \times F$ agree (as topological vector spaces), this fact assures the continuity of the maps f_n and g_n of the above construction.

LEMMA 5.4. *Let M be a $C_{[S]}$-bicomodule which is a Fréchet $\mathbb{C}$-space. Then*

$$M \,\square_{C^e} C \cong M \,\square_{C^e_{[S]}} C_{[S]}.$$

Proof. Consider the exact diagram

$$\begin{tikzcd}
0 \arrow[r] & M \,\square_{C^e} C \arrow[r] & M \hat{\otimes}_\pi C \arrow[r] & M \hat{\otimes}_\pi C^e \hat{\otimes}_\pi C \\
0 \arrow[r] & M \,\square_{C^e[S]} C_{[S]} \arrow[u, dashed] \arrow[r] & M \otimes C_{[S]} \arrow[u, "\mathrm{id} \otimes \pi_C"] \arrow[r] & M \hat{\otimes}_\pi C^e_{[S]} \hat{\otimes}_\pi C_{[S]} \arrow[u, "\mathrm{id} \otimes \pi_{C^e} \otimes \pi_C"']
\end{tikzcd}$$

where the left vertical arrow is induced by the commutativity of the diagram.

To see that this map is an isomorphism (and obtain it explicitly) consider

$$M \square_{C^e} C \cong \left(M \square_{C^e_{[S]}} C^e_{[S]}\right) \square_{C^e} C \cong M \square_{C^e_{[S]}} \left(C^e_{[S]} \square_{C^e} C\right)$$
$$\cong M \square_{C^e_{[S]}} \left(C_{[S]} \square_C C \square_C C_{[S]}\right) \cong M \square_{C^e_{[S]}} C_{[S]},$$

i.e., the inverse of the morphism of the diagram is $\mathrm{id}_M \square_{C^e_{[S]}} (\mathrm{id}_{C_{[S]}} \otimes \epsilon_{C_{[S]}} \otimes \epsilon_C) \circ \rho_M \square_C \mathrm{id}_C$. The first three isomorphisms being canonical, the result holds not only for the projective topology used on the tensor product and Fréchet spaces, but for any class of spaces and any topology on their tensor product such that Lemma 4.8 is verified.

Under our hypothesis, we know that C and $C_{[S]}$ are $\mathbb{C}$-coflat as $\mathbb{C}$-topological comodules. As a C-comodule, $C_{[S]}$ does not need to be coflat.

However, there are certain conditions which assure that coflatness holds.

- The first one is the Mittag–Leffler condition. We have a countable collection of objects $C^{(j)} = C$. For each $k \in \mathbb{N}$, there exists $j \geq k$ such that the image $C^{(i)} \to C^{(k)}$ equals the image of $C^{(j)} \to C^{(k)}$, for all $i \geq j$.
- The second one is a topological condition. If the Mittag–Leffler condition holds, and given short exact sequences of comodules

$$0 \to X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} Z_n \to 0$$

with f_n, g_n compatible with the morphisms in the projective systems (X_n, α_{nm}), (Y_n, β_{nm}), (Z_n, γ_{nm}) and the topologies on X_n and Z_n are respectively the induced and the quotient one, then algebraically

$$0 \to \lim_{\leftarrow} X_n \xrightarrow{f} \lim_{\leftarrow} Y_n \xrightarrow{g} \lim_{\leftarrow} Z_n \to 0$$

is exact, but the topology of $\lim_{\leftarrow} Z_n$ may not be the one in $\lim_{\leftarrow} Y_n / \lim_{\leftarrow} X_n$ (which is in general finer).

In order to study when their topologies are coincident, let us consider the following situation (from now on, all topological vector spaces will be Fréchet).

Denote

$$X := \lim_{\leftarrow} X_n, \qquad \mathscr{X} := \prod_{n \in \mathbb{N}} X_n, \qquad G_X := \mathrm{Coker}(\mathscr{X} \to^{\delta_X} \mathscr{X}),$$

where $\delta_X(x_0, x_1, \ldots) = (x_0 - \alpha_{0,1}(x_1), x_1 - \alpha_{1,2}(x_2), \ldots)$ and similar for Y and Z.

Then, the following commutative diagram has exact rows and columns, and the maps are continuous:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{X} & \xrightarrow{\Pi f_n} & \mathscr{Y} & \xrightarrow{\Pi g_n} & \mathscr{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_Y} & & \downarrow{\scriptstyle \delta_Z} & & \\
0 & \longrightarrow & \mathscr{X} & \xrightarrow{\Pi f_n} & \mathscr{Y} & \xrightarrow{\Pi g_n} & \mathscr{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle p_Z} & & \\
 & & G_X & \xrightarrow{\bar{f}} & G_Y & \xrightarrow{\bar{g}} & G_Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The Snake Lemma gives an exact sequence:

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\partial} G_X \xrightarrow{\bar{f}} G_Y \xrightarrow{\bar{g}} G_Z \to 0.$$

So, what we need is having $\partial = 0$, which clearly holds, for example, if δ_X is an epimorphism (meaning of course that the topology is the quotient one).

However, even if this is the unique possibility when the Y_n are Banach spaces [2], we shall see in the examples of the last section that coflatness holds in some cases where δ_X is not surjective.

We finish this section by a lemma (analogue to Lemma 5.3) which will be used in order to study the behaviour of (topological) Hochschild homology with respect to localization of topological algebras.

Considering the homology theory related to the tensor product functor $- \otimes_{Ae} A$, we define the relative $Tor_*{}^A$ in a dual way to Definition 3.8. We have:

LEMMA 5.5 (Analogue to Proposition 2.1 of [11], but with a different definition of tensor product).

$$0 \to M \to N \to T \to 0$$

is an exact sequence of A^e-modules which is k-split, then the relative $Tor_*{}^{A^e|k}$ gives a long exact sequence

$$\cdots \to Tor_1^{A^e|k}(A, M) \to Tor_1^{A^e|k}(A, N) \to Tor_1^{A^c|k}(A, T) \to$$
$$\to M \bigotimes_{A^e} A \to N \bigotimes_{A^e} A \to T \bigotimes_{A^e} A \to 0.$$

Proof. Considering relative projective resolutions P_* and Q_* of M and T, respectively, we have a commutative diagram with exact arrows and exact first and third columns,

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d, dashed] & \vdots \arrow[d] & \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r] \arrow[d, dashed] & Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r] \arrow[d, dashed] & Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, dashed] & T \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

which can be completed in the middle column (with the same techniques as Lemma 5.3) in order to give a k-split relative projective resolution of N, and an exact diagram. Once this construction is achieved, the rest of the proof follows as in the algebraic case.

6. LOCALIZATION AND *Hoch** FUNCTOR

PROPOSITION 6.1. *Let C be a topological coalgebra, M a C-bicomodule, and S a multiplicative subset of $Z(C')$. If $C_{[S]}$ is C-coflat, then there is a natural transformation*

$$Hoch^*({}_S M_S, C_{[S]}) \xrightarrow{f_*{}^M} {}_S Hoch^*(M, C)_S$$

which is the identity in degree zero.

Remark. In this proposition and the next theorem by coflatness we mean absolute coflatness (i.e., not only k-relative coflatness).

Proof. Given a C-bicomodule M, we define a map between the standard complexes computing $Hoch^*({}_S M_S, C_{[S]})$ and ${}_S Hoch^*(M, C)_S$, natural in M. With the same notations of the previous section, recall that ${}_S M_S = Z_S \square_Z M \square_Z Z_S$.

$$\begin{array}{ccccccc}
{}_SM_S & \longrightarrow & {}_S(M \tilde{\otimes} C)_S & \longrightarrow & {}_S(M \tilde{\otimes} C \tilde{\otimes} C)_S & \longrightarrow & \cdots \\
\Big\| f_0^M & & f_1^M \Big\uparrow & & f_2^M \Big\uparrow & & \\
{}_SM_S & \longrightarrow & {}_SM_S \tilde{\otimes} C_{[S]} & \longrightarrow & {}_SM_S \tilde{\otimes} C_{[S]} \tilde{\otimes} C_{[S]} & \longrightarrow & \cdots .
\end{array}$$

The maps f_n^M are defined by restriction of the maps

$$F_n^M \colon Z_S \tilde{\otimes} M \tilde{\otimes} Z_S \tilde{\otimes} C_{[S]}^{\tilde{\otimes} n} \to Z_S \tilde{\otimes} \big(M \tilde{\otimes} C^{\tilde{\otimes} n} \big) \tilde{\otimes} Z_S$$

$$x \otimes m \otimes y \otimes z_1 \otimes \cdots \otimes z_n \mapsto x \otimes m \otimes \pi(z_1) \otimes \cdots \otimes \pi(z_n) \otimes y,$$

where $x, y \in Z_S$, $m \in M$, $z_k \in C$ $(k = 1, \dots, n)$ and $\pi \colon C_{[S]} \to C$ is the coalgebra map of the universal property of $C_{[S]}$ (we obtain F_n^M by extending the above formula by linearity and continuity).

We have to check that the restriction (called $f^M{}_*$) of $F^M{}_*$ to $Z_S \,\square_Z M \,\square_Z Z_S \tilde{\otimes} C_{[S]}^{\tilde{\otimes} n}$ has $Z_S \,\square_Z (M \tilde{\otimes} C^{\tilde{\otimes} n}) \square_Z Z_S$ as a target, but this is clear as $Z_S \,\square_Z (M \tilde{\otimes} C^{\tilde{\otimes} n}) \square_Z Z_S \cong (Z_S \,\square_Z M \,\square_Z Z_S) \tilde{\otimes} C^{\tilde{\otimes} n}$ and F_n^M is nothing but tensoring $\pi^{\tilde{\otimes} n}$ with the identity of $Z_S \,\square_Z M \,\square_Z Z_S$. Next we shall see that $f^M{}_*$ is a morphism of complexes.

The differential of the complex ${}_SM_S \tilde{\otimes} C_{[S]}^{\tilde{\otimes} *}$ is given by

$$\begin{aligned}
&d(x \otimes m \otimes y \otimes z_1 \otimes \cdots \otimes z_n) \\
&\qquad = \sum_{k=0}^{n+1} (-1)^k d_k(x \otimes m \otimes y \otimes z_1 \otimes \cdots \otimes z_n),
\end{aligned}$$

where $d_0 = \rho^+_{{}_SM_S} \otimes \mathrm{id}_{(C_{[S]})^{\tilde{\otimes} n}}$, for $0 < i \leq n$ $d_i = \mathrm{id}_{{}_SM_S} \otimes \Delta_i$, $d_{n+1} = \sigma_{n+1, n, \dots, 2, 1} \circ (\rho^-_{{}_SM_S} \otimes \mathrm{id}_{(C_{[S]})^{\tilde{\otimes}}{}_n})$, ρ^+ and ρ^- denote as usual right and left C-comodule structure maps of M and we extend d by continuity. The differential on $Z_S \,\square_Z (M \otimes C^{\tilde{\otimes} *}) \square_Z Z_S$ is the one induced by the differential on $M \otimes C^{\tilde{\otimes} *}$. It is clear that the d_i's commute with $f^M{}_*$ for $0 < i \leq n$ because π is a coalgebra map. We will now consider d_0 (the argument for d_{n+1} is analogous),

$$f_n^M d_0 = f_n^M \circ \big(\rho^+ \otimes \mathrm{id}_{Z_S} \big).$$

On the other side,

$$d_0 f_{n-1}^M = \sigma \left(\mathrm{id}_{Z_S} \,\square_Z\, \rho^+ \,\square_Z\, \mathrm{id}_{Z_S} \right) \otimes \mathrm{id}_{C^{\tilde{\otimes} n-1}}.$$

This completes the proof of the existence of a natural transformation. As $C_{[S]}$ is C-coflat, the homology of the first row is ${}_S Hoch^*(M, C)_S$. The morphism f_*^M clearly induces the identity in degree zero.

THEOREM 6.2. *Let C be a Fréchet nuclear coalgebra, S a multiplicative subset of $Z(C')$, and M a Fréchet nuclear $C_{[S]}$-bicomodule* (*hence a C-bicomodule*). *If $C_{[S]}$ is C-coflat, then the natural transformation of Proposition* 6.1 *is an isomorphism.*

Proof. Let us consider the $\mathbb{C}$-split short exact sequence of $C_{[S]}$-bicomodules

$$0 \to M \stackrel{\rho_M}{\to} M \hat{\otimes} C^e_{[S]} \to K \to 0,$$

where K is the cokernel of the structure map of M.

Then, looking at it first as a sequence of $C_{[S]}$-bicomodules and afterwards as a sequence of C-bicomodules we have two long exact sequences (Lemma 5.3) related by the natural transformation of the above proposition (notice that for a $C_{[S]}$-bicomodule M, $Hoch^*(M,C) = {}_S Hoch^*(M,C)_S$ because the standad complexes are coincident),

$$\begin{array}{ccccccccc}
\cdots \to & Hoch^{n-1}(M \hat{\otimes} C^e_{[S]}, C) & \to & Hoch^{n-1}(K,C) & \to & Hoch^n(M,C) & \to & Hoch^n(M \hat{\otimes} C^e_{[S]}, C) & \to \cdots \\
& f \uparrow & & f \uparrow & & f \uparrow & & f \uparrow & \\
\cdots \to & Hoch^{n-1}(M \hat{\otimes} C^e_{[S]}, C_{[S]}) & \to & Hoch^{n-1}(K, C_{[S]}) & \to & Hoch^n(M, C_{[S]}) & \to & Hoch^n(M \hat{\otimes} C^e_{[S]}, C_{[S]}) & \to \cdots .
\end{array}$$

As $M \hat{\otimes} C^e_{[S]}$ is $C^e_{[S]}$-relative injective, then $Hoch^*(M \hat{\otimes} C^e_{[S]}, C_{[S]}) = 0$ for $* \neq 0$. If we show that the groups $Hoch^*(M \hat{\otimes} C^e_{[S]}, C)$ are also zero for $* \neq 0$ then the theorem follows inductively as the case $* = 0$ is already proved in general in the above proposition.

To see that $Hoch^*(M \hat{\otimes} C^e_{[S]}, C) = 0$ for $* \neq 0$ it si enough to show that $M \hat{\otimes} C^e_{[S]}$ is C^e-coflat (implying then $\mathbb{C}$-relative coflatness) in the category of Fréchet C-bicomodules.

By associativity of the cotensor product (Proposition 3.6), the functor $(M \hat{\otimes} C^e_{[S]}) \square_{C^e} - \cong M \hat{\otimes} (C_{[S]} \square_C - \square_C C_{[S]})$ is the composition of $C_{[S]} \square_C - \square_C C_{[S]}$ and $M \hat{\otimes} -$. Considering an exact sequence of Fréchet C-bimodules and applying $C^e_{[S]} \square_{C^e}$-we obtain an exact sequence (not necessarily $\mathbb{C}$-split) of Fréchet $\mathbb{C}$-vector spaces. Then the exactness of $M \hat{\otimes} -$ follows from nuclearity of M and Proposition 5.2.

Remark. For a (DF) A_S-bimodule M, $M \otimes_{A^e} A = M \otimes_{A^e_S} A_S$, the isomorphism is given by $\overline{m \otimes a} \mapsto \overline{m \otimes 1/a}$, with inverse $\overline{m \otimes a/s} \mapsto \overline{m(1/s) \otimes a}$. These isomorphisms are in fact homeomorphisms in the category [11]. So, $H_0(A, M) = H_0(A_S, M)$. Also, with M as before, we have the following short $\mathbb{C}$-split exact sequence of A_S-bimodules

$$0 \to K \to A^e_S \hat{\otimes} M \to M \to 0.$$

Then, it is also a sequence of A-bimodules. To there are two long exact sequences (with commuting squares):

$$\begin{tikzcd}
& 0 \arrow[r] & H_1(A_S, M) \arrow[r] & H_0(A_S, K) \arrow[r] & H_0(A_S, A_S^e \hat{\otimes} M) \arrow[r] & H_0(A_S, M) \arrow[r] & 0 \\
\cdots \arrow[r] & H_1(A, A_S^e \hat{\otimes} M) \arrow[r] & H_1(A, M) \arrow[u, dashed] \arrow[r] & H_0(A, K) \arrow[u, equal] \arrow[r] & H_0(A, A_S^e \hat{\otimes} M) \arrow[u, equal] \arrow[r] & H_0(A, M) \arrow[u, equal] \arrow[r] & 0,
\end{tikzcd}$$

where in the first row we made use of the fact that, $A_S^e \hat{\otimes} M$ being A_S^e-(relatively)free, $H_1(A_S, A_S^e \hat{\otimes} M) = 0$.

If A_S is A-flat, as M is $\mathbb{C}$-flat then $A_S^e \hat{\otimes} M$ is A^e-flat, so $H_1(A, A_S^e \hat{\otimes} M) = 0$ and $H_1(A, M) = H_1(A_S, M)$. By induction and applying the same arguments, $H_i(A, M) = H_i(A_S, M)$ for $i \geq 0$.

Remarks. (1) The flatness condition on A_S with respect to A is the same condition imposed by [12, Definition 1.2].

(2) Since $(A^*_{[S]})' = A_S$ (see [10]), when $C = A^*$, Theorem 6.2 can be considered as the dual of the well-known results stating that Hochschild homology of commutative algebras localizes.

7. APPLICATIONS

Let $e \in C'$ be an idempotent and $S = \{1, e\}$. For a C-comodule M, $M_S = \operatorname{Ker}(\mathrm{Id} - .e : M \to M)$ is a direct summand of M. In this case, $C_{[S]}$ is C-coflat. One of the possible ways to see this is the following.

Given an exact sequence of C-comodules

$$0 \to X \to Y \to Z \to 0$$

consider the commutative diagram with exact columns and rows

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X_S \arrow[r, "i_S"] \arrow[d] & Y_S \arrow[r, "\pi_S"] \arrow[d] & Z_S \arrow[d] & \\
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "f_X"] & Y \arrow[r, "\pi"] \arrow[d, "f_Y"] & Z \arrow[r] \arrow[d, "f_Z"] & 0 \\
0 \arrow[r] & X \arrow[r, "i"] \arrow[d] & Y \arrow[r, "\pi"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
& \operatorname{Coker}(f_X) \arrow[r, "\bar{i}"] \arrow[d] & \operatorname{Coker}(f_Y) \arrow[r, "\bar{\pi}"] \arrow[d] & \operatorname{Coker}(f_Z) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

where $f_X = \mathrm{id}_X - .e$ and analogously for Y and Z. By the Snake lemma, the sequence of *Kernels* can be glued with the sequence of *Cokernels*, as a consequence, π_S is surjective if and only if $\bar{i}$ is a monomorphism. This is the case, because if $\bar{i}(\bar{x}) = 0$ then $i(x) = y + y.e$ for some $y \in Y$, and then $x.e = 0$, so $x = x + x.e$, or equivalently $\bar{x} = 0$.

An example of this type is $C = C^\infty(G)$, the coalgebra of infinitely derivable functions on a compact Lie group G, and $e \in C'$ the unique invariant measure such that $e(G) = 1$. In this case, $e \otimes e$ is an invariant measure on the group $G \times G$, and a Fubini argument together with the invariance of the measure shows that $(e \otimes e)(\Delta f) = \int_{G\times G} f(x.y)\,dx\,dy$ equals $\int_G f(x)\,dx = e(f)$, i.e., e is an idempontent of the algebra C'. Localizing with respect to $S = \{1, e\}$ one obtains $C_{[S]} = \mathbb{C}$. If C is a compact smooth manifold with a smooth left action of G, then $M = C^\infty(X)$ is a right $C^\infty(G)$ comodule, $M_S = \{f \in C^\infty(X)/f.e = f\} = \{f \in C^\infty(X)/f(x) = \int_G f(g.x)\,dg\}$, i.e., $f(x)$ depends only on the orbit of x, so $M_S = C^\infty(X)^G$.

Applying then Theorem 6.2, we obtain

$$Hoch^*(C^\infty(G))_{[S]} = Hoch^*(\mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } * = 0 \\ 0 & \text{if } * \neq 0. \end{cases}$$

A different version of this kind of example or $G = S^1$ is obtained by replacing $e(f) = (1/2\pi i)\int_{S^1} f(e^{it})\,dt$ by $\Psi(f) = (1/2\pi i)\int_{S^1} e^{it} f(e^{it})\,dt$. The set $S = \{1, \Psi., (\Psi.)^2, \dots\}$ is a multiplicatively closed subset of $Z(C')$ and as the action of Ψ is the projection over the subspace generated by the function e^{ix}, the Mittag–Leffler conditions are satisfied.

Let X be a smooth manifold and $C = \mathscr{D}(X)$ the distributions on X. It is a topological coalgebra as it is $C^\infty(X)'$. Let U be an open subset of X which is a cozero set (i.e., $\exists f \in C^\infty(X)$ such that $U = \{x \in X/f(x) \neq 0\}$) and $S_U = \{g \in C^\infty(X)/g(x) \neq 0 \ \forall x \in 0\}$. Then $S_U \subseteq C^\infty(X) = C^\infty(X)''$ $= \mathscr{D}(X)'$ is a multiplicatively closed subset of $\mathscr{D}(X)'$. Then $\mathscr{D}(X)_{[S_U]} = \lim_{\substack{\leftarrow \\ S_U}} \mathscr{D}(X)^{(g)} = \lim_{\substack{\leftarrow \\ S_U}} C^\infty(X)'^{(g)} = (\lim_{\substack{\rightarrow \\ S_U}} C^\infty(X)^{(g)})' = C^\infty(U)' = \mathscr{D}(U)$ where the third equality is just an equality of $\mathbb{C}$-vector spaces, and the natural map $\mathscr{D}(U) \to \mathscr{D}(X)_{[S_U]}$ is continuous.

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