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A two phase elliptic singular perturbation problem with a forcing term ☆

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Abstract

We study the following two phase elliptic singular perturbation problem:

$$\Delta u^\varepsilon = \beta_\varepsilon\big(u^\varepsilon\big) + f^\varepsilon,$$

in $\Omega \subset \mathbb{R}^N$, where $\varepsilon > 0$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, with β a Lipschitz function satisfying $\beta > 0$ in $(0,1)$, $\beta \equiv 0$ outside $(0,1)$ and $\int \beta(s)\,\mathrm{d}s = M$. The functions u^ε and f^ε are uniformly bounded. One of the motivations for the study of this problem is that it appears in the analysis of the propagation of flames in the high activation energy limit, when sources are present.

We obtain uniform estimates, we pass to the limit ($\varepsilon \to 0$) and we show that limit functions are solutions to the two phase free boundary problem:

$$\Delta u = f\chi_{\{u \not\equiv 0\}} \quad \text{in } \Omega \setminus \partial\{u > 0\},$$
$$\big|\nabla u^+\big|^2 - \big|\nabla u^-\big|^2 = 2M \quad \text{on } \Omega \cap \partial\{u > 0\},$$

where $f = \lim f^\varepsilon$, in a viscosity sense and in a pointwise sense at regular free boundary points.

In addition, we show that the free boundary is smooth and thus limit functions are classical solutions to the free boundary problem, under suitable assumptions.

Some of the results obtained are new even in the case $f^\varepsilon \equiv 0$.

The results in this paper also apply to other combustion models. For instance, models with nonlocal diffusion and/or transport. Several of these applications are discussed here and we get, in some cases, the full regularity of the free boundary.

Résumé

Nous étudions le problème de perturbations singulières elliptiques à deux phases suivant :

$$\Delta u^\varepsilon = \beta_\varepsilon\big(u^\varepsilon\big) + f^\varepsilon,$$

dans $\Omega \subset \mathbb{R}^N$, où $\varepsilon > 0$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, β fonction lipschitzienne qui satisfait $\beta > 0$ sur $(0,1)$, $\beta \equiv 0$ hors de $(0,1)$ et $\int \beta(s)\,\mathrm{d}s = M$. Les fonctions u^ε et f^ε sont uniformément bornées.

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Une des motivations pour l'étude de ce problème est qu'on le trouve dans l'analyse de la propagation des flammes à la limite des hautes énergies d'activation, en présence de sources.

Nous obtenons des estimations uniformes qui nous permettent de passer à la limite lorsque ($\varepsilon \to 0$) : nous montrons que les fonctions limites sont solution du problème de frontière libre,

$$\begin{aligned} &\Delta u = f\chi_{\{u\not\equiv 0\}} \quad \text{dans } \Omega \setminus \partial\{u>0\},\\ &\left|\nabla u^+\right|^2 - \left|\nabla u^-\right|^2 = 2M \quad \text{sur } \Omega \cap \partial\{u>0\}, \end{aligned}$$

où $f = \lim f^\varepsilon$, au sens de la viscosité et au sens ponctuel aux points réguliers de la frontière libre.

De plus, nous montrons la régularité de la frontière libre, d'où les fonctions limites sont solutions classiques de notre problème à frontière libre, sous certaines hypothèses.

Une partie des résultats obtenus est originale, même dans le cas $f^\varepsilon \equiv 0$.

Les résultats obtenus s'appliquent à d'autres modèles de combustion. Par exemple aux modèles avec diffusion non locale et/ou avec transport. D'autres applications sont considerées ici et nous obtenons, dans certains cas, la régularité globale de la frontière libre.

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1. Introduction

In [24] the following singular perturbation problem for a nonlocal evolution operator was considered: Study the uniform properties, and the limit as $\varepsilon \to 0$, of nonnegative solutions u^ε of the problem:

$$\begin{aligned} &\theta\Delta u^\varepsilon + (1-\theta)\big(J * u^\varepsilon - u^\varepsilon\big) - u^\varepsilon_t = \beta_\varepsilon\big(u^\varepsilon\big) \quad \text{in } \mathbb{R}^N \times (0,+\infty),\\ &u^\varepsilon(x,0) = u^\varepsilon_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where $0 < \theta \leqslant 1$, $\varepsilon > 0$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, with β a Lipschitz continuous function satisfying $\beta > 0$ in $(0,1)$, $\beta \equiv 0$ outside $(0,1)$ and $\int \beta(s)\,\mathrm{d}s = M$. The symbol $*$ denotes spatial convolution and $J = J(x)$ is an even nonnegative kernel with unit integral.

Problem (1.1) arises in the analysis of the propagation of flames in the high activation energy limit, when admitting nonlocal effects (for the model, see [24] and the references therein).

In [24] it was shown that the understanding of the nonlocal problem (1.1) reduces to the understanding of the local problem:

$$\Delta u^\varepsilon - u^\varepsilon_t = \beta_\varepsilon\big(u^\varepsilon\big) + f^\varepsilon. \tag{$P_\varepsilon(f^\varepsilon)$}$$

It is worth noticing that problem $P_\varepsilon(f^\varepsilon)$ appears in other situations as well. For instance, in the study of the combustion model with transport,

$$\Delta u^\varepsilon + a^\varepsilon(x,t)\cdot\nabla u^\varepsilon + c^\varepsilon(x,t)u^\varepsilon - u^\varepsilon_t = \beta_\varepsilon\big(u^\varepsilon\big), \tag{1.2}$$

when a^ε, ∇u^ε, c^ε and u^ε are uniformly bounded. Moreover, the elliptic version of $P_\varepsilon(f^\varepsilon)$, namely:

$$\Delta u^\varepsilon = \beta_\varepsilon\big(u^\varepsilon\big) + f^\varepsilon, \tag{$E_\varepsilon(f^\varepsilon)$}$$

also appears in the analysis of the travelling wave solutions to a combustion model studied in [3].

In [24] a family of nonnegative solutions $u^\varepsilon(x,t)$ of equations $P_\varepsilon(f^\varepsilon)$ in a domain $\mathcal{D} \subset \mathbb{R}^{N+1}$ is considered. It is assumed that both families u^ε and f^ε are uniformly bounded in L^∞ norm in $\mathcal{D}$. Uniform estimates are obtained for the family u^ε that allow the passage to the limit, as $\varepsilon \to 0$. It is also shown that the limit function u is a solution of the free boundary problem:

$$\begin{aligned} &\Delta u - u_t = f \quad \text{in } \mathcal{D} \cap \{u>0\},\\ &|\nabla u| = \sqrt{2M} \quad \text{on } \mathcal{D} \cap \partial\{u>0\}, \end{aligned}$$

in a parabolic viscosity sense and in a pointwise sense at regular free boundary points. Here $f = \lim f^\varepsilon$, M is as above and the free boundary is defined as $\mathcal{D} \cap \partial\{u > 0\}$.

In order to go further in the understanding of problem $P_\varepsilon(f^\varepsilon)$, we deal in the present paper with the elliptic version of it, i.e., with $E_\varepsilon(f^\varepsilon)$.

We here consider a family u^ε of solutions to $E_\varepsilon(f^\varepsilon)$ in a domain $\Omega \subset \mathbb{R}^N$, such that both families u^ε and f^ε are uniformly bounded in L^∞ norm in Ω, and we study the passage to the limit, as $\varepsilon \to 0$.

Our aim is twofold: we are interested, on one hand, in discussing the problem when there is no sign restriction on u^ε and, on the other hand, in studying the regularity of the free boundary for the limit functions—topics that remained unexplored in [24].

We point out that there is a vast literature on problem $E_\varepsilon(f^\varepsilon)$ (and on the parabolic version of it, $P_\varepsilon(f^\varepsilon)$) in the particular case that $f^\varepsilon \equiv 0$. A well studied free boundary problem is obtained in the limit; see, for instance, [3,7,12, 13,16,19,23,24,27]. However, the extension of the results holding for $E_\varepsilon(f^\varepsilon)$ when $f^\varepsilon \equiv 0$ to the case $f^\varepsilon \not\equiv 0$ is not immediate, in particular when dealing with two phase functions.

On one hand, new tools are required to obtain uniform estimates that allow the passage to the limit. We achieve here this purpose with the aid of the recent monotonicity formula of [8].

On the other hand, the presence of a forcing term in $E_\varepsilon(f^\varepsilon)$ which does not have a sign, introduces a new difficulty due to the occurrence of a free boundary $\Gamma^- := \Omega \cap (\partial\{u < 0\} \setminus \partial\{u > 0\})$, that did not appear in the two phase homogeneous case (see [12,13,23]).

In fact, we prove that the limit problem has two free boundaries: $\Gamma^+ := \Omega \cap \partial\{u > 0\}$ (i.e., the one already appearing in the homogeneous problem) and $\Gamma^- = \Omega \cap (\partial\{u < 0\} \setminus \partial\{u > 0\})$. We show that on Γ^- limit functions are solutions of an obstacle type problem and that on Γ^+ limit functions behave as those in the case $f^\varepsilon \equiv 0$.

More precisely, we first prove that any limit function u satisfies:

$$\Delta u - f\chi_{\{u \not\equiv 0\}} = \Lambda \quad \text{in } \Omega,$$

with Λ a Radon measure supported on $\Omega \cap \partial\{u > 0\}$ and $f = \lim f^\varepsilon$. This implies, in particular, that there is no jump of ∇u on Γ^-.

We then show that, under suitable assumptions, the limit function u is a solution of the free boundary problem:

$$\begin{aligned} &\Delta u = f\chi_{\{u \not\equiv 0\}} \quad \text{in } \Omega \setminus \partial\{u > 0\}, \\ &|\nabla u^+|^2 - |\nabla u^-|^2 = 2M \quad \text{on } \Omega \cap \partial\{u > 0\}, \end{aligned} \tag{$E(f)$}$$

in a pointwise sense at regular free boundary points, and in a viscosity sense. Here M and f are as above, $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. The key tools here are: the monotonicity formula of [8]—in the case of the pointwise sense result—and some asymptotic development results proven in [24] for nonnegative functions with bounded heat (or Laplacian) at boundary points with a tangent ball—in the case of the viscosity sense result.

We also prove that, under certain conditions, the free boundary $\Omega \cap \partial\{u > 0\}$ is locally a $C^{1,\alpha}$ surface and therefore, the free boundary condition,

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2M \quad \text{on } \Omega \cap \partial\{u > 0\}, \tag{1.3}$$

is satisfied in the classical sense. We obtain two different type of results. One of them, holding for one phase limits, in the lines of the regularity theory developed in [1] (and its extension to inhomogeneous problems in [20] and [22]) and other results in the lines of the regularity theory developed in [5,6] (and its recent extension to inhomogeneous problems in [9]).

We remark that there are limit functions u which do not satisfy the free boundary condition (1.3) in the classical sense on any portion of $\Omega \cap \partial\{u > 0\}$ (see examples in [24], Section 3). The hypotheses we assume here are necessary to rule out those examples. In particular, we need to assume some kind of nondegeneracy for u^+, and we thus devote a complete section to the discussion of conditions implying this nondegeneracy.

We point out that most of the regularity results we prove in this paper are new even when $f^\varepsilon \equiv 0$ (see discussion in Remark 9.7). This is the case, in particular, of Theorems 9.5, 9.6 and 9.7 which are obtained by applying a local monotonicity formula recently proved by the authors, as well as its consequences (see [25]).

We finally present applications of our results to the study of the regularity of the free boundary for the limit of different singular perturbation problems. Namely, for the limit of stationary solutions to the nonlocal combustion model studied in [24], for the limit of stationary solutions to (1.2), for the limit of the travelling wave solutions to a combustion model first studied in [3] and for the limit of minimizers of an energy functional that we construct in Proposition 2.2. In particular, in the last two examples we prove that there is an open and dense subset $\mathcal{R}$ of the free boundary that is a $C^{1,\alpha}$ surface and the reminder of the free boundary has $(N-1)$-dimensional Hausdorff measure zero. In dimension 2 we prove that, in both cases, the whole free boundary is $C^{1,\alpha}$ and we get the same result in dimension 3 in the case of minimizers (Theorems 10.1 and 10.2).

An outline of the paper is as follows. In Section 2 we obtain uniform estimates for our problem and also the first results on the passage to the limit $\varepsilon \to 0$. Section 3 contains some basic examples and Section 4 results on the behavior of limit functions near the free boundary. In Section 5 we prove nondegeneracy results for u^+. Next, in Section 6 we obtain results on the asymptotic development at regular free boundary points. In Section 7 we obtain other asymptotic development results and we deal with the concept of viscosity solution to problem $E(f)$. In Section 8 we analyze the behavior of limit functions which satisfy an additional uniform nondegeneracy assumption on u^+. In Section 9 we study the regularity of the free boundary and finally, in Section 10 we discuss applications of our results.

Notation and assumptions. Throughout the paper N will denote the spatial dimension. The set $\Omega \cap \partial\{u > 0\}$ will be referred to as the free boundary.

We will assume that the functions β_ε are defined by scaling of a single function $\beta : \mathbb{R} \to \mathbb{R}$ satisfying:

(i) β is a Lipschitz continuous function,
(ii) $\beta > 0$ in $(0, 1)$ and $\beta \equiv 0$ otherwise,
(iii) $\int_0^1 \beta(s)\,\mathrm{d}s = M$.

And then $\beta_\varepsilon(s) := \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$.

In addition, the following notation will be used:

- $|S|$ N-dimensional Lebesgue measure of the set S,
- $\mathcal{H}^{N-1}$ $(N-1)$-dimensional Hausdorff measure,
- $B_r(x_0)$ open ball of radius r and center x_0,
- $⨍_{B_r(x_0)} u = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u\,\mathrm{d}x$,
- $⨍_{\partial B_r(x_0)} u = \frac{1}{\mathcal{H}^{N-1}(\partial B_r(x_0))}\int_{\partial B_r(x_0)} u\,\mathrm{d}\mathcal{H}^{N-1}$,
- χ_S characteristic function of the set S,
- $u^+ = \max(u, 0)$, $u^- = \max(-u, 0)$,
- $\langle\cdot,\cdot\rangle$ scalar product in $\mathbb{R}^N$,
- $B_\varepsilon(s) = \int_0^s \beta_\varepsilon(\tau)\,\mathrm{d}\tau$.

2. Uniform estimates and passage to the limit

In this section we consider a given family of solutions $u^\varepsilon(x)$ of the equations $E_\varepsilon(f^\varepsilon)$:

$$\Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon,$$

in a domain $\Omega \subset \mathbb{R}^N$. We assume that both families u^ε and f^ε are uniformly bounded in L^∞ norm in Ω, and we obtain further uniform estimates on the family u^ε that allow the passage to the limit, as $\varepsilon \to 0$.

We then pass to the limit, and we show that the limit problem has two free boundaries: $\Gamma^+ := \Omega \cap \partial\{u > 0\}$ (i.e., the free boundary that already appeared in the case $f^\varepsilon \equiv 0$) and $\Gamma^- := \Omega \cap (\partial\{u < 0\} \setminus \partial\{u > 0\})$ (a new free boundary, which was not present in the case $f^\varepsilon \equiv 0$).

We here show that on Γ^- limit functions are solutions of an obstacle type problem and we also draw our first conclusions on the behavior of limit functions on Γ^+.

More precisely, we prove that any limit function u satisfies:

$$\Delta u - f\chi_{\{u \not\equiv 0\}} = \Lambda \quad \text{in } \Omega, \tag{2.1}$$

with Λ a Radon measure supported on $\Omega \cap \partial\{u > 0\}$ and $f = \lim f^\varepsilon$. This implies, in particular, that there is no jump of ∇u on Γ^-.

Finally, we conclude the section by presenting an example of a family of uniformly bounded solutions of $E_\varepsilon(f^\varepsilon)$ in general settings, which is obtained by minimization of energy functionals.

We start the section by proving a lemma that will be used throughout the paper (for the homogeneous version see [1], Remark 4.2).

Lemma 2.1. *Let v be a continuous nonnegative function in a domain $\Omega \subset \mathbb{R}^N$, $v \in H^1(\Omega)$, such that $\Delta v = g$ in $\{v > 0\}$ with $g \in L^\infty(\Omega)$. Then $\lambda_v := \Delta v - g\chi_{\{v>0\}}$ is a nonnegative Radon measure with support on $\Omega \cap \partial\{v > 0\}$.*

Proof. Let $\eta \in C_c^\infty(\Omega)$ be nonnegative and let:

$$\phi_k = \eta\big(1 - h(kv)\big),$$

where $h(s) = \max(\min(2 - s, 1), 0)$. Then,

$$\int_\Omega g\chi_{\{v>0\}}\phi_k = -\int_\Omega \nabla v \nabla \phi_k \leqslant -\int_\Omega \nabla v \nabla \eta\big(1 - h(kv)\big).$$

Then, letting $k \to \infty$, we obtain:

$$\int_\Omega g\chi_{\{v>0\}}\eta \leqslant -\int_\Omega \nabla v \nabla \eta,$$

which gives the desired result. □

We will next obtain uniform Lipschitz estimates for our family. Before doing so we state the following monotonicity result from [8] that will allow us to obtain these estimates and that will also be used at other stages of our work:

Theorem 2.1. *Let u_i, $i = 1, 2$, be nonnegative continuous functions in $B_1(0)$, which verify:*

(i) $\Delta u_i \geqslant -1$ *in the sense of distributions in* $B_1(0)$,
(ii) $u_1(x).u_2(x) = 0$ *for* $x \in B_1(0)$,
(iii) $u_1(0) = u_2(0) = 0$.

Set

$$\Phi(r) = \left(\frac{1}{r^2}\int_{B_r(0)} \frac{|\nabla u_1(x)|^2}{|x|^{N-2}}\,\mathrm{d}x\right)\left(\frac{1}{r^2}\int_{B_r(0)} \frac{|\nabla u_2(x)|^2}{|x|^{N-2}}\,\mathrm{d}x\right).$$

Then,

$$\Phi(r) \leqslant C\big(1 + \|u_1\|^2_{L^2(B_{1/2}(0))} + \|u_2\|^2_{L^2(B_{1/2}(0))}\big)^2, \quad 0 < r < 1/4,\ C = C(N).$$

Suppose, in addition, that

(iv) $u_i(x) \leqslant C|x|^\sigma$ *in* $B_1(0)$, *for some* $C > 0$, $\sigma > 0$.

Then, the limit $\lim_{r\to 0^+} \Phi(r)$ *exists.*

Proof. It follows from Theorems 1.3 and 1.4 and Remark 2.2 of [8]. □

As a consequence we obtain:

Theorem 2.2. *Let u^ε be a family of solutions to $E_\varepsilon(f^\varepsilon)$ in a domain $\Omega \subset \mathbb{R}^N$ such that $\|u^\varepsilon\|_{L^\infty(\Omega)} \leqslant \mathcal{A}_1$ and $\|f^\varepsilon\|_{L^\infty(\Omega)} \leqslant \mathcal{A}_2$ for some $\mathcal{A}_1 > 0$, $\mathcal{A}_2 > 0$. Let $K \subset \Omega$ be compact and let $\tau > 0$ be such that $B_\tau(x_0) \subset \Omega$, for every $x_0 \in K$. There exists a constant $L = L(N, \tau, \mathcal{A}_1, \mathcal{A}_2, \|\beta\|_\infty)$ such that*

$$\big|\nabla u^\varepsilon(x)\big| \leqslant L \quad \text{for } x \in K. \tag{2.2}$$

Proof. We will follow arguments similar to those in Theorem 3 in [7].

In fact, let us first obtain estimate (2.2) for $x_0 \in K \cap \{u^\varepsilon < 0\}$. For that purpose we define, for $x \in B_1(0)$,

$$u_1(x) = \frac{1}{\tau^2 \mathcal{A}_2}\big(u^\varepsilon(\tau x + x_0) - \lambda\big)^+, \qquad u_2(x) = \frac{1}{\tau^2 \mathcal{A}_2}\big(u^\varepsilon(\tau x + x_0) - \lambda\big)^-,$$

with $\lambda = u^\varepsilon(x_0)$. Then, using Lemma 2.1 we see that u_i are under the assumptions of Theorem 2.1. This implies that, for $0 < r < \frac{1}{4}$,

$$\Phi(r) \leqslant C_1,$$

and thus, $|\nabla u_1(0)|^2 |\nabla u_2(0)|^2 \leqslant C_2$, which gives (2.2) at x_0.

Let us now consider $x_0 \in K \cap \{0 \leqslant u^\varepsilon \leqslant 2\varepsilon\}$. Without loss of generality we can assume that $\varepsilon < 1$. For $x \in B_{\tau/2}(0)$ we define:

$$v^\varepsilon(x) = \frac{1}{\varepsilon} u^\varepsilon(\varepsilon x + x_0).$$

The estimate obtained in $\{u^\varepsilon < 0\}$ implies that $v^\varepsilon \geqslant -C_3$ in $B_{\tau/2}(0)$. By using Harnack inequality we get:

$$\big|\Delta v^\varepsilon\big| \leqslant C_4, \qquad \big|v^\varepsilon\big| \leqslant C_5,$$

in $B_{\tau/4}(0)$ and thus (2.2) holds at x_0.

Let us finally consider $x_0 \in K \cap \{u^\varepsilon > \varepsilon\}$. We take v^ε satisfying:

$$\begin{aligned} &\Delta v^\varepsilon = f^\varepsilon \quad \text{in } B_{\tau/2}(x_0), \\ &v^\varepsilon = 0 \quad \text{on } \partial B_{\tau/2}(x_0), \end{aligned}$$

and let $w^\varepsilon = u^\varepsilon - v^\varepsilon$. Since $\beta_\varepsilon(u^\varepsilon) = 0$ in $\{u^\varepsilon > \varepsilon\}$, we have:

$$\Delta w^\varepsilon = 0, \quad \big|w^\varepsilon\big| \leqslant C_6 \quad \text{in } B_{\tau/2}(x_0) \cap \big\{u^\varepsilon > \varepsilon\big\},$$
$$\big|\nabla w^\varepsilon\big| \leqslant C_7 \quad \text{on } B_{\tau/2}(x_0) \cap \partial\big\{u^\varepsilon > \varepsilon\big\}$$

(we have used the estimate obtained in $\{0 \leqslant u^\varepsilon \leqslant 2\varepsilon\}$).

We now fix $\varphi \in C_0^\infty(B_{\tau/2}(x_0))$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ in $B_{\tau/4}(x_0)$. Then the function,

$$\varphi^2 \big|\nabla w^\varepsilon\big|^2 + \lambda\big(w^\varepsilon\big)^2,$$

is subharmonic in $B_{\tau/2}(x_0) \cap \{u^\varepsilon > \varepsilon\}$ if we choose a constant λ large enough (depending only on φ). Therefore, $|\nabla w^\varepsilon| \leqslant C_8$ in $B_{\tau/4}(x_0) \cap \{u^\varepsilon > \varepsilon\}$, which gives (2.2) at x_0. The proof is complete. □

With the uniform estimate obtained in the previous result we can now pass to the limit as $\varepsilon \to 0$.

Lemma 2.2. *Let u^ε be a family of solutions to $E_\varepsilon(f^\varepsilon)$ in a domain $\Omega \subset \mathbb{R}^N$. Let us assume that $\|u^\varepsilon\|_{L^\infty(\Omega)} \leqslant \mathcal{A}_1$ and $\|f^\varepsilon\|_{L^\infty(\Omega)} \leqslant \mathcal{A}_2$ for some $\mathcal{A}_1 > 0$, $\mathcal{A}_2 > 0$. For every $\varepsilon_n \to 0$ there exist a subsequence $\varepsilon_{n'} \to 0$, a function u which is locally Lipschitz continuous in Ω and a function $f \in L^\infty(\Omega)$, such that*

(i) *$u^{\varepsilon_{n'}} \to u$ uniformly on compact subsets of Ω,*
(ii) *$\nabla u^{\varepsilon_{n'}} \to \nabla u$ in $L^2_{\text{loc}}(\Omega)$,*
(iii) *$f^{\varepsilon_{n'}} \to f$ $*$-weakly in $L^\infty(\Omega)$,*
(iv) *$\Delta u \geqslant f$ in the distributional sense in Ω.*
(v) *$\Delta u = f$ in $\{u > 0\} \cup \{u < 0\}$.*

Proof. The result follows arguing as in Lemma 3.1 in [12]. □

The previous result shows that the limit problem has two free boundaries: $\Gamma^+ = \Omega \cap \partial\{u > 0\}$ and $\Gamma^- = \Omega \cap (\partial\{u < 0\} \setminus \partial\{u > 0\})$. The next result will allow us to draw our first conclusions on the behavior of limit functions on these free boundaries.

Proposition 2.1. *Let u^{ε_j} be a family of solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Then,*

$$\Delta u^+ - f\chi_{\{u>0\}} = \lambda_u^+ \quad \text{in } \Omega,$$
$$\Delta u^- + f\chi_{\{u<0\}} = \lambda_u^- \quad \text{in } \Omega,$$

with λ_u^+ and λ_u^- nonnegative Radon measures supported on the free boundary $\Gamma^+ = \Omega \cap \partial\{u>0\}$. It follows that

$$\Delta u - f\chi_{\{u\not\equiv 0\}} = \Lambda \quad \text{in } \Omega,$$

with Λ a Radon measure supported on the free boundary $\Gamma^+ = \Omega \cap \partial\{u>0\}$. In particular, $u \in W^{2,p}_{\mathrm{loc}}$ in $\{u \leqslant 0\}^\circ$, $1 < p < \infty$.

Proof. From Lemma 2.1 we deduce that

$$\Delta u^+ - f\chi_{\{u>0\}} = \lambda_u^+ \quad \text{in } \Omega,$$
$$\Delta u^- + f\chi_{\{u<0\}} = \lambda_u^- \quad \text{in } \Omega,$$

with λ_u^+ and λ_u^- nonnegative Radon measures, λ_u^+ supported on $\Omega \cap \partial\{u>0\}$ and λ_u^- supported on $\Omega \cap \partial\{u<0\}$.

Let us see that λ_u^- is actually supported on $\Omega \cap \partial\{u>0\}$. In fact, let $x_0 \in \Omega \cap (\partial\{u<0\} \setminus \partial\{u>0\})$, and let $B_r(x_0) \subset \{u \leqslant 0\}^\circ$. On one hand there holds that

$$\Delta u \geqslant f \geqslant -\|f\|_{L^\infty} \quad \text{in } B_r(x_0).$$

On the other hand,

$$\Delta u^- + f\chi_{\{u<0\}} = \lambda_u^- \geqslant 0$$

so that

$$\Delta u = -\Delta u^- \leqslant \|f\|_{L^\infty} \quad \text{in } B_r(x_0).$$

It follows that $u \in W^{2,p}_{\mathrm{loc}}(B_r(x_0))$, $1 < p < \infty$, and thus,

$$\Delta u^- + f\chi_{\{u<0\}} = 0 \quad \text{in } B_r(x_0).$$

Therefore support $\lambda_u^- \subset \Omega \cap \partial\{u>0\}$. □

Remark 2.1. By different arguments from those in Proposition 2.1 we can deduce that

$$\Delta u - f = \mu \quad \text{in } \Omega \tag{2.3}$$

with μ a nonnegative Radon measure. In fact, reasoning in a similar way as in [12], Proposition 3.1, we can deduce that

$$\int_K \beta_{\varepsilon_j}\big(u^{\varepsilon_j}\big) \leqslant C_K, \quad \text{for every } K \Subset \Omega. \tag{2.4}$$

Therefore there exists a nonnegative Radon measure μ such that $\beta_{\varepsilon_j}(u^{\varepsilon_j}) \to \mu$ weakly in Ω and such that (2.3) holds.

Notice that, as in [24], (2.3) implies that $f \leqslant 0$ in $\{u \equiv 0\}^\circ$.

Remark 2.2. When $u^{\varepsilon_j} \geqslant 0$ we deduce the nonnegativity of the Radon measure Λ appearing in Proposition 2.1 from the fact that $\Lambda = \lambda_u^+$ in Ω.

Remark 2.3. Let us point out that when $f^{\varepsilon_j} \equiv 0$ there holds that $\Gamma^- := \Omega \cap (\partial\{u<0\} \setminus \partial\{u>0\}) = \emptyset$. If $f^{\varepsilon_j} \not\equiv 0$ the boundary Γ^- may appear but, as we showed in Proposition 2.1, there holds that $u \in W^{2,p}$ across it.

On the other hand, we know that $f \leqslant 0$ in $\{u \equiv 0\}^\circ$, so if f is continuous necessarily $f \leqslant 0$ in Γ^-.

If $x_0 \in \Gamma^-$ and $f < -c < 0$ in $B_\delta(x_0) \cap \{u<0\}$ then we have the well known obstacle problem in a smaller ball $B_{\delta'}(x_0)$.

Examples with $\Gamma^- \neq \emptyset$ can be easily constructed in one dimension.

Now we state two results that follow from the convergence result (Lemma 2.2) exactly as Lemmas 3.2 and 3.3 in [12].

Lemma 2.3. *Let u^{ε_j} be a family of solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$, and let $x_n \in \Omega \cap \partial\{u > 0\}$ be such that $x_n \to x_0$ as $n \to \infty$. Let $\lambda_n \to 0$, $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_n + \lambda_n x)$, and $(u^{\varepsilon_j})_{\lambda_n}(x) = \frac{1}{\lambda_n} u^{\varepsilon_j}(x_n + \lambda_n x)$. Assume that $u_{\lambda_n} \to U$ as $n \to \infty$ uniformly on compact sets of $\mathbb{R}^N$. Then, there exists $j(n) \to +\infty$ such that for every $j_n \geqslant j(n)$ there holds that $\frac{\varepsilon_{j_n}}{\lambda_n} \to 0$, and*

(1) *$(u^{\varepsilon_{j_n}})_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^N$,*
(2) *$\nabla (u^{\varepsilon_{j_n}})_{\lambda_n} \to \nabla U$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.*

Also, we deduce that

(3) *$\nabla u_{\lambda_n} \to \nabla U$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.*

Lemma 2.4. *Let u^{ε_j} be a solution to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega_j \subset \mathbb{R}^N$ with $\Omega_j \subset \Omega_{j+1}$ and $\bigcup_j \Omega_j = \mathbb{R}^N$ such that $u^{\varepsilon_j} \to U$ uniformly on compact sets of $\mathbb{R}^N$, $f^{\varepsilon_j} \to 0$ $*$-weakly in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and $\varepsilon_j \to 0$. Let us assume that for some choice of positive numbers d_n and points $x_n \in \partial\{U > 0\}$, the sequence $U_{d_n}(x) = \frac{1}{d_n} U(x_n + d_n x)$ converges uniformly on compact sets of $\mathbb{R}^N$ to a function U_0. Let $(u^{\varepsilon_j})_{d_n}(x) = \frac{1}{d_n} u^{\varepsilon_j}(x_n + d_n x)$. Then, there exists $j(n) \to \infty$ such that for every $j_n \geqslant j(n)$, there holds that $\frac{\varepsilon_{j_n}}{d_n} \to 0$ and*

(1) *$(u^{\varepsilon_{j_n}})_{d_n} \to U_0$ uniformly on compact sets of $\mathbb{R}^N$,*
(2) *$\nabla (u^{\varepsilon_{j_n}})_{d_n} \to \nabla U_0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.*

We conclude the section by presenting an example of a family of uniformly bounded solutions of $E_\varepsilon(f^\varepsilon)$ in general settings. This family is obtained by minimization of energy functionals. We will come back to this family in forthcoming sections.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $\phi_\varepsilon \in H^1(\Omega)$ be such that $\|\phi_\varepsilon\|_{H^1(\Omega)} \leqslant \mathcal{A}_1$. Let $f^\varepsilon \in L^\infty(\Omega)$ such that $\|f^\varepsilon\|_{L^\infty(\Omega)} \leqslant \mathcal{A}_2$. There exists $u^\varepsilon \in H^1(\Omega)$ that minimizes the energy,*

$$J_\varepsilon(v) = \int_\Omega \frac{1}{2} |\nabla v|^2 + B_\varepsilon(v) + f^\varepsilon v,$$

among functions $v \in H^1(\Omega)$ such that $v = \phi_\varepsilon$ on $\partial\Omega$. Here $B_\varepsilon(s) = \int_0^s \beta_\varepsilon(\tau)\,\mathrm{d}\tau$.

Then, the functions u^ε satisfy:

$$\Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon \quad \text{in } \Omega,$$

and for every $\Omega' \Subset \Omega$ there exists $C = C(\Omega', \mathcal{A}_1, \mathcal{A}_2)$ such that

$$\|u^\varepsilon\|_{L^\infty(\Omega')} \leqslant C.$$

Proof. The proof of the existence of a minimizer of J_ε is standard and we omit it here. It is also standard the proof that a minimizer u^ε is a solution to $E_\varepsilon(f^\varepsilon)$. It is easy to see that $\|u^\varepsilon\|_{H^1(\Omega)} \leqslant C$ with C independent of ε.

Let us show that for every $\Omega' \Subset \Omega$ there exists $C = C(\Omega', \mathcal{A}_1, \mathcal{A}_2)$ such that

$$\|u^\varepsilon\|_{L^\infty(\Omega')} \leqslant C.$$

In fact, since u^ε is a solution to $E_\varepsilon(f^\varepsilon)$ in Ω there holds that $u^\varepsilon \in C^{1,\alpha}(\Omega)$. In particular, $\{u^\varepsilon < -1\}$ is open and $(u^\varepsilon + 1)^-$ is a nonnegative solution to

$$\Delta u = -f^\varepsilon \quad \text{in } \Omega \cap \{u^\varepsilon < -1\},$$
$$u = 0 \quad \text{on } \Omega \cap \partial\{u^\varepsilon < -1\},$$

with uniformly bounded $H^1(\Omega)$ norm. Thus,

$$\sup_{\Omega'} (u^\varepsilon + 1)^- \leqslant C.$$

In particular, u^ε is uniformly bounded from below. Now, $(u^\varepsilon + C + 1)$ is a nonnegative solution to,

$$\Delta u \geqslant f^\varepsilon \quad \text{in } \Omega,$$

with uniformly bounded $H^1(\Omega)$ norm. We deduce that

$$\sup_{\Omega'} (u^\varepsilon + C + 1) \leqslant \overline{C}.$$

So that the uniform boundedness of u^ε in Ω' follows. □

3. Basic limits

In this section we analyze some particular limits that are crucial in the understanding of general limits.

We need to prove first the following lemma:

Lemma 3.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ ∗-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Then there exists $\chi \in L^1_{\text{loc}}(\Omega)$ such that, for a subsequence, $B_{\varepsilon_j}(u^{\varepsilon_j}) \to \chi$ in $L^1_{\text{loc}}(\Omega)$, with $\chi \equiv M$ in $\{u > 0\}$, $\chi \equiv 0$ in $\{u < 0\}$, $\chi(x) \in \{0, M\}$ a.e. in Ω. If, in addition, $f^{\varepsilon_j} \to 0$ in $\{u \leqslant 0\}^\circ$, there holds that $\chi \equiv M$ or $\chi \equiv 0$ on every connected component of $\{u \leqslant 0\}^\circ$.*

Proof. We follow the same ideas as in Step IV in the proof of Theorem 3.1 in [23], where we had $f^\varepsilon \equiv 0$. If $f^\varepsilon \not\equiv 0$ we have, for every $K \Subset \Omega$,

$$\int_K \left|\nabla B_{\varepsilon_j}(u^{\varepsilon_j})\right| = \int_K \beta_{\varepsilon_j}(u^{\varepsilon_j})\left|\nabla u^{\varepsilon_j}\right| \leqslant C_K \int_K \beta_{\varepsilon_j}(u^{\varepsilon_j}), \tag{3.1}$$

where the last term is bounded by a constant C'_K due to estimate (2.4).

Since $0 \leqslant B_{\varepsilon_j}(u^{\varepsilon_j}) \leqslant M$, then, there exists $\chi \in L^1_{\text{loc}}(\Omega)$ such that, for a subsequence, $B_{\varepsilon_j}(u^{\varepsilon_j}) \to \chi$ in $L^1_{\text{loc}}(\Omega)$.

In order to see that necessarily $\chi = 0$ or $\chi = M$, we modify the argument in [23] as follows. Let $\rho_1, \rho_2 > 0$ and $K \Subset \Omega$. There exist $0 < \eta < 1$ and $\beta_\eta > 0$ such that

$$\begin{aligned} \left|\left\{x \in K \mid \rho_1 < B_{\varepsilon_j}(u^{\varepsilon_j}) < M - \rho_2\right\}\right| &\leqslant \left|\left\{x \in K \;\middle|\; \eta < \frac{u^{\varepsilon_j}}{\varepsilon_j} < 1 - \eta\right\}\right| \\ &\leqslant \left|\left\{x \in K \;\middle|\; \beta_{\varepsilon_j}(u^{\varepsilon_j}) \geqslant \frac{\beta_\eta}{\varepsilon_j}\right\}\right| \leqslant \frac{\varepsilon_j}{\beta_\eta} \int_K \beta_{\varepsilon_j}(u^{\varepsilon_j}) \to 0. \end{aligned}$$

This implies that

$$\left|\left\{x \in K \mid \rho_1 < \chi < M - \rho_2\right\}\right| = 0,$$

for every $\rho_1, \rho_2 > 0$ and $K \Subset \Omega$, so $\chi(x) \in \{0, M\}$ a.e. in Ω.

We now deduce that $\chi \equiv M$ in $\{u > 0\}$ and $\chi \equiv 0$ in $\{u < 0\}$ as in [13], Theorem 3.1.

Finally, in case $f^{\varepsilon_j} \to 0$ in $\{u \leqslant 0\}^\circ$, we take $K \Subset \{u \leqslant 0\}^\circ$ in (3.1), we observe that (as in [23]) the last term there goes to zero and the result follows. □

Proposition 3.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$. Let $x_0 \in \Omega$ and suppose u^{ε_j} converge to $u = \alpha(x - x_0)_1^+ - \gamma(x - x_0)_1^-$ uniformly on compact subsets of Ω, with $\alpha \geqslant 0, \gamma > 0$, $f^{\varepsilon_j} \to 0$ ∗-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Then*

$$\alpha^2 - \gamma^2 = 2M.$$

Proof. The proof follows as that of Proposition 5.1 in [12]. □

Proposition 3.2. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega\subset\mathbb{R}^N$. Let $x_0\in\Omega$ and suppose u^{ε_j} converge to $u=\alpha(x-x_0)_1^+$ uniformly on compact subsets of Ω, with $\alpha\in\mathbb{R}$, $f^{\varepsilon_j}\to 0$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j\to 0$. Then*

$$\alpha=0\quad or\quad \alpha=\sqrt{2M}.$$

Proof. First we see that necessarily $\alpha\geqslant 0$. In fact, if not we would have $u\leqslant 0$ in Ω, $u(x_0)=0$ and u subharmonic in Ω and thus $u\equiv 0$, which is a contradiction.

If $\alpha>0$ we deduce that $\alpha=\sqrt{2M}$ proceeding as in the proof of Proposition 5.1 in [24], but using in the present case Lemma 3.1 above. □

Proposition 3.3. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega\subset\mathbb{R}^N$. Let $x_0\in\Omega$ and suppose u^{ε_j} converge to $u=\alpha(x-x_0)_1^+ +\bar\alpha(x-x_0)_1^-$ uniformly on compact subsets of Ω, with $\alpha>0,\bar\alpha>0$, $f^{\varepsilon_j}\to 0$ uniformly on compact subsets of Ω and $\varepsilon_j\to 0$. Then*

$$\alpha=\bar\alpha\leqslant\sqrt{2M}.$$

Proof. The result was proven for the parabolic version of this problem in Proposition 5.3 in [12], for $f^\varepsilon\equiv 0$, and it was extended to the case $f^\varepsilon\not\equiv 0$ in Proposition 5.2 in [24], under the assumption that $u^\varepsilon\geqslant 0$. But the same proof in [24] is valid in the present case. □

Remark 3.1. We point out that all the situations present in Propositions 3.1, 3.2 and 3.3 can occur. We refer to Section 3 in [24] for examples of those situations. In particular, the analysis in [24] shows that for any given $\alpha\in[0,\sqrt{2M}\,]$ there are examples of families u^{ε_j} of solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in $\mathbb{R}^N$, with $f^{\varepsilon_j}\to 0$ uniformly on compact sets of $\mathbb{R}^N$ such that

$$u^{\varepsilon_j}\to u=\alpha x_1^+ +\alpha x_1^-,\quad \text{uniformly on compact sets of } \mathbb{R}^N.$$

4. Behavior of limit functions near the free boundary

In this section we analyze the behavior of limit functions $u=\lim u^\varepsilon$, with u^ε a family of solutions to problems $E_\varepsilon(f^\varepsilon)$.

The following result says that a limit function is, in a sense, a supersolution to the free boundary problem $E(f)$—this holding for any limit function, without imposing any additional hypothesis.

Theorem 4.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega\subset\mathbb{R}^N$ such that $u^{\varepsilon_j}\to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j}\to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j\to 0$. Let $x_0\in\Omega\cap\partial\{u>0\}$ and let $\gamma\geqslant 0$ be such that*

$$\limsup_{x\to x_0}\left|\nabla u^-(x)\right|\leqslant\gamma.$$

Then,

$$\limsup_{x\to x_0}\left|\nabla u^+(x)\right|\leqslant\sqrt{2M+\gamma^2}.$$

Proof. The proof follows as that of Theorem 6.1 in [12]. In fact, we define:

$$\alpha=\limsup_{\substack{x\to x_0\\ u(x)>0}}\left|\nabla u(x)\right|$$

and, proceeding as in [12], we assume that $\alpha>0$ and let $x_n\to x_0$ be such that $u(x_n)>0$ and $|\nabla u(x_n)|\to\alpha$. Then we let $z_n\in\Omega\cap\partial\{u>0\}$ be such that $d_n:=|x_n-z_n|=\operatorname{dist}(x_n,\partial\{u>0\})$. As in [12] we choose $\varepsilon_n^0:=\frac{\varepsilon_{j_n}}{d_n}\to 0$, and consider the sequences:

$$u_{d_n}(x)=\frac{1}{d_n}u(z_n+d_nx),\qquad u^{\varepsilon_n^0}(x)=\frac{1}{d_n}u^{\varepsilon_{j_n}}(z_n+d_nx).$$

There holds that the functions u_{d_n} satisfy (iv) and (v) in Lemma 2.2 with right-hand side f^n and that $u^{\varepsilon_n^0}$ are solutions to $E_{\varepsilon_n^0}(f^{\varepsilon_n^0})$, where the sequences,

$$f^n(x) = d_n f(z_n + d_n x), \qquad f^{\varepsilon_n^0}(x) = d_n f^{\varepsilon_{j_n}}(z_n + d_n x),$$

converge uniformly to 0 as $n \to \infty$ on compact sets, since $\|f^{\varepsilon_j}\|_\infty \leqslant C$. It follows that the blow-up limit $u_0 = \lim u_{d_n} = \lim u^{\varepsilon_n^0}$ is harmonic in the set $\{u_0 > 0\} \cup \{u_0 \leqslant 0\}^\circ$. Then we can argue as in [12]. In the present case we obtain (after a second blow up) a sequence $\varepsilon_n^{00} \to 0$ and solutions $u^{\varepsilon_n^{00}}$ to $E_{\varepsilon_n^{00}}(f^{\varepsilon_n^{00}})$ in $B_1(0)$ such that

$$u^{\varepsilon_n^{00}} \to u_{00} = \alpha x_1^+ + \mu x_1^- \quad \text{uniformly on compact subsets of } B_1(0),$$

with $f^{\varepsilon_n^{00}} \to 0$ uniformly on compact sets. Therefore, Propositions 3.1, 3.2 and 3.3 apply and we arrive at the conclusion as in [12]. □

Theorem 4.2. *Let u^{ε_j} be a solution to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega_j \subset \mathbb{R}^N$ such that $\Omega_j \subset \Omega_{j+1}$ and $\bigcup_j \Omega_j = \mathbb{R}^N$. Let us assume that u^{ε_j} converge to a function U uniformly on compact sets of $\mathbb{R}^N$, $f^{\varepsilon_j} \to 0$ $*$-weakly in $L^\infty_{\text{loc}}(\mathbb{R}^N)$ and $\varepsilon_j \to 0$. Assume, in addition, that U is Lipschitz continuous in $\mathbb{R}^N$ and $\partial\{U > 0\} \neq \emptyset$. If $\gamma \geqslant 0$ is such that $|\nabla U^-| \leqslant \gamma$ in $\mathbb{R}^N$ then,*

$$\left|\nabla U^+\right| \leqslant \sqrt{2M + \gamma^2} \quad \text{in } \mathbb{R}^N.$$

Proof. The proof follows as that of Theorem 6.2 in [12], since U is harmonic in the set $\{U > 0\} \cup \{U \leqslant 0\}^\circ$. □

5. Nondegeneracy results

At different stages of our work we will prove results for $u = \lim u^\varepsilon$, with u^ε solutions to $E_\varepsilon(f^\varepsilon)$, which hold under the assumption that u^+ satisfies some nondegeneracy condition at the free boundary (see Definition 5.1). The purpose of this section is to present results that imply some kind of nondegeneracy on u^+.

In particular we define the concept of minimal solution to problem $E_\varepsilon(f^\varepsilon)$ and we prove the uniform nondegeneracy of u^+ on the free boundary when u is the limit of any family of minimal solutions. We also prove the uniform nondegeneracy of u^+ on the free boundary when u is the limit of the minimizers to the energy functional constructed in Proposition 2.2.

We point out that, from Section 3 in [24], we know that there are examples where u^+ degenerates at the free boundary. Therefore, some additional assumption is required if one wants to guarantee the nondegeneracy of u^+ at a free boundary point.

Definition 5.1. Let $v \geqslant 0$ be a continuous function in a domain $\Omega \subset \mathbb{R}^N$.

We say that v is nondegenerate at a point $x_0 \in \Omega \cap \{v = 0\}$ if there exist $c > 0$ and $r_0 > 0$ such that one of the following conditions holds:

$$\fint_{\partial B_r(x_0)} v \geqslant cr \quad \text{for } 0 < r \leqslant r_0, \tag{5.1}$$

$$\fint_{B_r(x_0)} v \geqslant cr \quad \text{for } 0 < r \leqslant r_0, \tag{5.2}$$

$$\sup_{\partial B_r(x_0)} v \geqslant cr \quad \text{for } 0 < r \leqslant r_0, \tag{5.3}$$

$$\sup_{B_r(x_0)} v \geqslant cr \quad \text{for } 0 < r \leqslant r_0. \tag{5.4}$$

Otherwise, we say that v degenerates at x_0.

We say that v is uniformly nondegenerate on $\Gamma \subset \Omega \cap \{v = 0\}$ in the sense of (5.1) (resp. (5.2), (5.3) or (5.4)), if there exist $c > 0$ and $r_0 > 0$ such that (5.1) (resp. (5.2), (5.3) or (5.4)) holds for every $x_0 \in \Gamma$.

Remark 5.1. If $v \geqslant 0$ is locally Lipschitz continuous in a domain $\Omega \subset \mathbb{R}^N$ and $\Delta v \geqslant -C$ in Ω (which will be our case), the four concepts of nondegeneracy of Definition 5.1 are equivalent. In fact, this can be seen by arguing in a similar way as in Remark 3.1 in [23].

There holds the following result which will be applied to our limit functions

Proposition 5.1. *Let u be a locally Lipschitz continuous function in a domain $\Omega \subset \mathbb{R}^N$ satisfying that $\Delta u \geqslant -C$ in Ω. Assume that u^- is nondegenerate at $x_0 \in \Omega \cap \partial\{u > 0\}$ in the sense of* (5.2). *Then u^+ is nondegenerate at x_0 in the same sense.*

Proof. The result follows as Lemma 5.2 of [23], if we observe that in the present case $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ converges to u_0 with $\Delta u_0 \geqslant 0$. □

Our first result implying that u^+ is nondegenerate at a free boundary point is the following:

Theorem 5.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and assume that there exists $\nu \in \mathbb{R}^N$, with $|\nu| = 1$ such that*

$$\liminf_{r \to 0^+} \frac{|\{u > 0\} \cap \{\langle x - x_0, \nu\rangle > 0\} \cap B_r(x_0)|}{|B_r(x_0)|} = \alpha_1, \tag{5.5}$$

and

$$\liminf_{r \to 0^+} \frac{|\{u < 0\} \cap \{\langle x - x_0, \nu\rangle < 0\} \cap B_r(x_0)|}{|B_r(x_0)|} = \alpha_2 \tag{5.6}$$

with $\alpha_1 + \alpha_2 > 1/2$. Then, there exists a constant $C > 0$ such that, for every $r > 0$ small,

$$\sup_{\partial B_r(x_0)} u \geqslant Cr.$$

The constant C depends only on $\alpha_1 + \alpha_2$, N and the function β.

If, instead of (5.6)*, we have*

$$\begin{gathered}\liminf_{r \to 0^+} \frac{|\{u \leqslant 0\}^\circ \cap \{\langle x - x_0, \nu\rangle < 0\} \cap B_r(x_0)|}{|B_r(x_0)|} = \alpha_2,\\ \frac{u^{\varepsilon_j}}{\varepsilon_j} \to 0 \quad \text{a.e. in } \big(\{u \equiv 0\}^\circ \cup \partial\{u < 0\}\big) \cap \big\{\langle x - x_0, \nu\rangle < 0\big\} \cap B_{r_0}(x_0),\end{gathered} \tag{5.7}$$

we obtain the same conclusion.

Proof. *Case* $f^\varepsilon \equiv 0$. The proof was done in [12], Theorem 6.3 under assumption (5.6). Under assumption (5.7), the proof was done in [19], Proposition 4.1 and Remark 4.1, when $u^\varepsilon \geqslant 0$. It is not hard to see that the proof in [19] applies also under assumption (5.7) when there is no sign restriction on u^ε.

Case $f^\varepsilon \not\equiv 0$. The proof was done in [24], Theorem 6.2, under assumption (5.7), when $u^\varepsilon \geqslant 0$. The result in the statement, both for (5.6) or (5.7), follows as in the case $f^\varepsilon \equiv 0$ but treating the term f^ε as shown in [24]. □

Remark 5.2. If in Theorem 5.1, instead of (5.7), we have the alternative condition:

$$\begin{gathered}\liminf_{r \to 0^+} \frac{|\{u \leqslant 0\} \cap \{\langle x - x_0, \nu\rangle < 0\} \cap B_r(x_0)|}{|B_r(x_0)|} = \alpha_2,\\ \frac{u^{\varepsilon_j}}{\varepsilon_j} \to 0 \quad \text{a.e. in } \{u \equiv 0\} \cap \big\{\langle x - x_0, \nu\rangle < 0\big\} \cap B_{r_0}(x_0),\end{gathered}$$

we obtain the same conclusion.

Corollary 5.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that there exists an inward unit normal ν to $\partial\{u > 0\}$ at x_0 in the measure theoretic sense (see Definition 6.1), and assume that one of the following conditions holds*:

(1) $\liminf_{r\to 0^+} \frac{|\{u<0\}\cap B_r(x_0)|}{|B_r(x_0)|} > 0$,
(2) $\liminf_{r\to 0^+} \frac{|\{u<0\}\cap B_r(x_0)|}{|B_r(x_0)|} = 0$ *and* $\frac{u^{\varepsilon_j}}{\varepsilon_j} \to 0$ *a.e. in* $\{u \equiv 0\} \cap \{\langle x - x_0, \nu\rangle < 0\} \cap B_{r_0}(x_0)$.

Then, the same conclusion of Theorem 5.1 *holds.*

Proof. We first notice that there holds (5.5), with $\alpha_1 = 1/2$, and

$$\liminf_{r\to 0^+} \frac{|\{u \leqslant 0\} \cap \{\langle x - x_0, \nu\rangle < 0\} \cap B_r(x_0)|}{|B_r(x_0)|} = \frac{1}{2}.$$

Then, in case (1) holds the result is an immediate consequence of Theorem 5.1. In case (2) holds the result follows from Remark 5.2. □

Corollary 5.2. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that there exists a ball $B \subset \{u > 0\}$, with $x_0 \in \partial B$, and assume that one of the following conditions holds*:

(1) $\liminf_{r\to 0^+} \frac{|\{u<0\}\cap B_r(x_0)|}{|B_r(x_0)|} > 0$,
(2) $\liminf_{r\to 0^+} \frac{|\{u\leqslant 0\}^\circ\cap B_r(x_0)|}{|B_r(x_0)|} > 0$ *and* $\frac{u^{\varepsilon_j}}{\varepsilon_j} \to 0$ *a.e. in* $(\{u \equiv 0\}^\circ \cup \partial\{u < 0\}) \cap B^c \cap B_{r_0}(x_0)$.

Then, the same conclusion of Theorem 5.1 *holds.*

Proof. The result follows from Theorem 5.1, since (5.5) is satisfied with ν the inward unit normal to ∂B at x_0 and $\alpha_1 = 1/2$. □

Corollary 5.3. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that there exists a ball $B \subset \{u \leqslant 0\}^\circ$, with $x_0 \in \partial B$. Assume that $\liminf_{r\to 0^+} \frac{|\{u>0\}\cap B_r(x_0)|}{|B_r(x_0)|} > 0$, and that one of the following conditions holds*:

(1) $u < 0$ *in* B,
(2) $\frac{u^{\varepsilon_j}}{\varepsilon_j} \to 0$ *a.e. in* $\{u \equiv 0\} \cap B$.

Then, the same conclusion of Theorem 5.1 *holds.*

Proof. The result follows from Theorem 5.1, since either (5.6) or a condition equivalent to (5.7) are satisfied, with ν the outward unit normal to ∂B at x_0 and $\alpha_2 = 1/2$. □

The nondegeneracy of u^+ at a point $x_0 \in \partial\{u > 0\}$ can also be derived from Hopf's Principle under suitable assumptions on the smoothness of $\partial\{u > 0\}$ at x_0 and on the sign of f. In fact, we have:

Proposition 5.2. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$, and assume that one of the following conditions holds*:

(1) $\partial\{u > 0\}$ *satisfies a Dini interior condition at* x_0 *and* $f \leqslant 0$ *in* $B_{r_0}(x_0) \cap \{u > 0\}$,
(2) *there exists a ball* $B \subset \{u > 0\}$, *with* $x_0 \in \partial B$ *and* $f \leqslant 0$ *in* B,
(3) $\partial\{u > 0\}$ *satisfies a Dini exterior condition at* x_0, $\{u \equiv 0\}^\circ \cap B_{r_0}(x_0) = \emptyset$ *and* $f \geqslant 0$ *in* $B_{r_0}(x_0) \cap \{u < 0\}$,

(4) *there exists a ball* $B \subset \{u < 0\}$, *with* $x_0 \in \partial B$ *and* $f \geqslant 0$ *in* B.

Then u^+ *is nondegenerate at* x_0.

Proof. In case (1) or (2) hold, the result follows from the application of Hopf's Principle to u^+. In case (3) or (4) hold, it follows from the application of Hopf's Principle to u^- and from Proposition 5.1. □

We next define the concept of minimal solution to problem $E_\varepsilon(f^\varepsilon)$ and prove a nondegeneracy result for this kind of solutions. We will follow the lines of [3], Section 4.

Definition 5.2. Let u^ε be a solution to $E_\varepsilon(f^\varepsilon)$ in a domain $\Omega \subset \mathbb{R}^N$ with $f^\varepsilon \in L^\infty(\Omega)$. We say that u^ε is a minimal solution to $E_\varepsilon(f^\varepsilon)$ in Ω if whenever we have h^ε a strong supersolution to $E_\varepsilon(f^\varepsilon)$ in a bounded subdomain $\Omega' \Subset \Omega$, i.e.,

$$h^\varepsilon \in W^{2,p}(\Omega') \cap C\big(\overline{\Omega'}\big), \quad \Delta h^\varepsilon \leqslant \beta_\varepsilon\big(h^\varepsilon\big) + f^\varepsilon \quad \text{in } \Omega', \tag{5.8}$$

which satisfies, in addition,

$$h^\varepsilon \geqslant u^\varepsilon \quad \text{on } \partial\Omega',$$

then

$$h^\varepsilon \geqslant u^\varepsilon \quad \text{in } \Omega'.$$

Proposition 5.3. *Let* u^ε *be a minimal solution to* $E_\varepsilon(f^\varepsilon)$ *in a domain* $\Omega \subset \mathbb{R}^N$ *such that* $\|f^\varepsilon\|_{L^\infty(\Omega)} < \mathcal{A}$. *For every* $\Omega' \Subset \Omega$, *there exist positive constants* c_0, ρ *and* ε_0 *depending only on* N, $\mathcal{A}$, $\operatorname{dist}(\Omega', \partial\Omega)$ *and the function* β, *such that if* $\varepsilon \leqslant \varepsilon_0$ *and* $x \in \Omega'$, *then*

$$\big(u^\varepsilon\big)^+(x) \geqslant c_0 \operatorname{dist}\big(x, \big\{u^\varepsilon \leqslant \varepsilon\big\}\big) \quad \text{if } \operatorname{dist}\big(x, \big\{u^\varepsilon \leqslant \varepsilon\big\}\big) \leqslant \rho. \tag{5.9}$$

Proof. Our proof is a modification of Theorem 4.1 in [3]. In fact, let us fix $0 < a < b < 1$ and $\kappa > 0$ such that $\beta(s) > \kappa$ for $s \in [a, b]$. Let $x_0 \in \Omega'$ such that $u^\varepsilon(x_0) > \varepsilon$ and such that $2\delta = \operatorname{dist}(x_0, \{u^\varepsilon \leqslant \varepsilon\}) \leqslant \operatorname{dist}(\Omega', \partial\Omega)$ and $\delta < 1$. Without loss of generality we will assume that $x_0 = 0$.

In $B_{2\delta}(0)$ there holds that $\Delta u^\varepsilon = f^\varepsilon$. By the Harnack inequality there holds that

$$u^\varepsilon(x) \leqslant C u^\varepsilon(0) + C\delta^2\mathcal{A} \quad \text{in } B_\delta(0),$$

with $C = C(N) > 0$. We will exhibit a C^1 supersolution h^ε satisfying (5.8) in $B_\delta(0)$. In addition $h^\varepsilon = h^\varepsilon(r)$ will depend only on $r = |x|$ and will satisfy:

$$h^\varepsilon(0) = a\varepsilon < u^\varepsilon(0),$$

and also $h^\varepsilon(\delta) \geqslant \delta D^{-1}$ for some $D > 0$ depending only on $N, a, b, \kappa, \mathcal{A}$. By our hypothesis that u^ε is a minimal solution it follows that we cannot have $h^\varepsilon \geqslant u^\varepsilon$ everywhere on $\partial B_\delta(0)$. Hence

$$\frac{\delta}{D} \leqslant h^\varepsilon(\delta) \leqslant C u^\varepsilon(0) + C\delta^2\mathcal{A}$$

which gives:

$$u^\varepsilon(0) \geqslant c_0\delta,$$

if $\delta \leqslant \delta_0$, for constants c_0 and δ_0 depending only on $N, a, b, \kappa, \mathcal{A}$. This is, (5.9) holds.

We will take as h^ε the function constructed in [3], i.e.,

$$h^\varepsilon(r) = \begin{cases} \varepsilon a & \text{for } 0 \leqslant r \leqslant r_0, \\ \varepsilon a + \frac{k}{2}(r - r_0)^2 & \text{for } r_0 \leqslant r \leqslant \lambda, \\ H - \frac{A}{2}(r - \delta)^2 & \text{for } \lambda \leqslant r \leqslant \delta, \end{cases}$$

and we will show that we can choose the numbers r_0, λ, k, H and A so that h^ε has the desired properties for our problem, provided $\varepsilon \leqslant \varepsilon_0 = \varepsilon_0(N, a, b, \kappa, \mathcal{A})$.

As done in [3], we first ask that h^ε be C^1 and $h^\varepsilon(\lambda)=\varepsilon b$, this is,

$$\varepsilon b = H - \frac{A}{2}(\lambda-\delta)^2, \tag{5.10}$$

$$\varepsilon b = \varepsilon a + \frac{k}{2}(\lambda - r_0)^2, \tag{5.11}$$

$$k(\lambda - r_0) = A(\delta - \lambda). \tag{5.12}$$

We now take

$$\lambda = (1-\mu_0)\delta, \tag{5.13}$$

$$r_0 = \lambda - \widetilde{C}\varepsilon(b-a), \tag{5.14}$$

for some $0<\mu_0<1$ and $\widetilde{C}>0$ to be fixed later (notice that in order to have $r_0>0$ we need $\varepsilon<C\delta$). We now obtain k, A and H from (5.11), (5.12) and (5.10), resp.

Let us verify that in $B_\delta(0)$,

$$\Delta h^\varepsilon \leqslant \beta_\varepsilon\big(h^\varepsilon\big) + f^\varepsilon. \tag{5.15}$$

In fact, in $0\leqslant r\leqslant r_0$,

$$\beta_\varepsilon\big(h^\varepsilon\big) + f^\varepsilon \geqslant \frac{\kappa}{\varepsilon} - \mathcal{A}, \tag{5.16}$$

so (5.15) holds provided $\varepsilon \leqslant \varepsilon_0(\kappa,\mathcal{A})$.

Next, in $r_0\leqslant r\leqslant\lambda$, also (5.16) holds, so we have:

$$\beta_\varepsilon\big(h^\varepsilon\big) + f^\varepsilon \geqslant \frac{\kappa}{2\varepsilon},$$

if we take $\varepsilon_0(\kappa,\mathcal{A})$ smaller. Now

$$\Delta h^\varepsilon \leqslant k\left(1+(N-1)\frac{\lambda-r_0}{r_0}\right),$$

and we can make $\frac{\lambda-r_0}{r_0}\leqslant 1$ provided $\varepsilon\leqslant C\delta$, for some C depending on $\widetilde{C}$, μ_0, b and a. Then

$$\Delta h^\varepsilon \leqslant kN = \frac{2N}{\widetilde{C}^2\varepsilon(b-a)} \leqslant \frac{\kappa}{2\varepsilon},$$

if choose $\widetilde{C}$ big depending on N,b,a,κ.

It remains to verify (5.15) in $\lambda\leqslant r\leqslant\delta$. Here

$$\beta_\varepsilon\big(h^\varepsilon\big) + f^\varepsilon \geqslant -\mathcal{A},$$

and

$$\Delta h^\varepsilon = -A\left(1-(N-1)\frac{\delta-r}{r}\right) \leqslant -A\left(1-(N-1)\frac{\mu_0}{1-\mu_0}\right) \leqslant -\frac{A}{2},$$

if we take μ_0 small depending on N. Replacing A gives,

$$\Delta h^\varepsilon \leqslant -\frac{1}{\widetilde{C}\mu_0\delta} \leqslant -\mathcal{A},$$

for appropriate $\mu_0=\mu_0(N,a,b,\kappa,\mathcal{A})$. This shows that (5.15) holds in $B_\delta(0)$.

We have to see now that $h^\varepsilon(\delta)\geqslant\frac{\delta}{D}$. In fact,

$$h^\varepsilon(\delta) = H \geqslant \frac{A}{2}(\lambda-\delta)^2 = \frac{\mu_0}{\widetilde{C}}\delta,$$

and thus, a constant $D=D(N,a,b,\kappa,\mathcal{A})$ with the desired property exists.

We finally notice that the construction above fails when $\varepsilon\geqslant C\delta$, for $C=C(N,a,b,\kappa,\mathcal{A})$, but the result is immediate in this case since $u^\varepsilon(0)>\varepsilon$. The proof is now complete. □

As a consequence we obtain:

Corollary 5.4. *Let u^{ε_j} be a family of minimal solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ with $\|f^{\varepsilon_j}\|_{L^\infty(\Omega)} < \mathcal{A}$ and $\varepsilon_j \to 0$. For every $\Omega' \Subset \Omega$, there exist positive constants c_0 and ρ depending only on N, $\mathcal{A}$, $\operatorname{dist}(\Omega', \partial\Omega)$ and the function β, such that if $x \in \Omega'$, then*

$$u^+(x) \geqslant c_0 \operatorname{dist}\bigl(x, \{u \leqslant 0\}\bigr) \quad \text{if } \operatorname{dist}\bigl(x, \{u \leqslant 0\}\bigr) \leqslant \rho. \tag{5.17}$$

Proof. Let $x_0 \in \Omega'$ such that $u(x_0) > 0$ and such that $2\delta = \operatorname{dist}(x_0, \{u \leqslant 0\}) \leqslant \operatorname{dist}(\Omega', \partial\Omega)$. Then $u > 0$ in $B_{2\delta}(x_0)$. Moreover, if $0 < 2\sigma < 2\delta$, there holds that

$$u^\varepsilon > \varepsilon \quad \text{in } B_{2\sigma}(x_0), \tag{5.18}$$

if ε is small enough (we have dropped the subscript j).

From the proof of Proposition 5.3, we know that (5.18) implies that

$$u^\varepsilon(x_0) > c_0\sigma, \tag{5.19}$$

if $\sigma \leqslant \rho$ and $\varepsilon \leqslant \varepsilon_0$, for some constants c_0, ρ and ε_0 depending only on N, $\mathcal{A}$, $\operatorname{dist}(\Omega', \partial\Omega)$ and the function β.

Then, letting $\varepsilon \to 0$ in (5.19) first and then $\sigma \to \delta$, we get:

$$u(x_0) \geqslant c_0\delta,$$

which gives the desired result. □

Next, we prove the nondegeneracy of the limit of the minimizers constructed in Proposition 2.2. First, we follow closely the proof of Theorem 1.6 in [15] and we obtain:

Proposition 5.4. *Let u^ε be a minimizer to J_ε in the set of functions in $H^1(\Omega)$ that are equal to ϕ_ε on $\partial\Omega$ where $\|\phi_\varepsilon\|_{H^1(\Omega)} \leqslant C$ and $\|f^\varepsilon\|_{L^\infty(\Omega)} \leqslant \mathcal{A}$ with $C, \mathcal{A}$ independent of ε. Then, for every $\Omega' \Subset \Omega$, there exist positive constants c_0, ρ and ε_0 depending only on N, $\mathcal{A}$, $\operatorname{dist}(\Omega', \partial\Omega)$ and the function β, such that if $\varepsilon \leqslant \varepsilon_0$ and $x \in \Omega'$, then*

$$\bigl(u^\varepsilon\bigr)^+(x) \geqslant c_0 \operatorname{dist}\bigl(x, \bigl\{u^\varepsilon \leqslant \varepsilon\bigr\}\bigr) \quad \text{if } \operatorname{dist}\bigl(x, \bigl\{u^\varepsilon \leqslant \varepsilon\bigr\}\bigr) \leqslant \rho.$$

Proof. Let $x_0 \in \Omega'$ such that $u^\varepsilon(x_0) > \varepsilon$ and let us call $d_0 = \operatorname{dist}\{x_0, \{u^\varepsilon \leqslant \varepsilon\}\}$ and $w(x) = \frac{1}{d_0}u^\varepsilon(x_0 + d_0x)$. Then, in $B_1(0)$,

$$\Delta w = d_0 f^\varepsilon(x_0 + d_0x), \qquad w(x) > \frac{\varepsilon}{d_0}.$$

Let $\psi \in C^\infty(\overline{B}_1)$ such that $\psi \equiv 0$ in $B_{1/4}$, $\psi \equiv 1$ in $\overline{B}_1 \setminus B_{1/2}$. Let $\Omega' \Subset \Omega'' \Subset \Omega$, $L \geqslant \|\nabla u^\varepsilon\|_{L^\infty(\Omega'')}$ and assume that $B_{d_0}(x_0) \subset \Omega''$. By Harnack inequality there exists a constant $\bar{c} > 0$ such that

$$w(x) \leqslant \bar{c}w(0) + C_0d_0 \quad \text{in } B_{1/2}$$

for a certain constant C_0 depending on $\mathcal{A}$. Let $\alpha > 0$ be such that $u^\varepsilon(x_0) = \alpha d_0$. With this notation me have $\alpha = w(0)$. We want to prove that there exist $c, \rho > 0$ such that

$$\alpha \geqslant c \quad \text{if } d_0 \leqslant \rho.$$

Let

$$z(x) = \begin{cases} \min(w(x), (\bar{c}\alpha + C_0d_0)\psi) & \text{in } B_{1/2}, \\ w(x) & \text{outside } B_{1/2}. \end{cases}$$

Then, $z \in H^1(B_1)$ and z coincides with w on ∂B_1 so that, since w is a local minimizer of the functional,

$$\tilde{J}(v) = \int\limits_{B_1} \left[\frac{1}{2}|\nabla v|^2 + B_{\varepsilon/d_0}(v) + d_0 f^\varepsilon(x_0 + d_0x)v\right] \mathrm{d}x,$$

there holds that $\tilde{J}(z) \geqslant \tilde{J}(w)$.

Let $\mathcal{D} = B_{1/2} \cap \{w > (\bar{c}\alpha + C_0 d_0)\psi\}$. Observe that $B_{1/4} \subset \mathcal{D}$ and $B_{\varepsilon/d_0}(w) = M$ in $B_{1/4}$ whereas $z = 0$ in $B_{1/4}$. Therefore,

$$\int_{\mathcal{D}} \left\{B_{\varepsilon/d_0}(w) - B_{\varepsilon/d_0}(z)\right\} \mathrm{d}x \geqslant M|B_{1/4}|.$$

Thus,

$$M|B_{1/4}| - \mathcal{A}d_0 \int_{\mathcal{D}} \left[w - (\bar{c}\alpha + C_0 d_0)\psi\right] \mathrm{d}x \leqslant \int_{\mathcal{D}} |\nabla\psi|^2 (\bar{c}\alpha + C_0 d_0)^2 \leqslant C(\bar{c}\alpha + C_0 d_0)^2.$$

So that,

$$M|B_{1/4}| - \mathcal{A}d_0|B_{1/2}|(\bar{c}\alpha + C_0 d_0) \leqslant C(\bar{c}\alpha + C_0 d_0)^2. \tag{5.20}$$

Now, since $u^\varepsilon(x_0) > \varepsilon$ there holds that $\alpha > \frac{\varepsilon}{d_0}$. Therefore, if $\frac{\varepsilon}{d_0} \geqslant 1$ there is nothing to prove. Thus, we may assume that $\frac{\varepsilon}{d_0} \leqslant 1$. Thus, since there is a point $\bar{x}$ on $\partial B_{d_0}(x_0)$ such that $u^\varepsilon(\bar{x}) = \varepsilon$,

$$\alpha = \frac{u^\varepsilon(x_0)}{d_0} \leqslant \frac{\varepsilon + L d_0}{d_0} = \frac{\varepsilon}{d_0} + L \leqslant 1 + L.$$

Going back to (5.20) we have for $d_0 \leqslant \rho \leqslant 1$,

$$0 < k \leqslant C(\bar{c}\alpha + C_0 d_0) \leqslant C\bar{c}\alpha + \frac{k}{2},$$

if ρ is small enough. Therefore, $\alpha \geqslant c > 0$ and the proposition is proved. □

Then, proceeding as in Corollary 5.4 we get,

Corollary 5.5. *Let $u = \lim u^{\varepsilon_j}$ with $\varepsilon_j \to 0$, where u^{ε_j} are minimizers of J_{ε_j} in the set of functions in $H^1(\Omega)$ that coincide with ϕ_{ε_j} on $\partial\Omega$ where $\|\phi_{\varepsilon_j}\|_{H^1(\Omega)} \leqslant C$ and $\|f^{\varepsilon_j}\|_{L^\infty(\Omega)} \leqslant \mathcal{A}$ with $C, \mathcal{A}$ independent of ε_j. Then, for every $\Omega' \Subset \Omega$, there exist positive constants c_0 and ρ depending only on N, $\mathcal{A}$, $\operatorname{dist}(\Omega', \partial\Omega)$ and the function β, such that if $x \in \Omega'$, then*

$$u^+(x) \geqslant c_0 \operatorname{dist}\bigl(x, \{u \leqslant 0\}\bigr) \quad \textit{if } \operatorname{dist}\bigl(x, \{u \leqslant 0\}\bigr) \leqslant \rho.$$

Finally we prove a result, which will be applied to our limit functions, that relates the nondegeneracy in the sense of (5.17) with the four concepts of nondegeneracy of Definition 5.1 (recall Remark 5.1).

Proposition 5.5. *Let u be a locally Lipschitz continuous function in a domain $\Omega \subset \mathbb{R}^N$ satisfying that $\Delta u \geqslant -C$ in Ω. Assume that u^+ is locally uniformly nondegenerate in the sense that* (5.17) *holds on every compact subset of Ω. Then u^+ is locally uniformly nondegenerate on $\Omega \cap \partial\{u > 0\}$ in the sense of* (5.4) *and consequently in the sense of* (5.1), (5.2) *and* (5.3).

Proof. The proof was done in Lemma 2.15 in [22] for the case in which $C = 2$. For arbitrary C we proceed exactly as in [22], considering in the proof the auxiliary subharmonic function $v(x) = u(x) + \frac{C|x - x_1|^2}{2N}$. □

6. Asymptotic development at regular free boundary points

In this section we consider $u = \lim u^\varepsilon$, with u^ε solutions to problems $E_\varepsilon(f^\varepsilon)$, and we prove that the free boundary condition,

$$\left|\nabla u^+\right|^2 - \left|\nabla u^-\right|^2 = 2M, \tag{6.1}$$

is satisfied in a pointwise sense at any point $x_0 \in \partial\{u > 0\}$ that has an inward unit normal in the measure theoretic sense (see Definition 6.1). The result holds if u^+ satisfies a nondegeneracy condition at the point (see Definition 5.1).

We remark that, as shown by the examples in Section 3 in [24], an assumption that guarantees nondegeneracy of u^+ is essential in order to get the free boundary condition (6.1). We refer to Section 5 for a discussion on conditions under which u^+ is nondegenerate at a free boundary point x_0.

A key tool in this section is the monotonicity formula of [8] (see Theorem 2.1).

Definition 6.1. We say that ν is the inward unit normal to the free boundary $\partial\{u>0\}$ at a point $x_0 \in \partial\{u>0\}$ in the measure theoretic sense, if $\nu \in \mathbb{R}^N$, $|\nu|=1$ and

$$\lim_{r\to 0}\frac{1}{r^N}\int\limits_{B_r(x_0)} |\chi_{\{u>0\}} - \chi_{\{x/\langle x-x_0,\nu\rangle>0\}}|\,\mathrm{d}x = 0. \tag{6.2}$$

Definition 6.2. We say that a point $x_0 \in \partial\{u>0\}$ is regular if there exists an inward unit normal to $\partial\{u>0\}$ at x_0 in the measure theoretic sense.

We will need the following lemma:

Lemma 6.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that u^{ε_j} converge to u uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u>0\}$ and let $u_\lambda(x) = \frac{1}{\lambda}u(x_0+\lambda x)$ for $\lambda>0$. There exists $\delta \geqslant 0$ such that if, for a sequence $\lambda_n \to 0$, $u_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^N$, then there holds*:

$$\Phi_U(r) := \left(\frac{1}{r^2}\int\limits_{B_r(0)}\frac{|\nabla U^+(x)|^2}{|x|^{N-2}}\,\mathrm{d}x\right)\left(\frac{1}{r^2}\int\limits_{B_r(0)}\frac{|\nabla U^-(x)|^2}{|x|^{N-2}}\,\mathrm{d}x\right) \equiv \delta,$$

for every $r>0$.

Proof. We will assume without loss of generality that $x_0=0$ and that $B_1(0) \Subset \Omega$. Since $\Delta u^+ \geqslant -\|f\|_{L^\infty}$ and $\Delta u^- \geqslant -\|f\|_{L^\infty}$ (recall Proposition 2.1), we can apply Theorem 2.1 with $u_1=u^+$ and $u_2=u^-$.

For $r>0$, let

$$\Phi_u(r) := \left(\frac{1}{r^2}\int\limits_{B_r(0)}\frac{|\nabla u^+(x)|^2}{|x|^{N-2}}\,\mathrm{d}x\right)\left(\frac{1}{r^2}\int\limits_{B_r(0)}\frac{|\nabla u^-(x)|^2}{|x|^{N-2}}\,\mathrm{d}x\right).$$

Since u^+ and u^- are locally Lipschitz continuous, Theorem 2.1 implies, in particular, that there exists

$$\delta := \lim_{r\searrow 0}\Phi_u(r).$$

Noticing that there holds:

$$\Phi_{u_\lambda}(r) = \Phi_u(\lambda r),$$

we deduce that there exists $\lim_{\lambda\searrow 0}\Phi_{u_\lambda}(r)$ and it coincides with δ, for every $r>0$.

Let now $\lambda_n \to 0$ be such that $u_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^N$, and let $r>0$ be fixed.

By Lemma 2.3 we know that $\nabla u_{\lambda_n} \to \nabla U$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. So that for a subsequence, that we still call λ_n, we have $\nabla u_{\lambda_n} \to \nabla U$ a.e. in $\mathbb{R}^N$. Also $|\nabla u_{\lambda_n}(x)| \leqslant L$ for $|x| < \frac{r_0}{\lambda_n}$, where L is the bound of $|\nabla u|$ in some $B_{r_0}(0)$.

Consequently, we may pass to the limit in the expression of $\Phi_{u_{\lambda_n}}(r)$ to conclude that

$$\Phi_{u_{\lambda_n}}(r) \to \left(\frac{1}{r^2}\int\limits_{B_r(0)}\frac{|\nabla U^+(x)|^2}{|x|^{N-2}}\,\mathrm{d}x\right)\left(\frac{1}{r^2}\int\limits_{B_r(0)}\frac{|\nabla U^-(x)|^2}{|x|^{N-2}}\,\mathrm{d}x\right).$$

So that the lemma is proved with $\delta = \lim_{r\searrow 0}\Phi_u(r)$ independent of the sequence λ_n. □

The main result in the section is

Theorem 6.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that u^{ε_j} converge to u uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be a regular point. Assume that u^+ is nondegenerate at x_0. Then, there exist $\alpha > 0$ and $\gamma \geqslant 0$ such that*

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ - \gamma\langle x - x_0, \nu\rangle^- + o\big(|x - x_0|\big),$$

with

$$\alpha^2 - \gamma^2 = 2M,$$

where ν is the inward unit normal to $\partial\{u > 0\}$ at x_0 in the measure theoretic sense.

Proof. We will assume, without loss of generality, that $x_0 = 0$ and $\nu = e_1$. Let

$$u_\lambda(x) = \frac{1}{\lambda}u(\lambda x),$$

and let $r > 0$ be such that $B_r(0) \Subset \Omega$. We have that u_λ is Lipschitz continuous in $B_{r/\lambda}(0)$ uniformly in λ, and $u_\lambda(0) = 0$. Therefore, for every $\lambda_n \to 0$, there exists a subsequence, that we still call λ_n, and a function U, Lipschitz continuous in $\mathbb{R}^N$, such that $u_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^N$.

By (6.2), it follows that for every $k > 0$,

$$\big|\{u_\lambda > 0\} \cap \{x_1 < 0\} \cap B_k(0)\big| \to 0 \quad \text{as } \lambda \to 0,$$

and

$$\big|\{u_\lambda \leqslant 0\} \cap \{x_1 > 0\} \cap B_k(0)\big| \to 0 \quad \text{as } \lambda \to 0.$$

It follows that U is nonnegative in $\{x_1 > 0\}$ and harmonic in $\{U > 0\}$ and that U is nonpositive in $\{x_1 < 0\}$ and harmonic in $\{U < 0\}$ (recall Lemma 2.2(v)). So that U is superharmonic in $\{x_1 < 0\}$. On the other hand, from Lemma 2.2(iv) we deduce that U is subharmonic in $\mathbb{R}^N$. Thus, U is harmonic in $\{x_1 < 0\}$ and necessarily

$$U(x) = -\gamma x_1 \quad \text{in } \{x_1 < 0\},$$

for some $\gamma \geqslant 0$.

On the other hand, since $\{U > 0\} \subset \{x_1 > 0\}$, by Lemma A.1 in [6], there exists $\alpha \geqslant 0$ such that

$$U(x) = \alpha x_1^+ + o\big(|x|\big) \quad \text{in } \{x_1 > 0\}. \tag{6.3}$$

The nondegeneracy assumption of u^+ at x_0 implies that necessarily $\alpha > 0$.

Let us now show that

$$\alpha^2 - \gamma^2 = 2M. \tag{6.4}$$

By Lemma 2.3 there exists a subsequence ε_{j_n} such that $\delta_n := \frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ and

$$u^{\delta_n}(x) := \frac{1}{\lambda_n}u^{\varepsilon_{j_n}}(\lambda_n x),$$

$$u^{\delta_n} \to U \quad \text{uniformly on compact sets of } \mathbb{R}^N.$$

Let $f^{\delta_n}(x) := \lambda_n f^{\varepsilon_{j_n}}(\lambda_n x)$. Then, $f^{\delta_n} \to 0$ uniformly on compact sets of $\mathbb{R}^N$ and u^{δ_n} is a solution to $E_{\delta_n}(f^{\delta_n})$.

Now let $U_\lambda(x) = \frac{1}{\lambda}U(\lambda x)$. Then for a sequence $\lambda_k \to 0$,

$$U_{\lambda_k} \to \alpha x_1^+ - \gamma x_1^-,$$

uniformly on compact subsets.

As before, there exists a subsequence δ_{n_k} such that $\bar\delta_k := \frac{\delta_{n_k}}{\lambda_k} \to 0$ and that $u^{\bar\delta_k}(x) := \frac{1}{\lambda_k}u^{\delta_{n_k}}(\lambda_k x)$ satisfies that

$$u^{\bar\delta_k} \to \alpha x_1^+ - \gamma x_1^+,$$

uniformly on compact subsets.

Since $u^{\bar{\delta}_k}$ are solutions to $E_{\bar{\delta}_k}(\bar{f}^{\bar{\delta}_k})$ with $\bar{f}^{\bar{\delta}_k} \to 0$ (they are rescalings of the functions $f^{\delta_{n_k}}$) uniformly on compact sets of $\mathbb{R}^N$, we may apply Proposition 3.1, if $\gamma > 0$, or Proposition 3.2, if $\gamma = 0$, and deduce that $\alpha^2 - \gamma^2 = 2M$.

Let us show that we actually have:

$$U(x) = \alpha x_1^+ - \gamma x_1^-. \tag{6.5}$$

In fact, $\partial\{U > 0\} \neq \emptyset$, $|\nabla U^-| \leqslant \gamma$ and thus by Theorem 4.2 we have $|\nabla U^+| \leqslant \sqrt{2M + \gamma^2} = \alpha$. Using that $U \equiv 0$ in $\{x_1 = 0\}$ we deduce that

$$U \leqslant \alpha x_1 \quad \text{in } \{x_1 > 0\}.$$

Since U is globally subharmonic and satisfies (6.3) the application of Hopf's Principle yields

$$U = \alpha x_1 \quad \text{in } \{x_1 > 0\},$$

which gives (6.5).

Finally we observe that, by Lemma 6.1, there exists $\delta \geqslant 0$ independent of the sequence λ_n such that

$$\delta \equiv \Phi_U(r) \equiv C_N \alpha^2 \gamma^2. \tag{6.6}$$

So that (6.5) holds with $\alpha > 0$, $\gamma \geqslant 0$ satisfying (6.4) and (6.6). In particular, α and γ are independent of the sequence λ_n. The theorem is proved. □

Remark 6.1. We point out that, from Section 3 in [24], we know that there are examples where u^+ degenerates at x_0, and such that the conclusion in Theorem 6.1 does not hold.

We recall that in Section 5 we gave conditions under which u^+ is nondegenerate at a free boundary point x_0.

7. Viscosity solutions to the free boundary problem

In this section we consider $u = \lim u^\varepsilon$, with u^ε solutions to problems $E_\varepsilon(f^\varepsilon)$, and we prove that, under suitable assumptions, u is a viscosity solution of the free boundary problem $E(f)$ (Corollaries 7.1 and 7.2).

First, we prove results on asymptotic developments at free boundary points in which there is a tangent ball contained either in $\{u > 0\}$ or in $\{u \leqslant 0\}^\circ$ (Theorems 7.1 and 7.2). The corollaries follow as an immediate consequence.

Some of these results hold if u^+ satisfies a suitable nondegeneracy condition (we refer to Section 5 for conditions implying the nondegeneracy of u^+).

Definition 7.1. Let Ω be a domain in $\mathbb{R}^N$. For any function u on Ω we define:

$$\Omega^+(u) := \Omega \cap \{u > 0\}, \tag{7.1}$$

$$\Omega^-(u) := \Omega \cap \{u \leqslant 0\}^\circ, \tag{7.2}$$

and

$$F(u) = \Omega \cap \partial\{u > 0\}. \tag{7.3}$$

Definition 7.2. Let u be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. We say that a point $x_0 \in F(u)$ is a regular point from the right if there is a tangent ball at x_0 from $\Omega^+(u)$ for (i.e., there is a ball $B \subset \{u > 0\}$, with $x_0 \in \partial B$).

Analogously, we say that a point $x_0 \in F(u)$ is a regular point from the left if there is a tangent ball at x_0 from $\Omega^-(u)$ (i.e., there is a ball $B \subset \{u \leqslant 0\}^\circ$, with $x_0 \in \partial B$).

Definition 7.3. Let u be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. Let $f \in L^\infty(\Omega)$. Then u is called a viscosity supersolution of $E(f)$ in Ω if:

(i) $\Delta u \leqslant f\chi_{\{u \not\equiv 0\}}$ in $\Omega^+(u)$.
(ii) $\Delta u \leqslant f\chi_{\{u \not\equiv 0\}}$ in $\Omega^-(u)$.

(iii) Along $F(u)$, u satisfies the condition,

$$\left(u_\nu^+\right)^2 - \left(u_\nu^-\right)^2 \leqslant 2M,$$

in the following weak sense. If $x_0 \in F(u)$ is a regular point from the right with touching ball B, and

$$u(x) \geqslant \alpha\langle x - x_0, \nu\rangle^+ + \mathrm{o}\big(|x - x_0|\big) \quad \text{in } B,$$

with $\alpha \geqslant 0$ and ν the inward unit normal to ∂B at x_0, then

$$u(x) < -\gamma\langle x - x_0, \nu\rangle^- + \mathrm{o}\big(|x - x_0|\big) \quad \text{in } B^c,$$

for any $\gamma \geqslant 0$ such that $\alpha^2 - \gamma^2 > 2M$.

Definition 7.4. Let u be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. Let $f \in L^\infty(\Omega)$. Then u is called a viscosity subsolution of $E(f)$ in Ω if:

(i) $\Delta u \geqslant f\chi_{\{u\not\equiv 0\}}$ in $\Omega^+(u)$.
(ii) $\Delta u \geqslant f\chi_{\{u\not\equiv 0\}}$ in $\Omega^-(u)$.
(iii) Along $F(u)$, u satisfies the condition,

$$\left(u_\nu^+\right)^2 - \left(u_\nu^-\right)^2 \geqslant 2M,$$

in the following weak sense. If $x_0 \in F(u)$ is a regular point from the left with touching ball B, and

$$u(x) \leqslant -\gamma\langle x - x_0, \nu\rangle^- + \mathrm{o}\big(|x - x_0|\big) \quad \text{in } B,$$

with $\gamma \geqslant 0$ and ν the outward unit normal to ∂B at x_0, then

$$u(x) > \alpha\langle x - x_0, \nu\rangle^+ + \mathrm{o}\big(|x - x_0|\big) \quad \text{in } B^c,$$

for any $\alpha \geqslant 0$ such that $\alpha^2 - \gamma^2 < 2M$.

Definition 7.5. We say that u is a viscosity solution of $E(f)$ in a domain $\Omega \subset \mathbb{R}^N$ if it is both a viscosity subsolution and a viscosity supersolution of $E(f)$ in Ω.

We first prove the following result on asymptotic developments at regular points from the right:

Theorem 7.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Let $x_0 \in F(u)$ be a regular point from the right with touching ball B. Let ν be the inward unit normal to ∂B at x_0. Then,*

(1) *If u^- degenerates at x_0, u has the following asymptotic development in the ball B:*

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ + \mathrm{o}\big(|x - x_0|\big) \quad \textit{in } B,$$

with $0 \leqslant \alpha \leqslant \sqrt{2M}$.
(2) *If u^- is nondegenerate at x_0, u has the following asymptotic development:*

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ - \gamma\langle x - x_0, \nu\rangle^- + \mathrm{o}\big(|x - x_0|\big),$$

with $\alpha^2 - \gamma^2 = 2M$, $\alpha > 0$, $\gamma > 0$.

Proof. By Lemma A.1 in [24] we know that $u(x) = \alpha\langle x - x_0, \nu\rangle + \mathrm{o}(|x - x_0|)$ in B with $\alpha \geqslant 0$.

Since $u^- \geqslant 0$, $\Delta u^- = -f$ in $\{u^- > 0\}$, we apply Lemma 2.1 and deduce that $\Delta u^- \geqslant -C$ in Ω. On the other hand, $u^- \equiv 0$ in the ball B. Thus, by Lemma A.2 in [24] there holds that

$$u^-(x) = \gamma\langle x - x_0, \nu\rangle^- + \mathrm{o}\big(|x - x_0|\big),$$

with $\gamma \geqslant 0$.

Now, if u^- is nondegenerate at x_0 we have $\gamma > 0$, and

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ - \gamma\langle x - x_0, \nu\rangle^- + o(|x - x_0|).$$

Let $u_\lambda(x) = \frac{1}{\lambda}u(x_0 + \lambda x)$. Then, for a subsequence, $u_{\lambda_n} \to u_0$ uniformly on compact subsets of $\mathbb{R}^N$ with $u_0(x) = \alpha\langle x, \nu\rangle^+ - \gamma\langle x, \nu\rangle^-$. Since u_0 is the limit of a sequence $u^{\delta_n} = (u^{\varepsilon_{j_n}})_{\lambda_n}$ of solutions to,

$$\Delta v = \beta_{\delta_n}(v) + f^{\delta_n},$$

with $f^{\delta_n} \to 0$ and $\delta_n \to 0$, by Proposition 3.1,

$$\alpha^2 - \gamma^2 = 2M.$$

This ends the proof in the case (2).

If u^- degenerates at x_0 we have $\gamma = 0$. If $\alpha = 0$ there is nothing to prove. So let us assume that $\alpha > 0$. Let us consider again a blow up limit u_0. We know that in this case $u_0 \geqslant 0$ in $\mathbb{R}^N$ and $u_0 \equiv 0$ on the hyperplane $\langle x, \nu\rangle = 0$. Let us consider the function $v = u_0\chi_H$ where H is the half-space $\langle x, \nu\rangle < 0$. There holds that its positivity set is contained in H, $\Delta v = 0$ in $\{v > 0\}$ and v is Lipschitz continuous in $\mathbb{R}^N$. Applying Lemma A.1 of [6] we find that

$$u_0(x) = \bar{\alpha}\langle x, \nu\rangle^- + o(|x|) \quad \text{in } \langle x, \nu\rangle < 0,$$

with $\bar{\alpha} \geqslant 0$. Thus,

$$u_0(x) = \alpha\langle x, \nu\rangle^+ + \bar{\alpha}\langle x, \nu\rangle^- + o(|x|).$$

We take a new blow up limit $u_{00} = \lim (u_0)_{\lambda_k}$. There holds that $u_{00} = \lim u^{\tilde{\delta}_k}$ with $u^{\tilde{\delta}_k}$ with the same properties as u^{δ_n} above.

On the other hand,

$$u_{00}(x) = \alpha\langle x, \nu\rangle^+ + \bar{\alpha}\langle x, \nu\rangle^-.$$

Thus, we deduce from Proposition 3.2 (if $\bar{\alpha} = 0$) or Proposition 3.3 (if $\bar{\alpha} > 0$) that

$$0 \leqslant \alpha \leqslant \sqrt{2M}.$$

The theorem is proved. □

Next, we prove a result on asymptotic developments at regular points from the left.

Theorem 7.2. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ *-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Let $x_0 \in F(u)$ be a regular point from the left with touching ball B and assume that u^+ is nondegenerate at x_0. Then, there exist $\alpha > 0$ and $\gamma \geqslant 0$ such that the following asymptotic development holds*:

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ - \gamma\langle x - x_0, \nu\rangle^- + o(|x - x_0|),$$

with

$$\alpha^2 - \gamma^2 = 2M,$$

where ν is the outward unit normal to ∂B at x_0.

Proof. Since $u^+ \geqslant 0$, $\Delta u^+ = f$ in $\{u^+ > 0\}$, there holds by Lemma 2.1 that $\Delta u^+ \geqslant -C$ in Ω. On the other hand, $u^+ \equiv 0$ in B. Thus, by Lemma A.2 in [24],

$$u^+(x) = \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|),$$

with $\alpha \geqslant 0$. Since u^+ is nondegenerate at x_0, there holds that $\alpha > 0$.

Let us consider a blow-up limit u_0. Since $\alpha > 0$ and $u \leqslant 0$ in B,

$$u_0^+(x) = \alpha\langle x, \nu\rangle^+,$$

and in $\langle x,\nu\rangle<0$, $u_0\leqslant 0$ and $\Delta u_0=0$. So that necessarily,

$$u_0(x)=-\gamma\langle x,\nu\rangle^- \quad \text{in } \langle x,\nu\rangle<0$$

with $\gamma\geqslant 0$.

Summing up,

$$u_0(x)=\alpha\langle x,\nu\rangle^+-\gamma\langle x,\nu\rangle^-.$$

As in Theorem 7.1, we use that $u_0=\lim u^{\delta_n}$ with u^{δ_n} solutions to problems $E_{\delta_n}(f^{\delta_n})$ with f^{δ_n} converging to 0 and $\delta_n\to 0$, and deduce, by applying Proposition 3.1 if $\gamma>0$ or Proposition 3.2 if $\gamma=0$ that

$$\alpha^2-\gamma^2=2M. \tag{7.4}$$

Since α is independent of the blow-up sequence, (7.4) gives that also γ is independent of the blow-up sequence. Therefore,

$$u(x)=\alpha\langle x-x_0,\nu\rangle^+-\gamma\langle x-x_0,\nu\rangle^-+o\big(|x-x_0|\big).$$

The theorem is proved. □

As a corollary we obtain:

Corollary 7.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega\subset\mathbb{R}^N$ such that $u^{\varepsilon_j}\to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j}\to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j\to 0$. Then u is a viscosity supersolution to $E(f)$ in Ω.*

Proof. The proof follows immediately from Theorem 7.1. □

We also obtain:

Corollary 7.2. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega\subset\mathbb{R}^N$ such that $u^{\varepsilon_j}\to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j}\to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j\to 0$. Assume that u^+ is nondegenerate at every regular point from the left in $F(u)$. Then u is a viscosity subsolution to $E(f)$ in Ω.*

Proof. The proof follows immediately from Theorem 7.2. □

Remark 7.1. We point out that, from Section 3 in [24], we know that there are examples where u^+ degenerates at x_0, and such that the conclusions in Theorem 7.2 and Corollary 7.2 do not hold.

Remark 7.2. Let $u=\alpha x_1^+ + \alpha x_1^-$ be as in Remark 3.1. Then, from Corollaries 7.1 and 7.2 it follows that u is a viscosity solution to $E(f)$ in $\mathbb{R}^N$ (with $f=0$).

Remark 7.3. We have chosen to work with the notion of viscosity solution introduced in this section, because it is a natural extension to the inhomogeneous problem of the notion of weak solution introduced in [6] for the homogeneous problem. Notice that the results in this section also hold replacing Definitions 7.3, 7.4 and 7.5 by Definition 4.4 in [8].

8. Uniformly nondegenerate limit functions

In this section we analyze the behavior of limit functions which satisfy the additional hypothesis that u^+ is uniformly nondegenerate on $\Omega\cap\partial\{u>0\}$ (we refer to Section 5 for conditions implying the uniform nondegeneracy of u^+).

Remark 8.1. Let u be a continuous function in a domain $\Omega\subset\mathbb{R}^N$. If we have $\mathcal{H}^{N-1}(\Omega\cap\partial\{u>0\})<\infty$, then $\{u>0\}$ is a set of finite perimeter in Ω (see [18]). In this situation we will call, as usual, reduced boundary (and denote $\partial_{\mathrm{red}}\{u>0\}$), the subset of points in $\partial\{u>0\}$ which have an inward unit normal in the measure theoretic sense (see Definition 6.1).

We will next prove a representation formula for u which holds when u^+ is locally uniformly nondegenerate. We will denote by $\mathcal{H}^{N-1}\lfloor\partial\{u>0\}$ the measure $\mathcal{H}^{N-1}$ restricted to the set $\partial\{u>0\}$.

Theorem 8.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega\subset\mathbb{R}^N$ such that u^{ε_j} converge to a function u uniformly on compact subsets of Ω, $f^{\varepsilon_j}\to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j\to 0$. Let us assume that u^+ is locally uniformly nondegenerate on $\Omega\cap\partial\{u>0\}$ in the sense of* (5.1). *Then,*

(1) $\mathcal{H}^{N-1}(\Omega'\cap\partial\{u>0\})<\infty$, *for every* $\Omega'\Subset\Omega$.
(2) *There exist Borelian functions q_u^+ and q_u^- defined on $\Omega\cap\partial\{u>0\}$ such that*

$$\Delta u^+ - f\chi_{\{u>0\}} = q_u^+\mathcal{H}^{N-1}\lfloor\partial\{u>0\},$$
$$\Delta u^- + f\chi_{\{u<0\}} = q_u^-\mathcal{H}^{N-1}\lfloor\partial\{u>0\},$$

and thus,

$$\Delta u - f\chi_{\{u\neq 0\}} = \big(q_u^+ - q_u^-\big)\mathcal{H}^{N-1}\lfloor\partial\{u>0\}. \tag{8.1}$$

(3) *For every $\Omega'\Subset\Omega$ there exist $C>0$, $c>0$ and $r_1>0$ such that*

$$cr^{N-1}\leqslant\mathcal{H}^{N-1}\big(B_r(x_0)\cap\partial\{u>0\}\big)\leqslant Cr^{N-1}$$

for every $x_0\in\Omega'\cap\partial\{u>0\}$, $0<r<r_1$ and, in addition,
(4) $0<c\leqslant q_u^+\leqslant C$ *and* $0\leqslant q_u^-\leqslant C$ *in* $\Omega'\cap\partial\{u>0\}$, $q_u^-=0$ *in* $\partial\{u>0\}\setminus\partial\{u<0\}$.
(5) *u has the following asymptotic development at $\mathcal{H}^{N-1}$-almost every point x_0 in $\partial_{\rm red}\{u>0\}$ (this is, at $\mathcal{H}^{N-1}$-almost every point x_0 such that $\partial\{u>0\}$ has an inward unit normal ν in the measure theoretic sense),*

$$u(x) = q_u^+(x_0)\langle x-x_0,\nu\rangle^+ - q_u^-(x_0)\langle x-x_0,\nu\rangle^- + o\big(|x-x_0|\big).$$

Proof. From Proposition 2.1 we know that

$$\Delta u^+ - f\chi_{\{u>0\}} = \lambda_u^+,\qquad \Delta u^- + f\chi_{\{u<0\}} = \lambda_u^-,$$

with λ_u^+ and λ_u^+ nonnegative Radon measures supported on $\Omega\cap\partial\{u>0\}$.

On the other hand, since u is locally Lipschitz, for every $\Omega'\Subset\Omega$ there exist $C>0$ and $r_1>0$ such that

$$\fint_{\partial B_r(x)} u^+\leqslant Cr,$$

for any $x\in\Omega'\cap\partial\{u>0\}$, $0<r\leqslant r_1$. Therefore, for some $c>0$ and $r_2>0$,

$$c\leqslant\frac{1}{r}\fint_{\partial B_r(x)} u^+\leqslant C \tag{8.2}$$

for any $x\in\Omega'\cap\partial\{u>0\}$, $0<r\leqslant r_2$.

A suitable modification in the proof of Theorem 4.3 in [1] shows that there exist $c, C>0$ and $r_3>0$ such that

$$cr^{N-1}\leqslant\int_{B_r(x)} d\lambda_u^+\leqslant Cr^{N-1}, \tag{8.3}$$

for any $x\in\Omega'\cap\partial\{u>0\}$, $0<r\leqslant r_3$. In fact, we get for almost all $r<r_3$,

$$\int_{B_r(x)} d\lambda_u^+ = \int_{\partial B_r(x)}\nabla u^+.\nu\, d\mathcal{H}^{N-1} - \int_{B_r(x)} f\chi_{\{u>0\}}\leqslant Cr^{N-1},$$

which proves the second inequality in (8.3).

In order to obtain the first inequality in (8.3), we proceed as in Theorem 4.3 in [1], working with our measure λ_u^+ instead of the measure λ appearing there. In our case we need to use that if $x\in\partial\{u>0\}$, y is such that $u^+(y)>0$, with $|x-y|=\kappa r$, $0<\kappa<1$ and G_y the positive Green function for the Laplacian in $B_r(x)$ with pole y, there holds that

$$\int_{B_r(x)} G_y\,\mathrm{d}\lambda_u^+ = -u^+(y) + \int_{\partial B_r(x)} u^+\partial_{-\nu}G_y\,\mathrm{d}\mathcal{H}^{N-1} - \int_{B_r(x)} G_y f\chi_{\{u>0\}},$$

where we have the following bound for the last term:

$$\left|\int_{B_r(x)} G_y f\chi_{\{u>0\}}\right| \leqslant C(\kappa)\|f\|_{L^\infty} r^2.$$

Thus we get the first inequality in (8.3).

Then, arguing exactly as in Theorem 4.5 in [1] we deduce that (1) in the statement holds and that there exists a Borelian function q_u^+ defined on $\Omega\cap\partial\{u>0\}$ such that

$$\Delta u^+ - f\chi_{\{u>0\}} = q_u^+\mathcal{H}^{N-1}\lfloor\partial\{u>0\}.$$

Also we deduce as in [1] that (3) in the statement as well as estimate $0<c\leqslant q_u^+\leqslant C$ in $\Omega'\cap\partial\{u>0\}$ hold.

On the other hand, the same argument employed above and the fact that u^- is locally Lipschitz show that there exist $C>0$ and $r_4>0$ such that

$$\int_{B_r(x)} \mathrm{d}\lambda_u^- \leqslant Cr^{N-1},$$

for any $x\in\Omega'\cap\partial\{u>0\}$, $0<r\leqslant r_4$.

This implies (see Remark 4.6 in [1]) that there exists a Borelian function q_u^- defined on $\Omega\cap\partial\{u>0\}$ such that

$$\Delta u^- + f\chi_{\{u<0\}} = q_u^-\mathcal{H}^{N-1}\lfloor\partial\{u>0\},$$

with $0\leqslant q_u^-\leqslant C$ holding in $\Omega'\cap\partial\{u>0\}$. Since $u^-=0$ in a neighborhood of every point in $\partial\{u>0\}\setminus\partial\{u<0\}$, there holds that $q_u^-=0$ in $\partial\{u>0\}\setminus\partial\{u<0\}$. Thus, (4) follows.

In order to prove (5) in the statement we first apply similar arguments as those in Theorem 4.8 and Remark 4.9 in [1] to the function u^+ to deduce that

$$u^+(x) = q_u^+(x_0)\langle x-x_0,\nu\rangle^+ + \mathrm{o}\big(|x-x_0|\big),$$

for $\mathcal{H}^{N-1}$-almost every x_0 in $\partial_{\mathrm{red}}\{u>0\}$. We need to use here that any blow-up limit u_0 satisfies that $\Delta u_0=0$ in $\{u_0>0\}=\{\langle x,\nu\rangle>0\}$.

Proceeding again as Theorem 4.8 and Remark 4.9 in [1] now with the function u^- we can also deduce that

$$u^-(x) = q_u^-(x_0)\langle x-x_0,\nu\rangle^- + \mathrm{o}\big(|x-x_0|\big),$$

for $\mathcal{H}^{N-1}$-almost every x_0 in $\partial_{\mathrm{red}}\{u>0\}$. In this case we need to use that any blow-up limit u_0 satisfies that $\Delta u_0=0$ in $\{u_0\leqslant 0\}^\circ=\{\langle x,\nu\rangle<0\}$.

Then (5) follows and the theorem is proved. □

Remark 8.2. Notice that we had already shown in Proposition 2.1 that there holds, for any general limit u, that $\Delta u - f\chi_{\{u\not\equiv 0\}} = \Lambda$, with Λ a Radon measure supported on $\Omega\cap\partial\{u>0\}$. In Theorem 8.1 we characterize Λ in the particular case that u^+ is locally uniformly nondegenerate on $\Omega\cap\partial\{u>0\}$.

On the other hand, under the assumptions of Theorem 8.1, we have that Theorem 6.1 applies at every point x_0 in the reduced boundary. Therefore, the constants α and γ in Theorem 6.1 verify that $\alpha=q_u^+(x_0)$ and $\gamma=q_u^-(x_0)$ where q_u^+ and q_u^- are the Borelian functions in (2) in Theorem 8.1. In particular, $(q_u^+(x_0))^2-(q_u^-(x_0))^2=2M$ and thus, the function $q_u^+-q_u^-$ appearing in (8.1) is strictly positive at $\mathcal{H}^{N-1}$-almost every point on $\partial_{\mathrm{red}}\{u>0\}$.

9. Regularity of the free boundary

In this section we study the regularity of the free boundary $\Omega \cap \partial\{u > 0\}$. We recall that there are examples where u^+ degenerates at the free boundary as well as examples where there is no portion of $\{u \leqslant 0\}^\circ$ at the free boundary (like $u = \alpha x_1^+ + \alpha x_1^-$, $\alpha > 0$, see Remark 7.2). Thus, in order to prove that a limit function is a classical solution to $E(f)$ these situations need to be ruled out.

We here prove that, under suitable assumptions, there is a subset of the free boundary which is locally a $C^{1,\alpha}$ surface and u is a classical solution to the free boundary problem $E(f)$ there.

We refer to Remark 9.7 for a discussion on the different results obtained.

We first obtain, for nonnegative limit functions, the following result on the regularity of the free boundary

Theorem 9.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Assume that $u \geqslant 0$ in Ω,*

(i) *u is locally uniformly nondegenerate on $\Omega \cap \partial\{u > 0\}$ in the sense of* (5.2),
(ii) $\limsup_{r\to 0} \frac{|B_r(\bar{x}) \cap \{u \equiv 0\}|}{|B_r(\bar{x})|} > 0$ *at $\mathcal{H}^{N-1}$-almost every $\bar{x} \in \Omega \cap \partial\{u > 0\}$.*

Then, there is a subset $\mathcal{R}$ of the free boundary $\Omega \cap \partial\{u > 0\}$ ($\mathcal{R} = \partial_{\mathrm{red}}\{u > 0\}$) which is locally a $C^{1,\alpha}$ surface and u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of $\mathcal{R}$. Moreover, $\mathcal{R}$ is open and dense in $\Omega \cap \partial\{u > 0\}$ and the remainder of the free boundary has $(N-1)$-dimensional Hausdorff measure zero.

Proof. Let us first observe that, since Theorem 8.1 applies (we need to argue as in Remark 5.1), the free boundary $\Omega \cap \partial\{u > 0\}$ has locally finite $(N-1)$-dimensional Hausdorff measure and therefore, $\{u > 0\}$ has locally finite perimeter in Ω.

On the other hand, we observe that under our hypotheses, we have for $\mathcal{H}^{N-1}$-almost every point $\bar{x} \in \Omega \cap \partial\{u > 0\}$,

$$\limsup_{r\to 0} \frac{|B_r(\bar{x}) \cap \{u > 0\}|}{|B_r(\bar{x})|} > 0, \qquad \limsup_{r\to 0} \frac{|B_r(\bar{x}) \cap \{u \equiv 0\}|}{|B_r(\bar{x})|} > 0,$$

and therefore, Lemma 1 in [17], Section 5.8, gives that $\mathcal{H}^{N-1}$-almost all $\bar{x} \in \Omega \cap \partial\{u > 0\}$ is in the reduced boundary.

It follows from Theorem 8.1 and Remark 8.2 that u is a nonnegative function satisfying:

$$\Delta u - f\chi_{\{u>0\}} = \sqrt{2M}\mathcal{H}^{N-1} \lfloor \partial_{\mathrm{red}}\{u > 0\}.$$

In addition, u is locally Lipschitz continuous and satisfies (8.2) locally on the free boundary.

Under these assumptions, but with $f \equiv 0$, it was shown in [1] that $\partial_{\mathrm{red}}\{u > 0\}$ is locally a $C^{1,\alpha}$ surface. When $f \in L^\infty$, the proofs in [1] can be modified as done in [20] and [22] and the same conclusion holds.

Finally, Theorem 8.1(3) implies that the reduced boundary is dense in $\Omega \cap \partial\{u > 0\}$. Thus, the theorem is proved. □

Remark 9.1. Putting together the results in Corollaries 7.1 and 7.2, we will derive other results on the regularity of the free boundary for limit functions, when we do not impose that the limit functions be nonnegative. In fact, results of this kind were obtained in [23], for the particular case that $f^\varepsilon \equiv 0$, with the aid the regularity results in [5] and [6]—which apply to the homogeneous version of our problem. The extension of some results in [5] and [6] for the inhomogeneous problem is carried out in [9].

Before obtaining regularity results for two-phase limits we need a preliminary result.

Proposition 9.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Let $x_0 \in \Omega \cap \partial\{u > 0\}$, and let $\lambda_n > 0$ be a sequence such that $\lambda_n \to 0$. Consider the functions $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ and assume that $u_{\lambda_n} \to U$ as $n \to \infty$ uniformly on compact sets of $\mathbb{R}^N$. If u^- is nondegenerate at x_0 in the sense of* (5.2), *then*

$$U(x) = \alpha\langle x, \nu\rangle^+ - \gamma\langle x, \nu\rangle^- \quad \text{in } \mathbb{R}^N,$$

where ν is a unit vector, and α, γ are positive constants satisfying $\alpha^2 - \gamma^2 = 2M$.

Proof. Let us consider, for $r > 0$,

$$\Phi_U(r) := \left(\frac{1}{r^2}\int\limits_{B_r(0)} \frac{|\nabla U^+(x)|^2}{|x|^{N-2}}\,\mathrm{d}x\right)\left(\frac{1}{r^2}\int\limits_{B_r(0)} \frac{|\nabla U^-(x)|^2}{|x|^{N-2}}\,\mathrm{d}x\right).$$

From Lemma 6.1 it follows that there exists $\delta \geqslant 0$ independent of the sequence λ_n such that

$$\Phi_U(r) \equiv \delta \quad \text{for } r > 0. \tag{9.1}$$

Let us see that we necessarily have $\delta > 0$. In fact, assume that

$$\frac{1}{r^2}\int\limits_{B_r(0)} \frac{|\nabla U^-(x)|^2}{|x|^{N-2}}\,\mathrm{d}x = 0$$

for some $r > 0$. Then, $U^- \equiv 0$ in $B_r(0)$ and therefore,

$$0 = \lim_{n\to\infty} \frac{1}{r} \fint\limits_{B_r(0)} u_{\lambda_n}^- = \lim_{n\to\infty} \frac{1}{\lambda_n r} \fint\limits_{B_{\lambda_n r}(x_0)} u^-,$$

which contradicts the nondegeneracy of u^- at x_0 in the sense of (5.2). Since also u^+ is nondegenerate at x_0 in the same sense (recall Proposition 5.1), we proceed analogously with U^+.

That is, we have shown that (9.1) holds with $\delta > 0$.

We will now deduce that

$$U(x) = \alpha\langle x, \nu\rangle^+ - \gamma\langle x, \nu\rangle^- \quad \text{in } \mathbb{R}^N,$$

with $\alpha > 0$, $\gamma > 0$ and ν a unit vector.

In fact, this follows from the application of the monotonicity formula in [2] to the functions U^+ and U^-, which are harmonic where positive and satisfy (9.1) with $\delta \neq 0$ (see [2], Lemmas 5.1, 6.6 and Remark 6.1, and [4]).

Now, as done in previous results, we use that $U = \lim u^{\delta_n}$ with u^{δ_n} solutions to problems $E_{\delta_n}(f^{\delta_n})$ with f^{δ_n} converging to 0 and $\delta_n \to 0$ and deduce, by applying Proposition 3.1, that $\alpha^2 - \gamma^2 = 2M$. The proof is complete. □

Next, we obtain the following results for general two-phase limits.

Theorem 9.2. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ *-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Assume that u^+ is locally uniformly nondegenerate in the sense that* (5.17) *holds on every compact subset of Ω. If $x_0 \in \Omega \cap \partial\{u > 0\}$ is such that $\partial\{u > 0\}$ has at x_0 an inward unit normal in the measure theoretic sense then, the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0. Moreover, u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of x_0.*

Proof. From Corollaries 7.1 and 7.2 we deduce that u is a viscosity solution to $E(f)$ in Ω (in order to apply Corollary 7.2 we use Proposition 5.5).

On the other hand, since the free boundary has at x_0 an inward unit normal ν in the measure theoretic sense we can apply Theorem 6.1 to deduce that

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ - \gamma\langle x - x_0, \nu\rangle^- + o\big(|x - x_0|\big),$$

with $\alpha^2 - \gamma^2 = 2M$, $\alpha > 0$, $\gamma \geqslant 0$.

Then, given $\lambda_n \to 0$, the sequence $u_{\lambda_n}(x) = \frac{1}{\lambda_n}u(x_0 + \lambda_n x)$ converges uniformly on compact sets of $\mathbb{R}^N$ to $u_0(x) = \alpha\langle x, \nu\rangle^+ - \gamma\langle x, \nu\rangle^-$.

It is not hard to see that for any $\varepsilon > 0$ small, there holds that

$$u_{\lambda_n} > 0 \quad \text{in } B_1(0) \cap \big\{\langle x, \nu\rangle > \varepsilon\big\}, \tag{9.2}$$

$$u_{\lambda_n} \leqslant 0 \quad \text{in } B_1(0) \cap \big\{\langle x, \nu\rangle < -\varepsilon\big\}, \tag{9.3}$$

if n is large enough. Indeed, (9.2) follows easily and the same happens with (9.3) in case $\gamma > 0$. In case $\gamma = 0$, (9.3) follows from the nondegeneracy of u^+.

Therefore, if $f \equiv 0$, u falls under the hypotheses of Theorem 2′ in [6] for small balls around x_0. This eventually implies that $\partial\{u>0\}$ is a $C^{1,\alpha}$ surface in a neighborhood of x_0.

If $f \not\equiv 0$ the same conclusion follows from the application of the results in [9]. □

Remark 9.2. We point out that in the proof of Theorem 9.2 we only use the fact that the free boundary has at x_0 a normal in the measure theoretic sense to deduce, for a certain blow-up limit $u_0(x) = \lim \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$ with $\lambda_n \to 0$, that $u_0(x) = \alpha \langle x, \nu\rangle^+ - \gamma \langle x, \nu\rangle^-$ for some unit vector ν, with $\alpha > 0$ and $\gamma \geqslant 0$.

Theorem 9.3. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Assume that u^+ is locally uniformly nondegenerate in the sense that* (5.17) *holds on every compact subset of Ω. If u^- is nondegenerate at $x_0 \in \Omega \cap \partial\{u>0\}$ in the sense of* (5.2) *then, the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0. Moreover, u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of x_0.*

Proof. We argue in a similar way as in Theorem 9.2, but we apply in the present situation Proposition 9.1 instead of Theorem 6.1. □

As a consequence we obtain:

Corollary 9.1. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Assume that u^+ is locally uniformly nondegenerate in the sense that* (5.17) *holds on every compact subset of Ω. Let $x_0 \in \Omega \cap \partial\{u>0\}$ and let*:

$$\delta_C(x_0) := \lim_{r\to 0}\left(\frac{1}{r^2}\int\limits_{B_r(x_0)} \frac{|\nabla u^+(x)|^2}{|x-x_0|^{N-2}}\,\mathrm{d}x\right)\left(\frac{1}{r^2}\int\limits_{B_r(x_0)} \frac{|\nabla u^-(x)|^2}{|x-x_0|^{N-2}}\,\mathrm{d}x\right).$$

If $\delta_C(x_0) > 0$ then, the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0. Moreover, u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of x_0.

Proof. It follows from Lemma 6.1 that if $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x) \to U(x)$ with $\lambda_n \to 0$ then,

$$\delta_C(x_0) = \left(\frac{1}{r^2}\int\limits_{B_r(0)} \frac{|\nabla U^+(x)|^2}{|x|^{N-2}}\,\mathrm{d}x\right)\left(\frac{1}{r^2}\int\limits_{B_r(0)} \frac{|\nabla U^-(x)|^2}{|x|^{N-2}}\,\mathrm{d}x\right)$$

for every $r > 0$. Therefore, since $\delta_C(x_0) > 0$, there holds that u^- is nondegenerate at x_0 and Theorem 9.3 applies. □

Remark 9.3. We point out that there are strictly two phase limits u, which are classical solutions to the free boundary problem $E(f)$ in a neighborhood of $x_0 \in \Omega \cap \partial\{u>0\}$, for which $\delta_C(x_0) = 0$. This is possible when $f \not\equiv 0$.

We also obtain the next regularity result which is new even in the case that $f^\varepsilon \equiv 0$.

Theorem 9.4. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Assume that u^+ is locally uniformly nondegenerate in the sense that* (5.17) *holds on every compact subset of Ω. If $x_0 \in \Omega \cap \partial\{u>0\}$ is a regular point from the left then, the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0. Moreover, u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of x_0.*

Proof. We obtain the result as a consequence of Theorem 7.2, arguing once more in a similar way as in Theorem 9.2. □

Remark 9.4. The previous result proves that, under suitable nondegeneracy assumptions, limit functions are classical solutions to problem $E(f)$ in a neighborhood of a free boundary point x_0 which is regular from the left. We point out that such a result is not true if we have instead that the point x_0 is regular from the right (see Remark 3.1).

Some extra assumption at the point x_0 is needed if one wants to get a result of this kind. We achieve this purpose in the next two results. The key tool used to obtain them is the local monotonicity formula proven by the authors in [25].

From the results in [25] it follows that if u^{ε_j} are solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in $B_R(x_0)$, $u = \lim u^{\varepsilon_j}$ uniformly on $B_R(x_0)$, $f = \lim f^{\varepsilon_j}$ $*$-weakly in $L^\infty(B_R(x_0))$, $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$-weakly in $L^\infty(B_R(x_0))$, $\varepsilon_j \to 0$ and $x_0 \in \partial\{u > 0\}$, then there exists $\delta_W(x_0) \in \mathbb{R}$ such that

$$\delta_W(x_0) = \lim_{r\to 0^+} \frac{1}{r^2} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^N} \left(\left|\nabla(u\psi)\right|^2 + 2\psi^2\chi + \frac{1}{2}\frac{(u\psi)^2}{t} \right) G(x - x_0, -t)\,\mathrm{d}x\,\mathrm{d}t, \tag{9.4}$$

where $G(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp(-\frac{|x|^2}{4t})$ and ψ is any function satisfying that $\psi \in C_0^\infty(B_R(x_0))$, $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ in $B_{R/2}(x_0)$.

Moreover, the results in [25] show that $0 \leqslant \delta_W(x_0) \leqslant 6M$ (see also (9.5) below).

When $\psi \equiv 1$, $\alpha > 0$ and $u = \alpha x_1^+$, there holds that $\delta_W(0) = 3M$ (see the proof of Theorem 9.7); and when $\psi \equiv 1$, $\alpha > 0$ and $u = \alpha|x_1|$ there holds that $\delta_W(0) = 6M$. These values, $3M$ and $6M$, play a major role in the next theorems.

The next two theorems, which are new even when $f^\varepsilon \equiv 0$, deal with the case of a point that is regular from the right.

Theorem 9.5. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Assume that u^+ is locally uniformly nondegenerate in the sense that* (5.17) *holds on every compact subset of Ω. If $x_0 \in \Omega \cap \partial\{u > 0\}$ is a regular point from the right and $\delta_W(x_0) < 6M$ ($\delta_W(x_0)$ as in* (9.4)*) then, the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0. Moreover, u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of x_0.*

If $N = 2$ the same result holds without assuming that x_0 is a regular point from the right.

Proof. Assume x_0 is a regular point from the right. Let $\lambda_n \to 0$ such that there exist $u_0 = \lim u_{\lambda_n}$ uniformly on compact sets of $\mathbb{R}^N$ and $\chi_0 = \lim \chi^{\lambda_n}$ $*$-weakly in $L^\infty_{\text{loc}}(\mathbb{R}^N)$, where $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$, $\chi^{\lambda_n}(x) = \chi(x_0 + \lambda_n x)$ and $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$-weakly in $L^\infty(\Omega)$. Then, as proved in [25],

$$\delta_W(x_0) = \int_{-4}^{-1} \int_{\mathbb{R}^N} 2\chi_0 G(x, -t)\,\mathrm{d}x\,\mathrm{d}t. \tag{9.5}$$

Since $0 \leqslant \chi_0 \leqslant M$, our hypothesis on $\delta_W(x_0)$ implies that $|\{\chi_0 < M\}| > 0$. On the other hand, since there exists a ball $B \subset \{u > 0\}$ tangent to $\partial\{u > 0\}$ at x_0, there holds that $\chi \equiv M$ in B. (Here we choose a coordinate system such that $x_0 = 0$ and e_1 is the direction from x_0 to the center of the ball B.) Thus, $\chi_0 \equiv M$ in $\{x_1 > 0\}$.

By Theorem 3.1 in [25], there exist $\alpha > 0$ and $\sigma \in \mathbb{R}$ such that

$$u(x) = \alpha x_1^+ + \sigma x_1^- + o\big(|x|\big) \tag{9.6}$$

with one of the following situations:

(1) $\sigma \leqslant 0$ and $\alpha^2 - \sigma^2 = 2M$,
(2) $\sigma = \alpha > 0$.

But, if $\sigma > 0$ there holds that $\chi_0 \equiv M$ in $\{x_1 < 0\}$. Since this is a contradiction, there holds that

$$u(x) = \alpha x_1^+ - \gamma x_1^- + o\big(|x|\big) \quad \text{with } \gamma \geqslant 0 \text{ and } \alpha^2 - \gamma^2 = 2M.$$

Thus, we are again in a situation in which we can apply Corollaries 7.1 and 7.2 and the results in [9] to deduce that $\partial\{u > 0\}$ is $C^{1,\alpha}$ in a neighborhood of x_0.

Now assume that $N = 2$ and let x_0 be any free boundary point. If u^- is nondegenerate at x_0, the result follows from the application of Theorem 9.3. So let us assume that u^- degenerates at x_0. Then, in this case, the sequence λ_n above can be chosen so that

$$\frac{1}{\lambda_n} \fint_{B_{\lambda_n}(x_0)} u^- \to 0, \quad \text{as } n \to \infty. \tag{9.7}$$

From Corollary 2.1 in [25] it follows that

$$u_0(rx) = ru_0(x) \quad \text{for } r > 0,\ x \in \mathbb{R}^N. \tag{9.8}$$

We also observe that (9.7) together with (9.8) implies that $u_0 \geqslant 0$ in $\mathbb{R}^N$. The nondegeneracy assumption on u^+ implies that $u_0 \not\equiv 0$.

Now consider $\mathcal{A}$ a connected component of $\{u_0 > 0\}$. Then, (9.8) gives that, in some system of coordinates, either $\mathcal{A} \subset \{x_1 > 0\}$ or else $\{x_1 > 0\} \subset \mathcal{A}$. In the first case, Lemma A1 in [6] implies that $u_0(x) = \alpha x_1^+ + o(|x|)$ in $\{x_1 > 0\}$, with $\alpha \geqslant 0$ and then (9.8) yields:

$$u_0(x) = \alpha x_1^+ \quad \text{in } \{x_1 > 0\} \text{ and } \alpha > 0.$$

Now, with a similar analysis in $\{x_1 < 0\}$ we conclude that

$$u_0(x) = \alpha x_1^+ + \bar{\alpha} x_1^-, \quad \alpha > 0,\ \bar{\alpha} \geqslant 0. \tag{9.9}$$

The case in which $\{x_1 > 0\} \subset \mathcal{A}$ gives, with the same arguments, that again (9.9) holds.

By Lemma 2.3 there exists a subsequence ε_{j_n} such that $\delta_n := \frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ and $u^{\delta_n}(x) := \frac{1}{\lambda_n} u^{\varepsilon_{j_n}}(x_0 + \lambda_n x) \to u_0(x)$ uniformly on compact sets of $\mathbb{R}^N$ (u^{δ_n} thus is a solution to $E_{\delta_n}(f^{\delta_n})$ with $f^{\delta_n}(x) := \lambda_n f^{\varepsilon_{j_n}}(x_0 + \lambda_n x) \to 0$ uniformly on compact sets of $\mathbb{R}^N$). By arguing as in Theorem 3.1 in [25], we can choose the subsequence ε_{j_n} in such a way that we also have $B_{\delta_n}(u^{\delta_n}) \to \chi_0$ $*$-weakly in $L^\infty_{\text{loc}}(\mathbb{R}^N)$ (χ_0 as above).

Then, $\bar{\alpha} > 0$ in (9.9) would imply $\chi_0 \equiv M$, which contradicts the fact that $\delta_W(x_0) < 6M$ (recall (9.5)). Therefore

$$u_0(x) = \alpha x_1^+, \quad \alpha > 0,$$

(and thus $\alpha = \sqrt{2M}$) and now the conclusion follows as in the previous results. □

In the next theorem we guarantee that the density of the nonpositive set at the free boundary point x_0 is positive in a different way.

Theorem 9.6. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Assume that u^+ is locally uniformly nondegenerate in the sense that* (5.17) *holds on every compact subset of Ω. If $x_0 \in \Omega \cap \partial\{u > 0\}$ is a regular point from the right and* $\limsup_{r\to 0} \frac{|B_r(x_0)\cap\{u\leqslant 0\}|}{|B_r(x_0)|} > 0$ *then, the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0. Moreover, u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of x_0.*

If $N = 2$ the same result holds without assuming that x_0 is a regular point from the right.

Proof. Assume x_0 is a regular point from the right. Then, we proceed as in the proof of Theorem 9.5. This time we deduce that $\sigma \leqslant 0$ in (9.6) since $\sigma > 0$ implies that

$$\frac{|B_r(x_0) \cap \{u \leqslant 0\}|}{|B_r(x_0)|} \to 0 \quad \text{as } r \to 0, \tag{9.10}$$

contradicting our hypotheses.

Now, assume $N = 2$ and let x_0 be any free boundary point. Let $\lambda_n \to 0$ be such that $\lim_{n\to\infty} \frac{|B_{\lambda_n}(x_0)\cap\{u\leqslant 0\}|}{|B_{\lambda_n}(x_0)|} > 0$ and such that $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x) \to u_0(x)$. So that

$$\frac{|B_1(0) \cap \{u_0 \leqslant 0\}|}{|B_1(0)|} > 0. \tag{9.11}$$

Let us see that, in a certain coordinate system,

$$u_0(x) = \alpha x_1^+ - \gamma x_1^- \quad \text{with } \alpha > 0 \text{ and } \gamma \geqslant 0. \tag{9.12}$$

In fact, proceeding as in Theorem 9.5 with u_0^+ we deduce that

$$u_0^+(x) = \alpha x_1^+ + \bar{\alpha} x_1^- \quad \text{with } \alpha > 0 \text{ and } \bar{\alpha} \geqslant 0.$$

Actually, (9.11) implies that $\bar{\alpha} = 0$ so that $u_0^+(x) = \alpha x_1^+$.

Now we proceed in a similar way with u_0^- and get, for a certain direction ν that

$$u_0^-(x) = \gamma \langle x, \nu\rangle^- + \bar{\gamma}\langle x, \nu\rangle^+ \quad \text{with } \gamma \text{ and } \bar{\gamma} \geqslant 0.$$

Since $\alpha > 0$, it follows that one of them, let's say $\bar{\gamma}$, is zero.

If $\gamma > 0$, there holds that $\nu = e_1$ and (9.12) follows. If $\gamma = 0$, then $u_0 = u_0^+$ and we get again (9.12).

Now, the conclusion follows as before. □

The following regularity result also uses the local monotonicity formula proven by the authors in [25] and it is also new even when $f^\varepsilon \equiv 0$.

Theorem 9.7. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Assume that u^+ is locally uniformly nondegenerate in the sense that* (5.17) *holds on every compact subset of Ω. Let $x_0 \in \Omega \cap \partial\{u > 0\}$. There holds that $\delta_W(x_0) = 3M$ ($\delta_W(x_0)$ as in* (9.4)) *if and only if the free boundary is $C^{1,\alpha}$ in a neighborhood of x_0. This implies that u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of x_0.*

Proof. Assume that the free boundary is $C^{1,\alpha}$ in a neighborhood of x_0. Then, using that u^+ is nondegenerate at x_0 we derive that $\delta_W(x_0) = 3M$ from Remark 3.1 in [25].

Now assume that $\delta_W(x_0) = 3M$. If u^- is nondegenerate at x_0, the result follows from the application of Theorem 9.3.

We will prove that the result also holds when u^- degenerates at x_0. In fact, in this case there exists a sequence $\lambda_n \to 0$ such that

$$\frac{1}{\lambda_n} \fint_{B_{\lambda_n}(x_0)} u^- \to 0, \quad \text{as } n \to \infty. \tag{9.13}$$

Now consider, for a subsequence, $u_0 = \lim u_{\lambda_n}$ uniformly on compact sets of $\mathbb{R}^N$ and $\chi_0 = \lim \chi^{\lambda_n}$ $*$-weakly in $L^\infty_{\text{loc}}(\mathbb{R}^N)$, where $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_0 + \lambda_n x)$, $\chi^{\lambda_n}(x) = \chi(x_0 + \lambda_n x)$ and $\chi = \lim B_{\varepsilon_j}(u^{\varepsilon_j})$ $*$-weakly in $L^\infty(\Omega)$.

Then, there exists a subsequence ε_{j_n} such that $\delta_n := \frac{\varepsilon_{j_n}}{\lambda_n} \to 0$, $u^{\delta_n}(x) := \frac{1}{\lambda_n} u^{\varepsilon_{j_n}}(x_0 + \lambda_n x) \to u_0(x)$ uniformly on compact sets of $\mathbb{R}^N$ (u^{δ_n} thus is a solution to $E_{\delta_n}(f^{\delta_n})$ with $f^{\delta_n}(x) := \lambda_n f^{\varepsilon_{j_n}}(x_0 + \lambda_n x) \to 0$ uniformly on compact sets of $\mathbb{R}^N$) and such that $B_{\delta_n}(u^{\delta_n}) \to \chi_0$ $*$-weakly in $L^\infty_{\text{loc}}(\mathbb{R}^N)$.

Let us show that

$$u_0(rx) = r u_0(x) \quad \text{for } r > 0,\ x \in \mathbb{R}^N, \tag{9.14}$$

$$\chi_0(rx) = \chi_0(x) \quad \text{for } r > 0,\ \text{a.e. } x \in \mathbb{R}^N. \tag{9.15}$$

In fact, (9.14) follows from Corollary 2.1 in [25]. We now observe that (9.13) together with (9.14) implies that $u_0 \geqslant 0$ in $\mathbb{R}^N$.

In order to see that (9.15) holds, we first apply Lemma 3.1 to $u_0 = \lim u^{\delta_n}$ to deduce that $\chi_0 \equiv M$ in $\{u_0 > 0\}$, and $\chi_0 \equiv M$ or $\chi_0 \equiv 0$ on every connected component of $\{u_0 \equiv 0\}^\circ$, each of these open sets being a cone as a consequence of (9.14). We thus obtain (9.15) because there holds that $|\partial\{u_0 > 0\}| = 0$ and this last assertion follows from the application of Theorem 8.1(1) to u_0 (notice that u_0 is locally uniformly nondegenerate on $\partial\{u_0 > 0\}$ in the sense of (5.1) because the same property holds for u^+).

In addition, the bounds in the proof of Lemma 3.1 imply that $\chi_0 \in BV_{\text{loc}}(\mathbb{R}^N)$.

Next, we define, as in Section 10 in [27],

$$H_N := \int_{-4}^{-1} \int_{\mathbb{R}^N} 2M \chi_{\{x_1 > 0\}} G(x, -t)\, \mathrm{d}x\, \mathrm{d}t = 3M$$

(in [27] it is assumed that $M = \int \beta(s)\,ds = \frac{1}{2}$) and using that $\delta_W(x_0) = 3M$, we obtain from Corollary 2.1 in [25] that

$$\int_{-4}^{-1}\int_{\mathbb{R}^N}\left(|\nabla u_0|^2 + 2\chi_0 + \frac{1}{2}\frac{u_0^2}{t}\right)G(x,-t)\,dx\,dt = 3M = H_N. \tag{9.16}$$

Moreover, Corollary 2.2 in [25] implies that

$$\int_{-4}^{-1}\int_{\mathbb{R}^N} 2\chi_0 G(x,-t)\,dx\,dt = 3M,$$

and thus, $\chi_0 \not\equiv 0$ and $\chi_0 \not\equiv M$.

We are now in a situation very similar to that of Proposition 10.1(1) in [27], and we deduce by a dimension reduction argument that the equality in (9.16) implies that

$$u_0(x) = \alpha\langle x, \nu\rangle^+ \quad \text{in } \mathbb{R}^N,$$

for some unit vector ν and $\alpha > 0$ (and therefore $\alpha = \sqrt{2M}$). We refer to [26] for the remaining details.

Finally, arguing as in the previous theorems we get the conclusion. □

We next include a proposition and we discuss its consequences on the regularity of the free boundary (see Remark 9.5)

Proposition 9.2. *Let u^{ε_j} be solutions to $E_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Assume that u^- is nondegenerate at $x_0 \in \Omega \cap \partial\{u > 0\}$ in the sense of* (5.2).

Then given $0 < \mu < 1$ and $\frac{\pi}{4} < \theta_0 < \frac{\pi}{2}$, there exists $\lambda > 0$ and $\nu \in \mathbb{R}^N$, $|\nu| = 1$, such that $u_\lambda(x) = \frac{1}{\lambda}u(x_0 + \lambda x)$ is μ-monotone in $B_1(0)$ in any direction τ of the cone $\Gamma(\theta_0, \nu) = \{\tau\colon \text{angle}(\tau, \nu) \leqslant \theta_0\}$ (i.e., $u_\lambda(x + r\tau) \geqslant u_\lambda(x)$ for any $1 \geqslant r \geqslant \mu$).

Proof. Let $\lambda_n > 0$ be a sequence such that $\lambda_n \to 0$ and such that $u_{\lambda_n}(x) = \frac{1}{\lambda_n}u(x_0 + \lambda_n x)$ converges to a function U as $n \to \infty$, uniformly on compact sets of $\mathbb{R}^N$.

From Proposition 9.1, it follows that

$$U(x) = \alpha\langle x, \nu\rangle^+ - \gamma\langle x, \nu\rangle^- \quad \text{in } \mathbb{R}^N,$$

with ν a unit vector, and α, γ positive constants satisfying $\alpha^2 - \gamma^2 = 2M$.

Therefore, given $0 < \mu < 1$ and $\frac{\pi}{4} < \theta_0 < \frac{\pi}{2}$, there exists n_0 such that, for any $n \geqslant n_0$, and $x \in B_1(0)$,

$$u_{\lambda_n}(x + r\tau) \geqslant u_{\lambda_n}(x) \quad \text{for any } 1 \geqslant r \geqslant \mu$$

with τ any direction of the cone,

$$\Gamma(\theta_0, \nu) = \big\{\tau\colon \text{angle}(\tau, \nu) \leqslant \theta_0\big\}.$$

This is, u_{λ_n} is μ-monotone in $B_1(0)$ in $\Gamma(\theta_0, \nu)$. □

Remark 9.5. Let $u = \lim u^\varepsilon$ (u^ε solutions to $E_\varepsilon(f^\varepsilon)$) be such that u^+ is nondegenerate at every regular point from the left in $\Omega \cap \partial\{u > 0\}$. Then, from the results in Section 7 it follows that u is a viscosity solution to $E(f)$ ($f = \lim f^\varepsilon$).

Assume that u^- is nondegenerate at $x_0 \in \Omega \cap \partial\{u > 0\}$. Then, in case $f \equiv 0$, we deduce from Proposition 9.2, Theorem 1 in [6] and the results in [5] that $\Omega \cap \partial\{u > 0\}$ is a $C^{1,\alpha}$ surface in a neighborhood of x_0 and u is a classical solution to the free boundary problem $E(f)$.

In case $f \not\equiv 0$ we expect the same conclusion to hold. In fact, we expect a result analogous to Theorem 1 in [6] to hold, implying the regularity of the free boundary (at least for a wide cone, which is what we get in Proposition 9.2).

Remark 9.6 *(Higher regularity).* In all the regularity theorems in this section, when $u \geqslant 0$, we get further regularity of the smooth portion $\mathcal{R}$ of the free boundary according to the regularity of the function f. In fact, from Theorem 2 in [21] it follows that

$$f \in C^{k,\alpha}_{\mathrm{loc}} \text{ (resp. analytic) implies } \mathcal{R} \in C^{k+2,\alpha}_{\mathrm{loc}} \text{ (resp. analytic).}$$

Let us finally summarize the results in this section:

Remark 9.7 *(Conclusion).* We know that there are examples where u^+ degenerates at the free boundary as well as examples where there is no portion of $\{u \leqslant 0\}^\circ$ at the free boundary (like $u = \alpha x_1^+ + \alpha x_1^-$, $\alpha > 0$). Thus, in order to prove that a limit function is a classical solution to $E(f)$ these situations need to be ruled out.

We have obtained different regularity results. In all of them the assumption that u^+ is nondegenerate on the free boundary is present. But it is also necessary to make some other assumption guaranteeing that there is some portion of $\{u \leqslant 0\}^\circ$ at the free boundary.

In the regularity results ranging from Theorem 9.2 to the end of the section, we guarantee this fact by means of hypotheses of a pointwise nature, and the conclusions obtained hold in a neighborhood of a point.

In contrast, in Theorem 9.1, we guarantee this fact—for nonnegative limit functions—by means of a hypothesis of a global nature, but with a hypothesis which is the weakest possible way to ensure that there is some portion of $\{u \leqslant 0\}^\circ$ at the free boundary. The conclusion obtained holds almost everywhere on $\partial\{u > 0\}$.

We refer to Section 5 for conditions implying the nondegeneracy of u^+.

We finally point out that Theorems 9.1, 9.4–9.7 are new even in the case that $f^\varepsilon \equiv 0$.

10. Some applications

In this section we discuss applications of our results to the study of the regularity of the free boundary for the limit of different singular perturbation problems. Namely, for the limit of stationary solutions to the nonlocal combustion model studied in [24], for the limit of stationary solutions to (1.2), for the limit of the travelling wave solutions to a combustion model first studied in [3] and for the limit of the minimizers to the energy functional constructed in Proposition 2.2.

Example 10.1. Consider u^ε a family of solutions to the following nonlocal combustion model:

$$\theta \Delta u^\varepsilon + (1-\theta)\big(J * u^\varepsilon - u^\varepsilon\big) - u^\varepsilon_t = \beta_\varepsilon\big(u^\varepsilon\big) + g^\varepsilon, \tag{10.1}$$

where $0 < \theta \leqslant 1$, β_ε as before, $*$ denotes spatial convolution, $J = J(x)$ is an even nonnegative kernel with unit integral and g^ε are given functions.

In [24] it was shown that if u^ε are bounded solutions to (10.1) in $\mathbb{R}^N \times (0,T)$, with $\|u^\varepsilon_0\|_{L^\infty(\mathbb{R}^N)} \leqslant C_1$ and $\|g^\varepsilon\|_{L^\infty(\mathbb{R}^N \times (0,T))} \leqslant C_2$ then $\|u^\varepsilon\|_{L^\infty(\mathbb{R}^N \times (0,T))} \leqslant C_3$.

Let us now consider any family u^ε of solutions to (10.1) with $\|u^\varepsilon\|_{L^\infty} \leqslant \widetilde{C}_1$ and $\|g^\varepsilon\|_{L^\infty} \leqslant \widetilde{C}_2$.

Then, defining $\bar{u}^\varepsilon(x,t) = u^\varepsilon(\sqrt{\theta}\,x, t)$, it follows that $\bar{u}^\varepsilon$ are solutions to $P_\varepsilon(f^\varepsilon)$, with

$$f^\varepsilon(x,t) = -(1-\theta)\big(J * u^\varepsilon - u^\varepsilon\big)\big(\sqrt{\theta}\,x, t\big) + g^\varepsilon\big(\sqrt{\theta}\,x, t\big)$$

and $\|f^\varepsilon\|_{L^\infty} \leqslant \widetilde{C}_3$. If moreover $\bar{u}^\varepsilon$ are stationary, then they are solutions to $E_\varepsilon(f^\varepsilon)$ and the results in this paper apply.

We refer to [24] for a discussion of this problem in the one phase evolution case.

Example 10.2. Consider u^ε a family of solutions to the following combustion model with transport:

$$\Delta u^\varepsilon + a^\varepsilon(x,t) \cdot \nabla u^\varepsilon + c^\varepsilon(x,t) u^\varepsilon - u^\varepsilon_t = \beta_\varepsilon\big(u^\varepsilon\big), \tag{10.2}$$

with β_ε as before. If $\|u^\varepsilon\|_{L^\infty} \leqslant C_1$, $\|a^\varepsilon\|_{L^\infty} \leqslant C_2$, $\|c^\varepsilon\|_{L^\infty} \leqslant C_3$, then local uniform $Lip(1, \frac{1}{2})$ bounds are obtained for such a family in [11] (for u^ε nonnegative and stationary these estimates also follow from the previous paper [3]).

Then, u^ε are solutions to $P_\varepsilon(f^\varepsilon)$, with

$$f^\varepsilon(x,t) = -a^\varepsilon(x,t) \cdot \nabla u^\varepsilon - c^\varepsilon(x,t) u^\varepsilon,$$

and $\|f^\varepsilon\|_{L^\infty} \leqslant C_4$. If moreover u^ε are stationary and the coefficients a^ε and c^ε are independent of the time t, the functions u^ε are solutions to $E_\varepsilon(f^\varepsilon)$ and the results in this paper apply.

Example 10.3. Let $x = (x_1, y) \in \Omega = \mathbb{R} \times \Sigma$, with $\Sigma \subset \mathbb{R}^{N-1}$ a smooth bounded domain, let a be a continuous positive function on $\overline{\Sigma}$ and let $0 < \sigma < 1$ be given.

Then, we consider travelling wave solutions to the following combustion model;

$$\Delta v^\varepsilon - a(y) v_t^\varepsilon = \beta_\varepsilon\bigl(v^\varepsilon\bigr), \tag{10.3}$$

where β_ε is as before. This is, we consider solutions to (10.3) of the form $v^\varepsilon(x,t) = u^\varepsilon(x_1 + c^\varepsilon t, y)$. The functions u^ε are solutions to

$$\begin{aligned} &\Delta u^\varepsilon - c^\varepsilon a(y) u_{x_1}^\varepsilon = \beta_\varepsilon\bigl(u^\varepsilon\bigr) \quad \text{in } \Omega,\\ &u^\varepsilon(-\infty, y) = (1-\sigma)^{-1}, \quad u^\varepsilon(+\infty, y) = 0 \quad \text{in } \Sigma,\\ &\frac{\partial u^\varepsilon}{\partial \eta} = 0 \quad \text{on } \mathbb{R} \times \partial\Sigma, \end{aligned} \tag{10.4}$$

for some suitable c^ε.

Problem (10.4) was studied in [3] (some further regularity was assumed on β near 0 and $\beta'(0) > 0$). In particular, uniform estimates in L^∞ norm were obtained for u^ε and for ∇u^ε, as well as uniform bounds for c^ε. This implies that u^ε are nonnegative solutions to $E_\varepsilon(f^\varepsilon)$ in Ω, with

$$f^\varepsilon = c^\varepsilon a(y) u_{x_1}^\varepsilon,$$

and f^ε uniformly bounded in L^∞ norm.

In addition, in [3] the authors proved the local uniform nondegeneracy of u^ε in the sense that (5.9) holds on every compact subset of Ω. This implies the uniform nondegeneracy of $u = \lim u^\varepsilon$ on the free boundary in the sense of (5.2) and also in the sense that (5.17) holds on every compact subset of Ω.

All the results in this paper apply to this family, in particular the results in Section 9. Moreover, following ideas in [14] and [27], and using results from [25] and [26] we can prove the positive density of the zero set at every free boundary point. Thus, one of the results we obtain for this family is the following:

Theorem 10.1. *Let $u = \lim u^{\varepsilon_j}$ ($\varepsilon_j \to 0$), with u^{ε_j} solutions to* (10.4). *Then, there is a subset $\mathcal{R}$ of the free boundary $\Omega \cap \partial\{u > 0\}$ ($\mathcal{R} = \partial_{\text{red}}\{u > 0\}$) which is locally a $C^{1,\alpha}$ surface and u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of $\mathcal{R}$ ($f = ca(y)u_{x_1}$ with $c = \lim c^{\varepsilon_j}$). Moreover, $\mathcal{R}$ is open and dense in $\Omega \cap \partial\{u > 0\}$ and the remainder of the free boundary has $(N-1)$-dimensional Hausdorff measure zero.*

In dimension 2 we have $\mathcal{R} = \Omega \cap \partial\{u > 0\}$.

In addition, in any dimension, if $a \in C^{k,\alpha}_{\text{loc}}$ (resp. analytic) then, $\mathcal{R} \in C^{k+2,\alpha}_{\text{loc}}$ (resp. analytic).

Proof. We will show that for every $x_0 \in \Omega \cap \partial\{u > 0\}$,

$$\liminf_{r\to 0} \frac{|\{u \equiv 0\} \cap B_r(x_0)|}{|B_r(x_0)|} > 0. \tag{10.5}$$

Then, we will be under the assumptions of Theorem 9.1. The global regularity result in case $N = 2$ will follow from Theorem 9.6.

So let us show that (10.5) holds. In fact, assume it does not hold at a point x_0. Without loss of generality we may assume that $x_0 = 0$. Then, there exists a sequence $\lambda_n \to 0$ such that

$$\lim_{n\to\infty} \frac{|\{u \equiv 0\} \cap B_{\lambda_n}(0)|}{|B_{\lambda_n}(0)|} = 0.$$

Let $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(\lambda_n x)$. Then,

$$\lim_{n\to\infty} \frac{|\{u_{\lambda_n} \equiv 0\} \cap B_1(0)|}{|B_1(0)|} = 0 \quad \text{so that,} \quad \lim_{n\to\infty} \frac{|\{u_{\lambda_n} > 0\} \cap B_1(0)|}{|B_1(0)|} = 1$$

and, since $\chi^{\lambda_n}(x) = \chi(\lambda_n x) = M$ for every x in the positivity set of u_{λ_n}, we deduce that

$$\lim_{n\to\infty} \frac{|\{\chi^{\lambda_n} = M\} \cap B_1(0)|}{|B_1(0)|} = 1.$$

Let $\chi_0 = \lim_{n\to\infty} \chi^{\lambda_n}$ (we may assume without loss of generality that this limit exists almost everywhere). Then, $\chi_0 = M$ almost everywhere (we use again that χ_0 is homogeneous). Now, proceeding as in Theorem 9.7, we see that we are in a situation very similar to that of Proposition 10.1(1) in [27] and we deduce, by a dimension reduction argument, that there exist $\gamma \geqslant 0$ and a unit vector ν such that $u_0(x) = \gamma|\langle x, \nu\rangle|$ (see [26] for the details).

Since u^+ is nondegenerate, the same property holds for u_0, so that $\gamma > 0$. The fact that this leads to a contradiction was proved in [14], Lemma 5.10. For the readers convenience, we will sketch the proof.

Let $\varepsilon_{j_n} \to 0$ be such that $\delta_n = \frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ and $u^{\delta_n} = (u^{\varepsilon_{j_n}})_{\lambda_n} \to u_0$. Let $\delta > 0$ then, if n is large enough,

$$u^{\delta_n}(x) > \big(\gamma\big|\langle x, \nu\rangle\big| - \delta\big)^+ \quad \text{in } B_1.$$

Let now φ_n be the solution to (recall the notation $x = (x_1, y)$):

$$\begin{cases} \Delta\varphi_n - c^{\varepsilon_{j_n}}\lambda_n a(\lambda_n y)\varphi_{n\,x_1} = 0 & \text{in } B_1, \\ \varphi_n = (\gamma|\langle x, \nu\rangle| - 2\delta)^+ & \text{on } \partial B_1. \end{cases} \tag{10.6}$$

Then, $\varphi_n = \varphi + \tilde{\varphi}_n$, where

$$\begin{cases} \Delta\varphi = 0 & \text{in } B_1, \\ \varphi = (\gamma|\langle x, \nu\rangle| - 2\delta)^+ & \text{on } \partial B_1 \end{cases} \tag{10.7}$$

and $\tilde{\varphi}_n \rightrightarrows 0$ and $|\nabla\tilde{\varphi}_n| \rightrightarrows 0$ in B_1. Thus, for every $\mu > 0$ and every sequence $\mu_n \to 0$ there holds that

$$\{0 < \varphi_n < \mu_n\} \subset \mu\text{-neighborhood of } \big\{\gamma\big|\langle x, \nu\rangle\big| < 2\delta\big\} \cap \partial B_1.$$

Moreover, for every $K > 0$ there exist $\delta_0, \mu > 0$ such that if $\delta < \delta_0$, $|\nabla\varphi| \geqslant 2K$ in the μ-neighborhood of $\{\gamma|\langle x, \nu\rangle| < 2\delta\} \cap \partial B_1$. So, let us assume that, from the beginning, we have chosen δ smaller that δ_0 so that, for n large enough

$$|\nabla\varphi_n| \geqslant K \quad \text{in a } \mu\text{-neighborhood of } \big\{\gamma\big|\langle x, \nu\rangle\big| < 2\delta\big\} \cap \partial B_1.$$

Let now

$$\Gamma_n(t) = \begin{cases} 0, & t \leqslant 0, \\ t^2/2\mu_n, & 0 \leqslant t \leqslant \mu_n, \\ t - \mu_n/2, & t \geqslant \mu_n, \end{cases}$$

and $\psi_n = \Gamma_n(\varphi_n)$. Then, if we take $\mu_n = 2\delta_n$ and K large enough,

$$\begin{aligned} \Delta\psi_n - c^{\varepsilon_{j_n}}\lambda_n a(\lambda_n y)\psi_{n\,x_1} &= \Gamma_n''(\varphi_n)|\nabla\varphi_n|^2 \\ &= \frac{1}{\mu_n}\chi_{\{0<\varphi_n<\mu_n\}}|\nabla\varphi_n|^2 \geqslant K^2\frac{1}{2\delta_n}\chi_{\{0<\psi_n<\delta_n\}} \geqslant \beta_{\delta_n}(\psi_n). \end{aligned}$$

Finally, let us show that $u^{\delta_n} \geqslant \psi_n$. Since $u^{\delta_n}_{x_1} \leqslant 0$, $u^{\delta_n}(x) > \psi_n(x)$ on ∂B_1 (recall that $u^{\delta_n} > 0$ in $\overline{B}_1$) and $u^{\delta_n} \to \frac{(1-\sigma)^{-1}}{\lambda_n}$ uniformly in B_1 as $x_1 \to -\infty$, there exists $s_0 > 0$ depending on n such that

$$u^{\delta_n}(x_1 - s_0, y) > \psi_n(x) \text{ in } \overline{B}_1 \quad \text{and} \quad u^{\delta_n}(x_1 - s, y) > \psi_n(x) \text{ on } \partial B_1 \quad \text{for } 0 \leqslant s \leqslant s_0.$$

Let $h > 0$ be a constant to be determined later and let $\eta > 0$ small such that

$$\begin{aligned} &u^{\delta_n}(x_1 - s_0, y) > \psi_n(x) + \eta e^{h|x|^2} \quad \text{in } B_1, \quad \text{and} \\ &u^{\delta_n}(x_1 - s, y) > \psi_n(x) + \eta e^{h|x|^2} \quad \text{on } \partial B_1 \quad \text{for } 0 \leqslant s \leqslant s_0. \end{aligned}$$

Finally, let

$$\bar{s} = \inf\big\{0 < s < s_0 / u^{\delta_n}(x_1 - s, y) > \psi_n(x) + \eta e^{h|x|^2} \text{ in } \overline{B}_1\big\}.$$

If $\bar{s} = 0$ there holds that $u^{\delta_n}(x) \geqslant \psi_n(x)$ and we are done. If $\bar{s} > 0$, there exists $\bar{x} \in B_1$ such that $u^{\delta_n}(\bar{x}_1 - \bar{s}, \bar{y}) = \psi_n(\bar{x}) + \eta e^{h|\bar{x}|^2}$ and $u^{\delta_n}(x_1 - \bar{s}, y) \geqslant \psi_n(x) + \eta e^{h|x|^2}$ in B_1. Thus, using the fact that $a(y)$ is bounded and the velocities c^{ε_j} are uniformly bounded, we get, for a universal constant c_1,

$$\begin{aligned}\beta_{\delta_n}\big(u^{\delta_n}(\bar x_1-\bar s,\bar y)\big)&=\Delta u^{\delta_n}(\bar x_1-\bar s,\bar y)-c^{\varepsilon_{j_n}}\lambda_n a(\lambda_n\bar y)u^{\delta_n}_{x_1}(\bar x_1-\bar s,\bar y)\\&\geqslant \Delta\psi_n(\bar x)-c^{\varepsilon_{j_n}}\lambda_n a(\lambda_n\bar y)\psi_{n\bar x_1}(\bar x)+2\eta h e^{h|\bar x|^2}\big(N+h|\bar x|^2-c_1|\bar x_1|\big)\\&\geqslant \beta_{\delta_n}\big(u(\bar x_1-\bar s,\bar y)\big)-\frac{1}{\delta_n^2}\|\nabla\beta\|_\infty\eta e^{h|\bar x|^2}+2\eta h e^{h|\bar x|^2}\big(N+h|\bar x|^2-c_1|\bar x_1|\big)\\&>\beta_{\delta_n}\big(u(\bar x_1-\bar s,\bar y)\big)\end{aligned}$$

if h is chosen big enough ($h>\max(\frac{\|\nabla\beta\|_\infty}{N\delta_n^2},2c_1^3)$). This is a contradiction. Therefore, $\bar s=0$ and $u^{\delta_n}\geqslant\psi_n$ in B_1. In particular,

$$\liminf_{n\to\infty}u^{\delta_n}(0)\geqslant\lim_{n\to\infty}\psi_n(0)=\varphi(0)>0.$$

Since this contradicts the fact that $u^{\delta_n}\to\gamma|\langle x,\nu\rangle|$ as $n\to\infty$, we deduce that it is impossible in the present situation that $u_{\lambda_n}\to\gamma|\langle x,\nu\rangle|$ with $\gamma>0$ for a sequence $\lambda_n\to 0$. Therefore (10.5) holds and the theorem is proved. □

Example 10.4. Let u^ε be the minimizers to the energy functional J_ε constructed in Proposition 2.2. As in that proposition we assume that the boundary data are uniformly bounded in H^1 norm and the functions f^ε are uniformly bounded in L^∞ norm. Let $u=\lim u^{\varepsilon_j}$ with $\varepsilon_j\to 0$. By Corollary 5.5 we know that u is locally uniformly nondegenerate in the sense that (5.17) holds on every compact subset of Ω. By Proposition 5.5, this implies the local uniform nondegeneracy of u^+ on the free boundary in the sense of (5.2).

All the results in this paper apply to this family, in particular the results in Section 9. Moreover, following some arguments in [15] we can prove that the density of the nonpositive set is positive at every free boundary point. Thus, one of the results we obtain for this family is the following:

Theorem 10.2. *Let $u=\lim u^{\varepsilon_j}$ and $f=\lim f^{\varepsilon_j}$ with $\varepsilon_j\to 0$, where u^{ε_j} are minimizers of J_{ε_j} in the set of functions in $H^1(\Omega)$ that coincide with ϕ_{ε_j} on $\partial\Omega$, where $\|\phi_{\varepsilon_j}\|_{H^1(\Omega)}\leqslant C$ and $\|f^{\varepsilon_j}\|_{L^\infty(\Omega)}\leqslant\mathcal A$, with $C,\mathcal A$ independent of ε_j. Then, there is a subset $\mathcal R$ of the free boundary $\Omega\cap\partial\{u>0\}$ ($\mathcal R=\partial_{\rm red}\{u>0\}$) which is locally a $C^{1,\alpha}$ surface and u is a classical solution to the free boundary problem $E(f)$ in a neighborhood of $\mathcal R$. Moreover, $\mathcal R$ is open and dense in $\Omega\cap\partial\{u>0\}$ and the remainder of the free boundary has $(N-1)$-dimensional Hausdorff measure zero.*

In dimensions 2 and 3 we have $\mathcal R=\Omega\cap\partial\{u>0\}$.

In addition, in any dimension, if $u\geqslant 0$ and $f\in C^{k,\alpha}_{\rm loc}$ (resp. analytic) then, $\mathcal R\in C^{k+2,\alpha}_{\rm loc}$ (resp. analytic).

Proof. First, let us see that if $x_0\in\Omega\cap\partial\{u>0\}$, then

$$\liminf_{r\to 0}\frac{|B_r(x_0)\cap\{u\leqslant 0\}|}{|B_r(x_0)|}>0. \tag{10.8}$$

In fact, assume this is not true and let $\lambda_n\to 0$ such that

$$\lim_{n\to\infty}\frac{|B_{\lambda_n}(x_0)\cap\{u\leqslant 0\}|}{|B_{\lambda_n}(x_0)|}=0.$$

Let $u_{\lambda_n}(x)=\frac{1}{\lambda_n}u(x_0+\lambda_n x)$. Thus,

$$\lim_{n\to\infty}\frac{|B_1(0)\cap\{u_{\lambda_n}\leqslant 0\}|}{|B_1(0)|}=0. \tag{10.9}$$

Moreover, we may assume that there exists $u_0=\lim_{n\to\infty}u_{\lambda_n}$.

Now, by the uniform nondegeneracy of u^+ in the sense of (5.17), the fact that (10.9) holds implies that $B_1(0)\cap\{u_0\leqslant 0\}^\circ=\emptyset$.

On the other hand, there exists a sequence $\delta_n\to 0$ such that $u_0=\lim u^{\delta_n}$ and u^{δ_n} are solutions to $E_{\delta_n}(f^{\delta_n})$ with $f^{\delta_n}\to 0$ uniformly on compact sets of $\mathbb R^N$. Then, following the arguments in [15], Theorem 1.16 we can prove that u_0 is a local minimizer of the functional,

$$J(v)=\int\left[\frac12|\nabla v|^2+M\chi_{\{v>0\}}\right]dx. \tag{10.10}$$

Since u^+ is nondegenerate there holds that $0 \in \partial\{u_0 > 0\}$. Thus, by Theorem 7.1 in [2],

$$\frac{|B_1(0) \cap \{u_0 \leqslant 0\}|}{|B_1(0)|} > c > 0.$$

In particular, $|B_1(0) \cap \{u_0 \leqslant 0\}| > 0$ which contradicts the fact that $B_1(0) \cap \{u_0 \leqslant 0\}^\circ = \emptyset$ and $|\partial\{u_0 > 0\}| = 0$ (see [2]). Therefore, (10.8) holds.

Since u^+ is locally uniformly nondegenerate, by applying (1) in Theorem 8.1, we have that the free boundary has locally finite $\mathcal{H}^{N-1}$ measure. Moreover,

$$\limsup_{r\to 0} \frac{|B_r(x_0) \cap \{u > 0\}|}{|B_r(x_0)|} > 0 \tag{10.11}$$

and, by (10.8) and (10.11), $\mathcal{H}^{N-1}$-a.e. point in the free boundary belongs to the reduced free boundary $\partial_{\mathrm{red}}\{u > 0\}$.

If $u \geqslant 0$, Theorem 9.1 applies. In the general case, Theorem 9.2 applies at $\mathcal{H}^{N-1}$ almost every point in $\Omega \cap \partial\{u > 0\}$ and thus the statement is proved in case $N \geqslant 4$.

In dimension 2 the regularity of the whole free boundary follows from the application of Theorem 9.6 (recall (10.8)).

Let us consider the case of dimension 3. Let $x_0 \in \Omega \cap \partial\{u > 0\}$. If u^- is nondegenerate at x_0, Theorem 9.3 applies and we deduce that the free boundary is $C^{1,\alpha}$ in a neighborhood of x_0. Let us now assume that u^- degenerates at x_0. Then, there is a blow up limit u_0 centered at x_0 that is nonnegative in $B_1(0)$ and since u_0 is homogeneous (see Corollary 2.1 in [25]), $u_0 \geqslant 0$ in $\mathbb{R}^N$. We will use Theorem 9.2 and Remark 9.2. In fact, we will show that, in a certain coordinate system,

$$u_0(x) = \sqrt{2M}\, x_1^+. \tag{10.12}$$

This will prove the regularity of the free boundary around the free boundary point x_0.

Indeed, the fact that (10.12) holds follows by direct application of [10] where the authors prove that this is true for any nonnegative homogeneous minimizer of (10.10). □

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