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Asymptotic behavior for nonlocal diffusion equations

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Abstract

We study the asymptotic behavior for nonlocal diffusion models of the form $u_t = J * u - u$ in the whole $\mathbb{R}^N$ or in a bounded smooth domain with Dirichlet or Neumann boundary conditions. In $\mathbb{R}^N$ we obtain that the long time behavior of the solutions is determined by the behavior of the Fourier transform of J near the origin, which is linked to the behavior of J at infinity. If $\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$ $(0 < \alpha \leqslant 2)$, the asymptotic behavior is the same as the one for solutions of the evolution given by the $\alpha/2$ fractional power of the Laplacian. In particular when the nonlocal diffusion is given by a compactly supported kernel the asymptotic behavior is the same as the one for the heat equation, which is yet a local model. Concerning the Dirichlet problem for the nonlocal model we prove that the asymptotic behavior is given by an exponential decay to zero at a rate given by the first eigenvalue of an associated eigenvalue problem with profile an eigenfunction of the first eigenvalue. Finally, we analyze the Neumann problem and find an exponential convergence to the mean value of the initial condition.

Résumé

Nous étudions le comportement asymptotique pour les modèles de diffusion non-locale de la forme $u_t = J * u - u$ dans $\mathbb{R}^N$ tout entier, ou dans un domaine borné régulier, avec conditions de Dirichlet ou de Neumann. Dans $\mathbb{R}^N$, nous obtenons que le comportement en temps grand des solutions est déterminé par le comportement de la tranformée de Fourier de J près de l'origine, lui-même relié au comportement de J à l'infini. Si $\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$ $(0 < \alpha \leqslant 2)$, le comportement asymptotique est le même que celui donné par les solutions de l'équation d'évolution avec laplacien fractionnaire d'ordre $\alpha/2$. En particulier, lorsque l'équation non-locale est donnée par un noyau ă support compact, le comportement asymptotique est le même que celui de l'équation de la chaleur, qui est pourtant un modèle local. Concernant le problème de Dirichlet pour le modèle non-local, nous montrons que le comportement asymptotique est donné par une décroissance exponentielle en relation avec la première valeur propre d'un problème associé, et le profil est donné par la première fonction propre. Enfin, nous analysons le problème de Neumann et nous obtenons une convergence exponentielle vers la valeur moyenne de la donnée initiale.

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1. Introduction

The aim of this paper is to study the asymptotic behavior of solutions of a nonlocal diffusion operator in the whole $\mathbb{R}^N$ or in a bounded smooth domain with Dirichlet or Neumann boundary conditions.

First, let us introduce what kind of nonlocal diffusion problems we consider. To this end, let $J:\mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial function with $\int_{\mathbb{R}^N} J(r)\,\mathrm{d}r = 1$. Nonlocal evolution equations of the form:

$$\begin{aligned} &u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,\mathrm{d}y - u(x,t),\\ &u(x,0) = u_0(x), \end{aligned} \tag{1}$$

and variations of it, have been recently widely used in the modelling of diffusion processes, see [1,3,6,9,11,16, 17,20–22]. As stated in [16], if $u(x,t)$ is thought of as the density of a single population at the point x at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $(J*u)(x,t) = \int_{\mathbb{R}^N} J(y-x)u(y,t)\,\mathrm{d}y$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)\,\mathrm{d}y$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies Eq. (1).

Eq. (1) is called nonlocal diffusion equation since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$, but on all the values of u in a neighborhood of x through the convolution term $J*u$. This equation shares many properties with the classical heat equation, $u_t = cu_{xx}$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and, even if J is compactly supported, perturbations propagate with infinite speed [16]. However, there is no regularizing effect in general. For instance, if J is rapidly decaying (or compactly supported) the singularity of the source solution, that is a solution of (1) with initial condition a delta measure, $u_0 = \delta_0$, remains with an exponential decay. In fact, this fundamental solution can be decomposed as $w(x,t) = \mathrm{e}^{-t}\delta_0 + v(x,t)$ where $v(x,t)$ is smooth, see Lemma 2.2. In this way we see that there is no regularizing effect since the solution u of (1) can be written as $u = w * u_0 = \mathrm{e}^{-t}u_0 + v * u_0$ with v smooth, which means that $u(t)$ is as regular as u_0 is, and no more (see again Lemma 2.2). For more information on this topic, we refer to Section 1.2 at the end of the introduction.

Let us also mention that our results have a probabilistic counterpart in the setting of Markov chains (we refer also to Section 1.2 for a brief exposition of this matter).

1.1. Main results

Let us now state our results concerning the asymptotic behavior for Eq. (1), for the Cauchy, Dirichlet and Neumann problems.

1.1.1. The Cauchy problem

We will understand a solution of (1) as a function $u \in C^0([0,+\infty); L^1(\mathbb{R}^N))$ that verifies (1) in the integral sense, see Theorem 2.1. Our first result states that the decay rate as t goes to infinity of solutions of this nonlocal problem is determined by the behavior of the Fourier transform of J near the origin. The asymptotic decays are the same as the ones that hold for solutions of the evolution problem with right hand side given by a power of the Laplacian.

In the sequel we denote by $\hat{f}$ the Fourier transform of f. Let us recall our hypotheses on J that we will assume throughout the paper:

(H) $J \in C(\mathbb{R}^N, \mathbb{R})$ is a nonnegative, radial function with $\int_{\mathbb{R}^N} J(r)\,\mathrm{d}r = 1$.

This means that J is a radial density probability which implies obviously that $|\hat{J}(\xi)| \leqslant 1$ with $\hat{J}(0) = 1$, and we shall assume that $\hat{J}$ has an expansion of the form $\hat{J}(\xi) = 1 - A|\xi|^\alpha + \mathrm{o}(|\xi|^\alpha)$ for $\xi \to 0$ $(A > 0)$. Remark that in this case, (H) implies also that $0 < \alpha \leqslant 2$ and $\alpha \neq 1$ if J has a first momentum (see Lemma 2.1).

Theorem 1. *Let u be a solution of* (1) *with $u_0, \hat{u}_0 \in L^1(\mathbb{R}^N)$. If there exist $A > 0$ and $0 < \alpha \leqslant 2$ such that*

$$\hat{J}(\xi) = 1 - A|\xi|^\alpha + \mathrm{o}\big(|\xi|^\alpha\big), \quad \xi \to 0, \tag{2}$$

then the asymptotic behavior of $u(x,t)$ is given by:

$$\lim_{t\to+\infty} t^{N/\alpha} \max_x \big|u(x,t) - v(x,t)\big| = 0,$$

where v is the solution of $v_t(x,t) = -A(-\Delta)^{\alpha/2} v(x,t)$ with initial condition $v(x,0) = u_0(x)$. Moreover, we have,

$$\big\|u(\cdot,t)\big\|_{L^\infty(\mathbb{R}^N)} \leqslant C t^{-N/\alpha},$$

and the asymptotic profile is given by:

$$\lim_{t\to+\infty} \max_y \big|t^{N/\alpha} u\big(y t^{1/\alpha}, t\big) - \|u_0\|_{L^1} G_A(y)\big| = 0,$$

where $G_A(y)$ satisfies $\widehat{G}_A(\xi) = \mathrm{e}^{-A|\xi|^\alpha}$.

In the special case $\alpha = 2$, the decay rate is $t^{-N/2}$ and the asymptotic profile is a Gaussian $G_A(y) = (4\pi A)^{N/2} \exp(-A|y|^2/4)$ with $A \cdot \mathrm{Id} = -(1/2) D^2 \hat{J}(0)$, see Lemma 2.1. Note that in this case (that occurs, for example, when J is compactly supported) the asymptotic behavior is the same as the one for solutions of the heat equation and, as happens for the heat equation, the asymptotic profile is a Gaussian.

The decay in L^∞ of the solutions together with the conservation of mass give the decay of the L^p-norms by interpolation. As a consequence of Theorem 1, we find that this decay is analogous to the decay of the evolution given by the fractional Laplacian, that is,

$$\big\|u(\cdot,t)\big\|_{L^p(\mathbb{R}^N)} \leqslant C t^{-\frac{N}{\alpha}(1-\frac{1}{p})},$$

see Corollary 2.2. We refer to [10] for the decay of the L^p-norms for the fractional Laplacian, see also [7,12,14] for finer decay estimates of L^p-norms for solutions of the heat equation.

Next we consider a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and impose boundary conditions to our model. From now on we assume that J is continuous.

1.1.2. The Dirichlet problem

We consider the problem:

$$\begin{aligned} &u_t(x,t) = \int_{\mathbb{R}^N} J(x-y) u(y,t)\,\mathrm{d}y - u(x,t), \quad x \in \Omega,\ t > 0,\\ &u(x,t) = 0, \quad x \notin \Omega,\ t > 0,\\ &u(x,0) = u_0(x), \quad x \in \Omega. \end{aligned} \tag{3}$$

In this model we have that diffusion takes place in the whole $\mathbb{R}^N$ but we impose that u vanishes outside Ω. This is the analogous of what is called Dirichlet boundary conditions for the heat equation. However, the boundary data is not understood in the usual sense, see Remark 3.1. As for the Cauchy problem we understand solutions in an integral sense, see Theorem 3.1.

In this case we find an exponential decay given by the first eigenvalue of an associated problem and the asymptotic behavior of solutions is described by the unique (up to a constant) associated eigenfunction. Let $\lambda_1 = \lambda_1(\Omega)$ be given by:

$$\lambda_1 = \inf_{u \in L^2(\Omega)} \frac{\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(x)-u(y))^2\,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega (u(x))^2\,\mathrm{d}x}, \tag{4}$$

and ϕ_1 an associated eigenfunction (a function where the infimum is attained).

Theorem 2. *For every $u_0 \in L^1(\Omega)$ there exists a unique solution u of* (3) *such that $u \in C([0,\infty); L^1(\Omega))$. Moreover, if $u_0 \in L^2(\Omega)$, solutions decay to zero as $t \to \infty$ with an exponential rate*

$$\big\|u(\cdot,t)\big\|_{L^2(\Omega)} \leqslant \|u_0\|_{L^2(\Omega)} \mathrm{e}^{-\lambda_1 t}. \tag{5}$$

If u_0 is continuous, positive and bounded then there exist positive constants C and C^* such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C\,\mathrm{e}^{-\lambda_1 t}, \tag{6}$$

and

$$\lim_{t\to\infty}\max_x \left|\mathrm{e}^{\lambda_1 t}u(x,t) - C^*\phi_1(x)\right| = 0. \tag{7}$$

1.1.3. The Neumann problem

Let us turn our attention to Neumann boundary conditions. We study:

$$\begin{aligned} &u_t(x,t) = \int_\Omega J(x-y)\big(u(y,t)-u(x,t)\big)\,\mathrm{d}y, \quad x\in\Omega,\ t>0,\\ &u(x,0) = u_0(x), \quad x\in\Omega. \end{aligned} \tag{8}$$

Again solutions are to be understood in an integral sense, see Theorem 4.1. In this model we have that the integral terms take into account the diffusion inside Ω. In fact, as we have explained the integral $\int J(x-y)(u(y,t)-u(x,t))\,\mathrm{d}y$ takes into account the individuals arriving or leaving position x from other places. Since we are integrating in Ω, we are imposing that diffusion takes place only in Ω. The individuals may not enter nor leave Ω. This is the analogous of what is called homogeneous Neumann boundary conditions in the literature.

Again in this case we find that the asymptotic behavior is given by an exponential decay determined by an eigenvalue problem. Let β_1 be given by:

$$\beta_1 = \inf_{u\in L^2(\Omega),\,\int_\Omega u=0} \frac{\frac12\int_\Omega\int_\Omega J(x-y)(u(y)-u(x))^2\,\mathrm{d}y\,\mathrm{d}x}{\int_\Omega (u(x))^2\,\mathrm{d}x}. \tag{9}$$

Concerning the asymptotic behavior of solutions of (8) our last result reads as follows:

Theorem 3. *For every $u_0\in L^1(\Omega)$ there exists a unique solution u of* (8) *such that $u\in C([0,\infty);L^1(\Omega))$. This solution preserves the total mass in Ω*:

$$\int_\Omega u(y,t)\,\mathrm{d}y = \int_\Omega u_0(y)\,\mathrm{d}y.$$

Moreover, let $\varphi = \frac{1}{|\Omega|}\int_\Omega u_0$, then the asymptotic behavior of solutions of (8) *is described as follows*: *if $u_0\in L^2(\Omega)$,*

$$\|u(\cdot,t)-\varphi\|_{L^2(\Omega)} \leqslant \mathrm{e}^{-\beta_1 t}\|u_0-\varphi\|_{L^2(\Omega)}, \tag{10}$$

and if u_0 is continuous and bounded there exist a positive constant C such that

$$\|u(\cdot,t)-\varphi\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\beta_1 t}. \tag{11}$$

1.2. Comments

We will now devote some lines to comment on our results from the qualitative viewpoint, in order to give a clearer picture of the situation.

1.2.1. Absence of regularization

As was said above, there is clearly NO regularizing effect as seen in Lemma 2.2, since the fundamental solution takes the form:

$$u(x,t) = \mathrm{e}^{-t}\delta_0(x) + v(x,t).$$

The function v has no point singularity at $x=0$. Moreover, if $\hat J\in L^1(\mathbb{R}^N)$ then $v\in C^\infty(\mathbb{R}^N\times\mathbb{R}_+)$. This phenomenon is in sharp contrast with what happens for the heat equation, for which an initial condition like δ_0 is automatically regularized and the corresponding solution is C^∞.

One could think that this situation is in some sense close to what happens in the subcritical fast-diffusion case: $u_t = \Delta(u^m)$, with $0 < m \leqslant (N-2)_+/N$. Indeed, it is proved in [5] that the solution with initial data $u_0 = \delta_0$ has a permanent singularity for all positive times, $u(x,t) = \delta_0(x) \otimes 1(t)$, which means that there is no diffusion at all for this special data.

But in fact, the nonlocal equation (1) is a little bit more interesting since some mass transfer occurs. Although the Dirac delta remains at $x = 0$, its mass decays exponentially fast. Thus, total conservation of mass implies that this mass is redistributed in all the surrounding space, through the function $v(x,t)$.

This may be seen as a radiation phenomena, which is a feature shared by the fast diffusion equation in the case $(N-2)_+/N < m < 1$. When considering strong singularities of the kind $\infty \cdot \delta_0$ (see [8]), there is an explicit solution which reads

$$u(x,t) = \left(\frac{Ct}{|x|^2}\right)^{1/(1-m)}.$$

Such a solution has also a standing singularity at $x = 0$, but nevertheless radiation occurs. The only difference is that, in the fast diffusion situation, the singularity has an infinite mass, and the amount of mass spread into the surrounding space will eventually lead to $u(x,t) \to +\infty$ as $t \to \infty$ everywhere.

1.2.2. Influence of the behavior of J

Let us first notice that in the Cauchy problem, if J is compactly supported in $\mathbb{R}^N$, then it has a second momentum, $\int_{\mathbb{R}^N} |x|^2 J(x)\,\mathrm{d}x < +\infty$, and since by symmetry the first momentum of J is null, we necessarily have:

$$\hat{J}(\xi) = 1 - c|\xi|^2 + \mathrm{o}\big(|\xi|^2\big), \quad \xi \to 0,$$

which implies an asymptotic behavior of heat equation type, which is quite surprising since the heat equation is a *local* equation.

The same happens even if J is not compactly supported, but decreases sufficiently fast at infinity (roughly speaking, faster than $|x|^{-(N+2)}$). A well-known example is provided by the Gaussian law, namely in 1-D,

$$J(x) = \mathrm{e}^{-x^2}, \quad \hat{J}(\xi) = \mathrm{e}^{-|\xi|^2} = 1 - |\xi|^2 + \mathrm{o}\big(|\xi|^2\big), \quad \xi \to 0.$$

In general, J may not have a second momentum, so that more general expansions may occur: $\hat{J}(\xi) = 1 - c|\xi|^\alpha + \mathrm{o}(|\xi|^\alpha)$ with $\alpha \in (0,2]$, like it is the case for stable laws of index α (see [13, p. 149]). A typical example (in 1-D) is the Cauchy law,

$$J(x) = \frac{1}{1+|x|^2}, \quad \text{where } \hat{J}(\xi) = 1 - |\xi| + \mathrm{o}\big(|\xi|\big),\ \xi \to 0.$$

Note that this example provides a J that does not have a first momentum but has nevertheless an expansion of the form $\hat{J}(\xi) = 1 - |\xi| + \mathrm{o}(|\xi|)$. In these cases $(0 < \alpha < 2)$, we obtain that the asymptotic behavior is given by the non-local fractional Laplace parabolic equation.

But more diffusions may be considered like for instance the case when

$$\hat{J}(\xi) \sim 1 + \xi^2 \ln \xi \quad \text{as } \xi \to 0.$$

This last case is really interesting since it can be shown (see Section 5) that the asymptotic behavior is still given by a solution of the heat equation, yet viewed in a different time scale. More precisely, if $\hat{J}$ is as above and v is the solution of the heat equation $v_t = (1/2)\Delta v$ with the same initial datum, then

$$\lim_{t\to+\infty} (t \ln t)^{N/2} \max_x \big|u(x,t) - v(x, t\ln t)\big| = 0.$$

1.2.3. On the diffusive effect of the equation

In the case when J has a moment of order 2, then $\hat{J}(\xi) = 1 - A|\xi|^2 + \mathrm{o}(|\xi|^2)$, where A is defined as follows (see Lemma 2.1):

$$-\frac{1}{2} D^2 \hat{J}(0) = \left(\frac{1}{2N}\int x^2 J(x)\,\mathrm{d}x\right)\mathrm{Id} = A \cdot \mathrm{Id}.$$

Since the first moment of J is null, its second moment measures the dispersion of J around its mean, which is zero. Now, the asymptotic behavior of solutions to (1) is related to those of the heat equation with speed $c = A^{1/2}$. This means that the more dispersed J is, the greater the speed.

This effect can be understood as follows: if J is not dispersed, then almost no diffusion occurs since $J * u \approx u$, the limit case being $J = \delta_0$ for which the equation becomes: $u_t = \delta_0 * u - u = 0$. Thus for concentrated J's, the diffusion effect is very small, which is also visible in the asymptotic behavior since the speed of the Gaussian profile is also small.

On the contrary, when J is very dispersed, $(J * u)(x_0, t)$ will take into accounts values of the density u situated at points "far" from x_0 so that a great diffusion effect occurs. This is reflected in the asymptotic Gaussian profile which has a great velocity.

1.2.4. *The frequency viewpoint*

A simple way to understand our results in the Cauchy problem is the following: the behavior (2) means that at low frequencies ($\xi \sim 0$), the operator is very much like the fractional Laplacian (usual Laplacian if $\alpha = 2$). Now, as time evolves, diffusion occurs and high frequencies of the initial data go to zero, this is reflected in the explicit frequency solution (see Theorem 2.1):

$$\hat{u}(\xi, t) = \mathrm{e}^{(\hat{J}(\xi)-1)t}\hat{u}_0(\xi).$$

Indeed, if J is a L^1 function, then it happens that $\hat{J}(\xi) \to 0$ as $\xi \to \infty$, so that for $|\xi| \gg 1$, the high frequencies of u_0 are multiplied by something decreasing exponentially fast in time (this could be different in the case when J is a measure, but we do not consider such a case here).

Thus, roughly speaking, only low frequencies of the solution will play an important role in the asymptotic behavior as $t \to \infty$, which explains why we obtain something similar to the fractional Laplacian equation (or heat equation) in the rescaled limit.

And in fact what we do in the proof of Theorem 1 is precisely to separate the low frequencies where we use the expansion (2) from the high frequencies that we control since they tend to zero fast enough in a suitable time scale.

1.2.5. *Asymptotics in bounded domains*

In the case of bounded domains, the asymptotic behavior of solutions is NOT related to the behavior of $\hat{J}$ near zero. Indeed, this case is similar to the case when J is compactly supported, since the operator will not take into account values of u at $|x| = +\infty$. The asymptotic behavior thus depends only on the eigenvalues of the operator (whether in Dirichlet or Neumann problems). However, if the domain is unbounded the behavior of J at infinity may enter into play (see Section 5).

1.2.6. *Probabilistic interpretation*

Recently, E. Lesigne and M. Peigné [19] turned our attention on the fact that the problem we study has a clear probabilistic interpretation, that we briefly explain below.

Let $(E, \mathcal{E})$ be a measurable space and $P : E \times \mathcal{E} \to [0, 1]$ be a probability transition on E. Then we define a Markovian transition function as follows: for any $x \in E$, $\mathcal{A} \in \mathcal{E}$, let

$$P_t(x, \mathcal{A}) = \mathrm{e}^{-t} \sum_{n=0}^{+\infty} \frac{t^n}{n!} P^{(n)}(x, \mathcal{A}), \quad t \in \mathbb{R}_+,$$

where $P^{(n)}$ denotes the nth iterate of P acting on the space of bounded measurable functions on E. The associated family of Markovian operators, $P_t f(x) = \int f(y) P_t(x, \mathrm{d}y)$ satisfies:

$$\frac{\partial}{\partial t} P_t f(x) = \int P_t f(y) P(x, \mathrm{d}y) - P_t f(x).$$

If we consider a Markov process $(Z_t)_{t \geqslant 0}$ associated to the transition function $(P_t)_{t \geqslant 0}$, and if we denote by μ_t the distribution of Z_t, then the family $(\mu_t)_{t \geqslant 0}$ satisfies also a linear partial differential equation:

$$\frac{\partial}{\partial t} \mu_t = \int P(y, \cdot) \mu_t(\mathrm{d}y) - \mu_t.$$

In particular, if $E = \mathbb{R}^N$, if the probability transition $P(x, \mathrm{d}y)$ has a density $y \mapsto J(x, y)$, and if μ_t has a density $y \mapsto u(y, t)$, then the following equation is satisfied,

$$\frac{\partial}{\partial t}u(x,t) = \int J(y,x)u(y,t)\,\mathrm{d}\lambda(y) - u(x,t). \tag{12}$$

With different particular choices of P we recover the equation studied in the Cauchy, the Dirichlet and the Neumann cases. For example, if $P(x, \mathrm{d}y) = J(y-x)\,\mathrm{d}y$ is the transition probability of a random walk, Eq. (12) is just Eq. (1). In this particular case, the asymptotic behavior described in Theorem 1 can be obtained as a consequence of the so-called "Local Limit Theorem for Random Walks" which is a classical result in probability theory (see Theorem 1 (p. 506) and Theorem 2 (p. 508) in [15]).

In the Dirichlet and Neumann cases, the results described in the present article give interesting information on the asymptotic behavior of some natural Markov process in the space.

1.3. Organization of the paper

The rest of the paper is organized as follows: in Section 2 we prove Theorem 1 and we also find the estimate of the decay of the L^p-norms; in Section 3 we deal with the Dirichlet problem; in Section 4 we analyze the behavior of the Neumann problem and finally in Section 5 we discuss some possible extensions of this work.

2. The Cauchy problem. Proof of Theorem 1

In this section, we shall make an extensive use of the Fourier transform in order to obtain explicit solutions in frequency formulation. Let us recall (see for instance [18]) that if $f \in L^1(\mathbb{R}^N)$ then $\hat{f}$ and $\check{f}$ are bounded and continuous, where $\hat{f}$ is the Fourier transform of f and $\check{f}$ its inverse Fourier transform. Moreover,

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \check{f}(x) = 0.$$

We begin by collecting some properties of the function J.

Lemma 2.1. *Let J satisfy hypotheses* (H). *Then,*

(i) $|\hat{J}(\xi)| \leqslant 1$, $\hat{J}(0) = 1$.
(ii) *If $\int_{\mathbb{R}^N} J(x)|x|\,\mathrm{d}x < +\infty$, then*

$$(\nabla_\xi \hat{J})_i(0) = -\mathrm{i}\int_{\mathbb{R}^N} x_i J(x)\,\mathrm{d}x = 0,$$

and if $\int_{\mathbb{R}^N} J(x)|x|^2\,\mathrm{d}x < +\infty$, then

$$(D^2\hat{J})_{ij}(0) = -\int_{\mathbb{R}^N} x_i x_j J(x)\,\mathrm{d}x,$$

therefore $(D^2\hat{J})_{ij}(0) = 0$ when $i \neq j$ and $(D^2\hat{J})_{ii}(0) \neq 0$. Hence the Hessian matrix of $\hat{J}$ at the origin is given by:

$$D^2\hat{J}(0) = -\left(\frac{1}{N}\int_{\mathbb{R}^N} |x|^2 J(x)\,\mathrm{d}x\right)\cdot \mathrm{Id}.$$

(iii) *If $\hat{J}(\xi) = 1 - A|\xi|^\alpha + \mathrm{o}(|\xi|)^\alpha$ then necessarily $\alpha \in (0, 2]$, and if J has a first momentum, then $\alpha \neq 1$. Finally, if $\alpha = 2$, then*

$$A\cdot \mathrm{Id} = -(1/2)(D^2\hat{J})_{ij}(0).$$

Proof. Points (i) and (ii) are rather straightforward (recall that J is radially symmetric). Now we turn to (iii). Let us recall a well-known probability lemma (see for instance Theorem 3.9 in [13]) that says that if $\hat{J}$ has an expansion of the form,

$$\hat{J}(\xi) = 1 + \mathrm{i}\langle a, \xi\rangle - \frac{1}{2}\langle \xi, B\xi\rangle + \mathrm{o}\big(|\xi|^2\big),$$

then J has a second momentum and we have:

$$a_i = \int x_i J(x)\,\mathrm{d}x, \qquad B_{ij} = \int x_i x_j J(x)\,\mathrm{d}x < \infty.$$

Thus if (iii) holds for some $\alpha > 2$, it would turn out that the second moment of J is null, which would imply that $J \equiv 0$, a contradiction. Finally, when $\alpha = 2$, then clearly $B_{ij} = -(D^2\hat{J})_{ij}(0)$ hence the result since by symmetry, the Hessian is diagonal. □

Now, we first prove existence and uniqueness of solutions using the Fourier transform.

Theorem 2.1. *Let $u_0 \in L^1(\mathbb{R}^N)$ such that $\hat{u_0} \in L^1(\mathbb{R}^N)$. There exists a unique solution $u \in C^0([0,\infty); L^1(\mathbb{R}^N))$ of* (1)*, and it is given by*:

$$\hat{u}(\xi, t) = \mathrm{e}^{(\hat{J}(\xi)-1)t}\hat{u}_0(\xi).$$

Proof. We have:

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,\mathrm{d}y - u(x,t).$$

Applying the Fourier transform to this equation, we obtain:

$$\hat{u}_t(\xi,t) = \hat{u}(\xi,t)\big(\hat{J}(\xi) - 1\big).$$

Hence,

$$\hat{u}(\xi, t) = \mathrm{e}^{(\hat{J}(\xi)-1)t}\hat{u}_0(\xi).$$

Since $\hat{u}_0 \in L^1(\mathbb{R}^N)$ and $\mathrm{e}^{(\hat{J}(\xi)-1)t}$ is continuous and bounded, the result follows by taking the inverse of the Fourier transform. □

Remark 2.1. One can also understand solutions of (1) directly in Fourier variables. This concept of solution is equivalent to the integral one in the original variables under our hypotheses on the initial condition.

Now we prove a lemma concerning the fundamental solution of (1).

Lemma 2.2. *Let $J \in \mathcal{S}(\mathbb{R}^N)$, the space of rapidly decreasing functions. The fundamental solution of* (1)*, that is the solution of* (1) *with initial condition $u_0 = \delta_0$, can be decomposed as*

$$w(x,t) = \mathrm{e}^{-t}\delta_0(x) + v(x,t), \tag{13}$$

with $v(x,t)$ smooth. Moreover, if u is a solution of (1) *it can be written as*

$$u(x,t) = (w * u_0)(x,t) = \int_{\mathbb{R}^N} w(x-z,t)u_0(z)\,\mathrm{d}z.$$

Proof. By the previous result, we have:

$$\hat{w}_t(\xi,t) = \hat{w}(\xi,t)\big(\hat{J}(\xi) - 1\big).$$

Hence, as the initial datum verifies $\hat{u}_0 = \hat{\delta}_0 = 1$,

$$\hat{w}(\xi,t) = \mathrm{e}^{(\hat{J}(\xi)-1)t} = \mathrm{e}^{-t} + \mathrm{e}^{-t}\big(\mathrm{e}^{\hat{J}(\xi)t} - 1\big).$$

The first part of the lemma follows applying the inverse Fourier transform in $\mathcal{S}(\mathbb{R}^N)$.

To finish the proof we just observe that $w * u_0$ is a solution of (1) (just use Fubini's theorem) with $(w * u_0)(x, 0) = u_0(x)$. □

Remark 2.2. The above proof together with the fact that $\hat{J}(\xi) \to 0$ (since $J \in L^1(\mathbb{R}^N)$) shows that if $\hat{J} \in L^1(\mathbb{R}^N)$ then the same decomposition (13) holds and the result also applies.

Next, we prove the first part of Theorem 1.

Theorem 2.2. *Let u be a solution of* (1) *with $u_0, \hat{u}_0 \in L^1(\mathbb{R}^N)$. If*

$$\hat{J}(\xi) = 1 - A|\xi|^\alpha + \mathrm{o}\big(|\xi|^\alpha\big), \quad \xi \to 0,$$

the asymptotic behavior of $u(x, t)$ is given by:

$$\lim_{t\to+\infty} t^{N/\alpha} \max_x \big|u(x, t) - v(x, t)\big| = 0,$$

where v is the solution of $v_t(x, t) = -A(-\Delta)^{\alpha/2} v(x, t)$ with initial condition $v(x, 0) = u_0(x)$.

Proof. As in the proof of the previous lemma, we have:

$$\hat{u}_t(\xi, t) = \hat{u}(\xi, t)\big(\hat{J}(\xi) - 1\big).$$

Hence,

$$\hat{u}(\xi, t) = \mathrm{e}^{(\hat{J}(\xi)-1)t} \hat{u}_0(\xi).$$

On the other hand, let $v(x, t)$ be a solution of

$$v_t(x, t) = -A(-\Delta)^{\alpha/2} v(x, t),$$

with the same initial datum $v(x, 0) = u_0(x)$. Solutions of this equation are understood in the sense that

$$\hat{v}(\xi, t) = \mathrm{e}^{-A|\xi|^\alpha t} \hat{u}_0(\xi).$$

Hence in Fourier variables,

$$\int_{\mathbb{R}^N} |\hat{u} - \hat{v}|(\xi, t)\,\mathrm{d}\xi = \int_{\mathbb{R}^N} \big|\big(\mathrm{e}^{t(\hat{J}(\xi)-1)} - \mathrm{e}^{-A|\xi|^\alpha t}\big)\hat{u}_0(\xi)\big|\,\mathrm{d}\xi$$
$$\leqslant \int_{|\xi|\geqslant r(t)} \big|\big(\mathrm{e}^{t(\hat{J}(\xi)-1)} - \mathrm{e}^{-A|\xi|^\alpha t}\big)\hat{u}_0(\xi)\big|\,\mathrm{d}\xi + \int_{|\xi|<r(t)} \big|\big(\mathrm{e}^{t(\hat{J}(\xi)-1)} - \mathrm{e}^{-A|\xi|^\alpha t}\big)\hat{u}_0(\xi)\big|\,\mathrm{d}\xi = I + II.$$

To get a bound for I we proceed as follows, we decompose it in two parts,

$$I \leqslant \int_{|\xi|\geqslant r(t)} \big|\mathrm{e}^{-A|\xi|^\alpha t} \hat{u}_0(\xi)\big|\,\mathrm{d}\xi + \int_{|\xi|\geqslant r(t)} \big|\mathrm{e}^{t(\hat{J}(\xi)-1)} \hat{u}_0(\xi)\big|\,\mathrm{d}\xi = I_1 + I_2.$$

First, we deal with I_1. We have:

$$t^{N/\alpha} \int_{|\xi|>r(t)} \mathrm{e}^{-A|\xi|^\alpha t} \big|\hat{u}_0(\xi)\big|\,\mathrm{d}\xi \leqslant \|\hat{u}_0\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta|>r(t)t^{1/\alpha}} \mathrm{e}^{-A|\eta|^\alpha} \to 0,$$

as $t \to \infty$ if we impose that

$$r(t)t^{1/\alpha} \to \infty \quad \text{as } t \to \infty. \tag{14}$$

Now, remark that from our hypotheses on J, we have that $\hat{J}$ verifies:

$$\hat{J}(\xi) \leqslant 1 - A|\xi|^\alpha + |\xi|^\alpha h(\xi),$$

where h is bounded and $h(\xi) \to 0$ as $\xi \to 0$. Hence there exists $D > 0$ such that

$$\hat{J}(\xi) \leqslant 1 - D|\xi|^{\alpha}, \quad \text{for } |\xi| \leqslant a,$$

and $\delta > 0$ such that

$$\hat{J}(\xi) \leqslant 1 - \delta, \quad \text{for } |\xi| \geqslant a.$$

Therefore, I_2 can be bounded by

$$\begin{aligned}\int\limits_{|\xi|\geqslant r(t)} \left|e^{t(\hat{J}(\xi)-1)}\hat{u}_0(\xi)\right| d\xi &\leqslant \int\limits_{a\geqslant|\xi|\geqslant r(t)} \left|e^{t(\hat{J}(\xi)-1)}\hat{u}_0(\xi)\right| d\xi + \int\limits_{|\xi|\geqslant a} \left|e^{t(\hat{J}(\xi)-1)}\hat{u}_0(\xi)\right| d\xi \\ &\leqslant \int\limits_{a\geqslant|\xi|\geqslant r(t)} \left|e^{t(\hat{J}(\xi)-1)}\hat{u}_0(\xi)\right| d\xi + Ce^{-\delta t}.\end{aligned}$$

Using this bound and changing variables, $\eta = \xi t^{1/\alpha}$,

$$\begin{aligned} t^{N/\alpha} I_2 &\leqslant C \int\limits_{at^{1/\alpha}\geqslant|\eta|\geqslant t^{1/\alpha}r(t)} \left|e^{-D|\eta|^{\alpha}}\hat{u}_0\big(\eta t^{-1/\alpha}\big)\right| d\eta + t^{N/\alpha}Ce^{-\delta t} \\ &\leqslant C \int\limits_{|\eta|\geqslant t^{1/\alpha}r(t)} e^{-D|\eta|^{\alpha}}\, d\eta + t^{N/\alpha}Ce^{-\delta t},\end{aligned}$$

and then

$$t^{N/\alpha} I_2 \to 0 \quad \text{as } t \to \infty,$$

if (14) holds.

Now we estimate II as follows:

$$t^{N/\alpha} \int\limits_{|\xi|<r(t)} \left|e^{(\hat{J}(\xi)-1+A|\xi|^{\alpha})t} - 1\right| e^{-A|\xi|^{\alpha}t} \left|\hat{u}_0(\xi)\right| d\xi \leqslant Ct^{N/\alpha} \int\limits_{|\xi|<r(t)} t|\xi|^{\alpha}h(\xi)e^{-A|\xi|^{\alpha}t}\, d\xi,$$

provided we impose

$$t\big(r(t)\big)^{\alpha} h\big(r(t)\big) \to 0 \quad \text{as } t \to \infty. \tag{15}$$

In this case, we have

$$t^{N/\alpha} II \leqslant C \int\limits_{|\eta|<r(t)t^{1/\alpha}} |\eta|^{\alpha} h\big(\eta/t^{1/\alpha}\big) e^{-A|\eta|^{\alpha}}\, d\eta,$$

and we use dominated convergence, $h(\eta/t^{1/\alpha}) \to 0$ as $t \to \infty$ while the integrand is dominated by $\|h\|_{\infty}|\eta|^{\alpha} \times \exp(-c|\eta|^{\alpha})$, which belongs to $L^1(\mathbb{R}^N)$.

This shows that

$$t^{N/\alpha}(I + II) \to 0 \quad \text{as } t \to \infty, \tag{16}$$

provided we can find a $r(t) \to 0$ as $t \to \infty$ which fulfills both conditions (14) and (15). This is done in Lemma 2.3, which is postponed just after the end of the present proof. To conclude, we only have to observe that from (16) we obtain:

$$t^{N/\alpha} \max_x \left|u(x,t) - v(x,t)\right| \leqslant t^{N/\alpha} \int\limits_{\mathbb{R}^N} \left|\hat{u} - \hat{v}\right|(\xi,t)\, d\xi \to 0, \quad t \to \infty,$$

which ends the proof of the theorem. □

The following lemma shows that there exists a function $r(t)$ satisfying (14) and (15), as required in the proof of the previous theorem.

Lemma 2.3. *Given a function $h \in C(\mathbb{R}, \mathbb{R})$ such that $h(\rho) \to 0$ as $\rho \to 0$ with $h(\rho) > 0$ for small ρ, there exists a function r with $r(t) \to 0$ as $t \to \infty$ which satisfies*:

$$\lim_{t\to\infty} r(t)t^{1/\alpha} = \infty$$

and

$$\lim_{t\to\infty} t\big(r(t)\big)^\alpha h\big(r(t)\big) = 0.$$

Proof. For fixed t large enough, we choose $r(t)$ as a small solution of

$$r\big(h(r)\big)^{1/(2\alpha)} = t^{-1/\alpha}. \tag{17}$$

This equation defines a function $r = r(t)$ which, by continuity arguments, goes to zero as t goes to infinity. Indeed, if there exists $t_n \to \infty$ with no solution of (17) for $r \in (0, \delta)$ then $h(r) \equiv 0$ in $(0, \delta)$ a contradiction. □

Remark 2.3. In the case when $h(t) = t^s$ with $s > 0$, we can look for a function h of power-type, $r(t) = t^\beta$ with $\beta < 0$ and the two conditions read as follows:

$$\beta + 1/\alpha > 0, \qquad 1 + \beta\alpha + s\beta < 0. \tag{18}$$

This implies that $\beta \in (-1/\alpha, -1/(\alpha + s))$ which is of course always possible.

As a consequence of Theorem 2.2, we obtain the following corollary which completes the results gathered in Theorem 1 in the Introduction.

Corollary 2.1. *If $\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$, $\xi \to 0$, $0 < \alpha \leqslant 2$, the asymptotic behavior of solutions of* (1) *is given by*:

$$\big\|u(\cdot,t)\big\|_{L^\infty(\mathbb{R}^N)} \leqslant \frac{C}{t^{N/\alpha}}.$$

Moreover, the asymptotic profile is given by:

$$\lim_{t\to+\infty} \max_y \big|t^{N/\alpha} u\big(yt^{1/\alpha}, t\big) - \|u_0\|_{L^1} G_A(y)\big| = 0,$$

where $G_A(y)$ satisfies $\widehat{G}_A(\xi) = e^{-A|\xi|^\alpha}$.

Proof. From Theorem 2.2 we obtain that the asymptotic behavior is the same as the one for solutions of the evolution given by the fractional Laplacian.

It is easy to check that this asymptotic behavior is exactly the one described in the statement of the corollary. Indeed, in Fourier variables we have for $t \to \infty$,

$$\hat{v}(t^{-1/\alpha}\eta, t) = e^{-A|\eta|^\alpha} \hat{u}_0\big(\eta t^{-1/\alpha}\big) \to e^{-A|\eta|^\alpha} \hat{u}_0(0) = e^{-A|\eta|^\alpha} \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Therefore

$$\lim_{t\to+\infty} \max_y \big|t^{N/\alpha} v\big(yt^{1/\alpha}, t\big) - \|u_0\|_{L^1} G_A(y)\big| = 0,$$

where $G_A(y)$ satisfies $\widehat{G}_A(\xi) = e^{-A|\xi|^\alpha}$. □

To end this section we find the decay rate in L^p of solutions of (1).

Corollary 2.2. *Let $1 < p < \infty$. If $\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$, $\xi \to 0$, $0 < \alpha \leqslant 2$, then, the decay of the L^p-norm of the solution of* (1) *is given by*:

$$\big\|u(\cdot,t)\big\|_{L^p(\mathbb{R}^N)} \leqslant Ct^{-\frac{N}{\alpha}(1-\frac{1}{p})}.$$

Proof. By interpolation, see [4], we have:

$$\|u\|_{L^p(\mathbb{R}^N)} \leqslant \|u\|_{L^1(\mathbb{R}^N)}^{1/p} \|u\|_{L^\infty(\mathbb{R}^N)}^{1-1/p}.$$

As (1) preserves the L^1 norm, the result follows from the previous results that give the decay in L^∞ of the solutions. □

3. The Dirichlet problem. Proof of Theorem 2

In this section we assume that J is continuous and verifies (H). Recall that a solution of the Dirichlet problem is defined as follows: $u \in C([0,\infty); L^1(\Omega))$ satisfying:

$$\begin{aligned} &u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,\mathrm{d}y - u(x,t), \quad x \in \Omega,\ t>0,\\ &u(x,t) = 0, \quad x \notin \Omega,\ t>0,\\ &u(x,0) = u_0(x), \quad x \in \Omega. \end{aligned} \tag{19}$$

Before studying the asymptotic behavior, we shall first derive existence and uniqueness of solutions, which is a consequence of Banach's fixed point theorem.

Fix $t_0 > 0$ and consider the Banach space:

$$X_{t_0} = \{ w \in C([0,t_0]; L^1(\Omega)) \},$$

with the norm

$$|||w||| = \max_{0 \leqslant t \leqslant t_0} \|w(\cdot,t)\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator $\mathcal{T} : X_{t_0} \to X_{t_0}$ defined by:

$$\begin{aligned} &\mathcal{T}_{w_0}(w)(x,t) = w_0(x) + \int_0^t \int_{\mathbb{R}^N} J(x-y)\big(w(y,s) - w(x,s)\big)\,\mathrm{d}y\,\mathrm{d}s,\\ &\mathcal{T}_{w_0}(w)(x,t) = 0, \quad x \notin \Omega. \end{aligned}$$

Lemma 3.1. *Let $w_0, z_0 \in L^1(\Omega)$ and $w, z \in X_{t_0}$, then there exists a constant C depending on J and Ω such that*

$$\big|\big|\big| \mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z) \big|\big|\big| \leqslant C t_0 |||w - z||| + \|w_0 - z_0\|_{L^1(\Omega)}.$$

Proof. We have:

$$\begin{aligned} \int_\Omega \big|\mathcal{T}_{w_0}(w)(x,t) - \mathcal{T}_{z_0}(z)(x,t)\big|\,\mathrm{d}x &= \int_\Omega |w_0 - z_0|(x)\,\mathrm{d}x \\ &+ \int_\Omega \left| \int_0^t \int_{\mathbb{R}^N} J(x-y)\big[\big(w(y,s) - z(y,s)\big) - \big(w(x,s) - z(x,s)\big)\big]\,\mathrm{d}y\,\mathrm{d}s \right| \mathrm{d}x. \end{aligned}$$

Hence, taking into account that w and z vanish outside Ω,

$$\big|\big|\big| \mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z) \big|\big|\big| \leqslant \|w_0 - z_0\|_{L^1(\Omega)} + C t_0 |||w - z|||,$$

as we wanted to prove. □

Theorem 3.1. *For every $u_0 \in L^1(\Omega)$ there exists a unique solution u, such that $u \in C([0,\infty); L^1(\Omega))$.*

Proof. We check first that $\mathcal{T}_{u_0}$ maps X_{t_0} into X_{t_0}. Taking z_0, $z \equiv 0$ in Lemma 3.1 we get that $\mathcal{T}(w) \in C([0,t_0]; L^1(\Omega))$.

Choose t_0 such that $Ct_0 < 1$. Now taking $z_0 \equiv w_0 \equiv u_0$ in Lemma 3.1 we get that $\mathcal{T}_{u_0}$ is a strict contraction in X_{t_0} and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem in the interval $[0,t_0]$. To extend the solution to $[0,\infty)$ we may take as initial data $u(x,t_0) \in L^1(\Omega)$ and obtain a solution up to $[0,2t_0]$. Iterating this procedure we get a solution defined in $[0,\infty)$. □

Next we look for steady states of (3).

Proposition 3.1. *$u \equiv 0$ is the unique stationary solution of* (3).

Proof. Let u be a stationary solution of (3). Then

$$0 = \int_{\mathbb{R}^N} J(x-y)\big(u(y)-u(x)\big)\,\mathrm{d}y, \quad x \in \Omega,$$

and $u(x) = 0$ for $x \notin \Omega$. Hence, using that $\int J = 1$ we obtain that for every $x \in \mathbb{R}^N$ it holds,

$$u(x) = \int_{\mathbb{R}^N} J(x-y)u(y)\,\mathrm{d}y.$$

This equation, together with $u(x) = 0$ for $x \notin \Omega$, implies that $u \equiv 0$. □

Now, let us analyze the asymptotic behavior of the solutions. As there exists a unique stationary solution, it is expected that solutions converge to zero as $t \to \infty$. Our main concern will be the rate of convergence.

First, let us look the eigenvalue given by (4), that is we look for the first eigenvalue:

$$u(x) - \int_{\mathbb{R}^N} J(x-y)u(y)\,\mathrm{d}y = \lambda_1 u(x). \tag{20}$$

This is equivalent to,

$$(1-\lambda_1)u(x) = \int_{\mathbb{R}^N} J(x-y)u(y)\,\mathrm{d}y. \tag{21}$$

Let $T : L^2(\Omega) \to L^2(\Omega)$ be the operator given by:

$$T(u)(x) := \int_{\mathbb{R}^N} J(x-y)u(y)\,\mathrm{d}y.$$

In this definition we have extended by zero a function in $L^2(\Omega)$ to the whole $\mathbb{R}^N$. Hence we are looking for the largest eigenvalue of T. Since T is compact this eigenvalue is attained at some function $\phi_1(x)$ that turns out to be an eigenfunction for our original problem (20).

By taking $|\phi_1|$ instead of ϕ_1 in (4) we may assume that $\phi_1 \geqslant 0$ in Ω. Indeed, one simply has to use the fact that $(a-b)^2 \geqslant (|a|-|b|)^2$.

Next, we analyze some properties of the eigenvalue problem (20).

Proposition 3.2. *Let λ_1 the first eigenvalue of* (20) *and denote by $\phi_1(x)$ a corresponding non-negative eigenfunction. Then $\phi_1(x)$ is strictly positive in Ω and λ_1 is a positive simple eigenvalue with $\lambda_1 < 1$.*

Proof. In what follows, we denote by $\bar{\phi}_1$ the natural continuous extension of ϕ_1 to $\bar{\Omega}$. We begin with the positivity of the eigenfunction ϕ_1. Assume for contradiction that the set $\mathbf{B} = \{x \in \Omega\colon\ \phi_1(x) = 0\}$ is non-void. Then, from the continuity of ϕ_1 in Ω, we have that $\mathbf{B}$ is closed. We next prove that $\mathbf{B}$ is also open, and hence, since Ω is connected, standard topological arguments allows to conclude that $\Omega \equiv \mathbf{B}$ yielding to a contradiction. Consider $x_0 \in \mathbf{B}$. Since

$\phi_1 \geqslant 0$, we obtain from (21) that $\Omega \cap B_1(x_0) \in \mathbf{B}$. Hence $\mathbf{B}$ is open and the result follows. Analogous arguments apply to prove that $\bar{\phi}_1$ is positive in $\bar{\Omega}$.

Assume now for contradiction that $\lambda_1 \leqslant 0$ and denote by M^* the maximum of $\bar{\phi}_1$ in $\bar{\Omega}$ and by x^* a point where such maximum is attained. Assume for the moment that $x^* \in \Omega$. From Proposition 3.1, one can choose x^* in such a way that $\phi_1(x) \neq M^*$ in $\Omega \cap B_1(x^*)$. By using (21) we obtain that,

$$M^* \leqslant (1-\lambda_1)\phi_1(x^*) = \int_{\mathbb{R}^N} J(x^*-y)\phi_1(y) < M^*$$

and a contradiction follows. If $x^* \in \partial\Omega$, we obtain a similar contradiction after substituting and passing to the limit in (21) on a sequence $\{x_n\} \in \Omega$, $x_n \to x^*$ as $n \to \infty$. To obtain the upper bound, assume that $\lambda_1 \geqslant 1$. Then, from (21) we obtain for every $x \in \Omega$ that

$$0 \geqslant (1-\lambda_1)\phi_1(x^*) = \int_{\mathbb{R}^N} J(x^*-y)\phi_1(y)$$

a contradiction with the positivity of ϕ_1.

Finally, to prove that λ_1 is a simple eigenvalue, let $\phi_1 \neq \phi_2$ be two different eigenfunctions associated to λ_1 and define

$$C^* = \inf\{C > 0\colon\ \bar{\phi}_2(x) \leqslant C\bar{\phi}_1(x),\ x \in \overline{\Omega}\}.$$

The regularity of the eigenfunctions and the previous analysis shows that C^* is nontrivial and bounded. Moreover from its definition, there must exists $x^* \in \bar{\Omega}$ such that $\bar{\phi}_2(x^*) = C^*\bar{\phi}_1(x^*)$. Define $\phi(x) = C^*\phi_1(x) - \phi_2(x)$. From the linearity of (20), we have that ϕ is a non-negative eigenfunction associated to λ_1 with $\bar{\phi}(x^*) = 0$. From he positivity of the eigenfunctions stated above, it must be $\phi \equiv 0$. Therefore, $\phi_2(x) = C^*\phi_1(x)$ and the result follows. This completes the proof. □

Remark 3.1. Note that the first eigenfunction ϕ_1 is strictly positive in Ω (with positive continuous extension to $\bar{\Omega}$) and vanishes outside Ω. Therefore a discontinuity occurs on $\partial\Omega$ and the boundary value is not taken in the usual "classical" sense.

Proof of Theorem 2. Using the symmetry of J, we have:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\int_{\Omega} u^2(x,t)\,\mathrm{d}x\right) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\big[u(y,t)-u(x,t)\big]u(x,t)\,\mathrm{d}y\,\mathrm{d}x \\ &= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\big[u(y,t)-u(x,t)\big]^2\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

From the definition of λ_1, (4), we get:

$$\frac{\partial}{\partial t}\int_{\Omega} u^2(x,t)\,\mathrm{d}x \leqslant -2\lambda_1\int_{\Omega} u^2(x,t)\,\mathrm{d}x.$$

Therefore

$$\int_{\Omega} u^2(x,t)\,\mathrm{d}x \leqslant \mathrm{e}^{-2\lambda_1 t}\int_{\Omega} u_0^2(x)\,\mathrm{d}x$$

and we have obtained (5).

We now establish the decay rate and the convergence stated in (6) and (7) respectively. Consider a nontrivial and non-negative continuous initial data $u_0(x)$ and let $u(x,t)$ be the corresponding solution to (1). We first note that $u(x,t)$ is a continuous function satisfying $u(x,t) > 0$ for every $x \in \Omega$ and $t > 0$, and the same holds for $\bar{u}(x,t)$, the unique natural continuous extension of $u(x,t)$ to $\overline{\Omega}$. This instantaneous positivity can be obtained by using analogous topological arguments to those in Proposition 3.2.

In order to deal with the asymptotic analysis, is more convenient to introduce the rescaled function $v(x,t) = \mathrm{e}^{\lambda_1 t} u(x,t)$. By substituting in (1), we find that the function $v(x,t)$ satisfies:

$$v_t(x,t) = \int_{\mathbb{R}^N} J(x-y)v(y,t)\,\mathrm{d}y - (1-\lambda_1)v(x,t). \tag{22}$$

On the other hand, we have that $C\phi_1(x)$ is a solution of (22) for every $C \in \mathbb{R}$ and moreover, it follows from the eigenfunction analysis above, that the set of stationary solutions of (22) is given by $\mathbf{S}^* = \{C\phi_1,\ C \in \mathbb{R}\}$.

Define now for every $t > 0$, the function:

$$C^*(t) = \inf\bigl\{C > 0\colon\ v(x,t) \leqslant C\phi_1(x),\ x \in \Omega\bigr\}.$$

By definition and by using the linearity of Eq. (22), we have that $C^*(t)$ is a non-increasing function. In fact, this is a consequence of the comparison principle applied to the solutions $C^*(t_1)\phi_1(x)$ and $v(x,t)$ for t larger than any fixed $t_1 > 0$. It implies that $C^*(t_1)\phi_1(x) \geqslant v(x,t)$ for every $t \geqslant t_1$, and therefore, $C^*(t_1) \geqslant C^*(t)$ for every $t \geqslant t_1$. In an analogous way, one can see that the function

$$C_*(t) = \sup\bigl\{C > 0\colon\ v(x,t) \geqslant C\phi_1(x),\ x \in \Omega\bigr\},$$

is non-decreasing. These properties imply that both limits exist,

$$\lim_{t\to\infty} C^*(t) = K^* \quad \text{and} \quad \lim_{t\to\infty} C_*(t) = K_*,$$

and also provides the compactness of the orbits necessary in order passing to the limit (after subsequences if needed) to obtain that $v(\cdot, t + t_n) \to w(\cdot, t)$ as $t_n \to \infty$ uniformly on compact subsets in $\Omega \times \mathbb{R}_+$ and that $w(x,t)$ is a continuous function which satisfies (22). We also have for every $g \in \omega(u_0)$ there holds,

$$K_*\phi_1(x) \leqslant g(x) \leqslant K^*\phi_1(x).$$

Moreover, $C^*(t)$ plays a role of a Lyapunov function and this fact allows to conclude that $\omega(u_0) \subset \mathbf{S}^*$ and the uniqueness of the convergence profile. In more detail, assume that $g \in \omega(u_0)$ does not belong to $\mathbf{S}^*$ and consider $w(x,t)$ the solution of (22) with initial data $g(x)$ and define:

$$C^*(w)(t) = \inf\bigl\{C > 0\colon\ w(x,t) \leqslant C\phi_1(x),\ x \in \Omega\bigr\}.$$

It is clear that $W(x,t) = K^*\phi_1(x) - w(x,t)$ is a non-negative continuous solution of (22) and it becomes strictly positive for every $t > 0$. This implies that there exists $t^* > 0$ such that $C^*(w)(t^*) < K^*$ and by the convergence, the same holds before passing to the limit. Hence, $C^*(t^* + t_j) < K^*$ if j is large enough and a contradiction with the properties of $C^*(t)$ follows. The same arguments allow to establish the uniqueness of the convergence profile. □

4. The Neumann problem. Proof of Theorem 3

As we did for the Dirichlet problem, we assume that J is continuous. Solutions of the Neumann problem are functions $u \in C([0,\infty); L^1(\Omega))$ which satisfy:

$$\begin{aligned} &u_t(x,t) = \int_{\Omega} J(x-y)\bigl(u(y,t) - u(x,t)\bigr)\,\mathrm{d}y, \quad x \in \Omega,\ t > 0,\\ &u(x,0) = u_0(x), \quad x \in \Omega. \end{aligned} \tag{23}$$

As in the previous section, see also [11], existence and uniqueness will be a consequence of Banach's fixed point theorem. The main arguments are basically the same but we repeat them here to make this section self-contained.

Fix $t_0 > 0$ and consider the Banach space,

$$X_{t_0} = C\bigl([0,t_0]; L^1(\Omega)\bigr),$$

with the norm

$$|||w||| = \max_{0\leqslant t\leqslant t_0} \bigl\|w(\cdot,t)\bigr\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator $\mathcal{T} : X_{t_0} \to X_{t_0}$ defined by:

$$\mathcal{T}_{w_0}(w)(x,t) = w_0(x) + \int_0^t \int_\Omega J(x-y)\big(w(y,s) - w(x,s)\big)\,dy\,ds. \tag{24}$$

The following lemma is the main ingredient in the proof of existence.

Lemma 4.1. *Let $w_0, z_0 \in L^1(\Omega)$ and $w, z \in X_{t_0}$, then there exists a constant C depending only on Ω and J such that*

$$\big|\!\big|\!\big|\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)\big|\!\big|\!\big| \leqslant Ct_0 |\!|\!| w - z |\!|\!| + \|w_0 - z_0\|_{L^1(\Omega)}.$$

Proof. We have:

$$\int_\Omega \big|\mathcal{T}_{w_0}(w)(x,t) - \mathcal{T}_{z_0}(z)(x,t)\big|\,dx \leqslant \int_\Omega |w_0 - z_0|(x)\,dx$$
$$+ \int_\Omega \left| \int_0^t \int_\Omega J(x-y)\big[\big(w(y,s) - z(y,s)\big) - \big(w(x,s) - z(x,s)\big)\big]\,dy\,ds \right| dx.$$

Hence

$$\int_\Omega \big|\mathcal{T}_{w_0}(w)(x,t) - \mathcal{T}_{z_0}(z)(x,t)v\big|\,dx$$
$$\leqslant \|w_0 - z_0\|_{L^1(\Omega)} + \int_0^t \int_\Omega \big|\big(w(y,s) - z(y,s)\big)\big|\,dy + \int_0^t \int_\Omega \big|\big(w(x,s) - z(x,s)\big)\big|\,dx.$$

Therefore, we obtain:

$$\big|\!\big|\!\big|\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)\big|\!\big|\!\big| \leqslant Ct_0 |\!|\!| w - z |\!|\!| + \|w_0 - z_0\|_{L^1(\Omega)},$$

as we wanted to prove. □

Theorem 4.1. *For every $u_0 \in L^1(\Omega)$ there exists a unique solution u of* (8) *such that $u \in C([0,\infty); L^1(\Omega))$. Moreover, the total mass in Ω verifies,*

$$\int_\Omega u(y,t)\,dy = \int_\Omega u_0(y)\,dy. \tag{25}$$

Proof. We check first that $\mathcal{T}_{u_0}$ maps X_{t_0} into X_{t_0}. From (24) we see that for $0 < t_1 < t_2 \leqslant t_0$,

$$\big\|\mathcal{T}_{u_0}(w)(t_2) - \mathcal{T}_{u_0}(w)(t_1)\big\|_{L^1(\Omega)} \leqslant 2\int_{t_1}^{t_2} \int_\Omega \big|w(y,s)\big|\,dx\,dy\,ds.$$

On the other hand, again from (24),

$$\big\|\mathcal{T}_{u_0}(w)(t) - w_0\big\|_{L^1(\Omega)} \leqslant Ct |\!|\!| w |\!|\!|.$$

These two estimates give that $\mathcal{T}_{u_0}(w) \in C([0,t_0]; L^1(\Omega))$. Hence $\mathcal{T}_{u_0}$ maps X_{t_0} into X_{t_0}.

Choose t_0 such that $Ct_0 < 1$. Now taking $z_0 \equiv w_0 \equiv u_0$, in Lemma 4.1 we get that $\mathcal{T}_{u_0}$ is a strict contraction in X_{t_0} and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem in the interval $[0,t_0]$. To extend the solution to $[0,\infty)$ we may take as initial data $u(x,t_0) \in L^1(\Omega)$ and obtain a solution up to $[0,2t_0]$. Iterating this procedure we get a solution defined in $[0,\infty)$.

We finally prove that if u is the solution, then the integral in Ω of u satisfies (25). Since

$$u(x,t) - u_0(x) = \int_0^t \int_\Omega J(x-y)\big(u(y,s) - u(x,s)\big)\,dy\,ds.$$

We can integrate in x and apply Fubini's theorem to obtain:

$$\int_\Omega u(x,t)\,dx - \int_\Omega u_0(x)\,dx = 0$$

and the theorem is proved. □

Now we study the asymptotic behavior as $t \to \infty$. We start by analyzing the corresponding stationary problem so we consider the equation:

$$0 = \int_\Omega J(x-y)\big(\varphi(y) - \varphi(x)\big)\,dy. \tag{26}$$

The only solutions are constants. In fact, in particular, (26) implies that φ is a continuous function. Set

$$K = \max_{x \in \overline{\Omega}} \varphi(x)$$

and consider the set

$$\mathcal{A} = \big\{x \in \overline{\Omega} \mid \varphi(x) = K\big\}.$$

The set $\mathcal{A}$ is clearly closed and non empty. We claim that it is also open in $\overline{\Omega}$. Let $x_0 \in \mathcal{A}$. We have then,

$$\varphi(x_0) = \left(\int_\Omega J(x_0 - y)\,dy\right)^{-1} \int_\Omega J(x_0 - y)\varphi(y)\,dy,$$

and $\varphi(y) \leqslant \varphi(x_0)$ this implies $\varphi(y) = \varphi(x_0)$ for all $y \in \Omega \cap B(x_0, d)$, and hence $\mathcal{A}$ is open as claimed. Consequently, as Ω is connected, $\mathcal{A} = \overline{\Omega}$ and φ is constant.

We have proved the following proposition:

Proposition 4. *Every stationary solution of* (8) *is constant in* Ω.

We end this section with a proof of the exponential rate of convergence to steady states of solutions in L^2.

Let us take β_1 as

$$\beta_1 = \inf_{u \in L^2(\Omega),\ \int_\Omega u = 0} \frac{\frac{1}{2}\int_\Omega \int_\Omega J(x-y)(u(y) - u(x))^2\,dy\,dx}{\int_\Omega (u(x))^2\,dx}. \tag{27}$$

It is clear that $\beta_1 \geqslant 0$. Let us prove that β_1 is in fact strictly positive. To this end we consider the subspace of $L^2(\Omega)$ given by the orthogonal to the constants, $H = \langle cts\rangle^\perp$ and the symmetric (self-adjoint) operator $T : H \mapsto H$ given by:

$$T(u) = \int_\Omega J(x-y)\big(u(x) - u(y)\big)\,dy = -\int_\Omega J(x-y)u(y)\,dy + A(x)u(x).$$

Note that T is the sum of an inversible operator and a compact operator. Since T is symmetric we have that its spectrum verifies $\sigma(T) \subset [m, M]$, where

$$m = \inf_{u \in H,\ \|u\|_{L^2(\Omega)} = 1} \langle Tu, u\rangle$$

and

$$M = \sup_{u \in H,\ \|u\|_{L^2(\Omega)} = 1} \langle Tu, u\rangle,$$

see [4]. Remark that

$$m = \inf_{u\in H,\ \|u\|_{L^2(\Omega)}=1} \langle Tu, u\rangle = \inf_{u\in H,\ \|u\|_{L^2(\Omega)}=1} \int_\Omega \int_\Omega J(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y\, u(x)\,\mathrm{d}x = \beta_1.$$

Then $m \geqslant 0$. Now we just observe that

$$m > 0.$$

In fact, if not, as $m \in \sigma(T)$ (see [4]), we have that $T : H \mapsto H$ is not inversible. Using Fredholm's alternative this implies that there exists a nontrivial $u \in H$ such that $T(u) = 0$, but then u must be constant in Ω. This is a contradiction with the fact that H is orthogonal to the constants.

To study the asymptotic behavior of the solutions we need an upper estimate on β_1.

Lemma 4.2. *Let β_1 be given by* (27)*, then*

$$\beta_1 \leqslant \min_{x\in\overline{\Omega}} \int_\Omega J(x-y)\,\mathrm{d}y. \tag{28}$$

Proof. Let

$$A(x) = \int_\Omega J(x-y)\,\mathrm{d}y.$$

Since $\overline{\Omega}$ is compact and A is continuous there exists a point $x_0 \in \overline{\Omega}$ such that

$$A(x_0) = \min_{x\in\overline{\Omega}} A(x).$$

For every ε small let us choose two disjoint balls of radius ε contained in Ω, $B(x_1,\varepsilon)$ and $B(x_2,\varepsilon)$ in such a way that $x_i \to x_0$ as $\varepsilon \to 0$. We use

$$u_\varepsilon(x) = \chi_{B(x_1,\varepsilon)}(x) - \chi_{B(x_2,\varepsilon)}(x)$$

as a test function in the definition of β_1, (27). Then we get that for every ε small it holds:

$$\begin{aligned}\beta_1 &\leqslant \frac{\frac{1}{2}\int_\Omega\int_\Omega J(x-y)(u_\varepsilon(y)-u_\varepsilon(x))^2\,\mathrm{d}y\,\mathrm{d}x}{\int_\Omega (u_\varepsilon(x))^2\,\mathrm{d}x}\\ &= \frac{\int_\Omega A(x)u_\varepsilon^2(x)\,\mathrm{d}x - \int_\Omega\int_\Omega J(x-y)u_\varepsilon(y)\,u_\varepsilon(x)\,\mathrm{d}y\,\mathrm{d}x}{\int_\Omega (u_\varepsilon(x))^2\,\mathrm{d}x}\\ &= \frac{\int_\Omega A(x)u_\varepsilon^2(x)\,\mathrm{d}x - \int_\Omega\int_\Omega J(x-y)u_\varepsilon(y)\,u_\varepsilon(x)\,\mathrm{d}y\,\mathrm{d}x}{2|B(0,\varepsilon)|}.\end{aligned}$$

Using the continuity of A and the explicit form of u_ε, we obtain:

$$\lim_{\varepsilon\to 0} \frac{\int_\Omega A(x)u_\varepsilon^2(x)\,\mathrm{d}x}{2|B(0,\varepsilon)|} = A(x_0),$$

and

$$\lim_{\varepsilon\to 0} \frac{\int_\Omega\int_\Omega J(x-y)u_\varepsilon(y)u_\varepsilon(x)\,\mathrm{d}y\,\mathrm{d}x}{2|B(0,\varepsilon)|} = 0.$$

Therefore, (28) follows. □

Now let us prove the exponential convergence of $u(x,t)$ to the mean value of the initial datum.

Theorem 4.2. *For every $u_0 \in L^2(\Omega)$ the solution $u(x,t)$ of* (8) *satisfies*:

$$\left\| u(\cdot\,,t) - \varphi \right\|_{L^2(\Omega)} \leqslant \mathrm{e}^{-\beta_1 t} \| u_0 - \varphi \|_{L^2(\Omega)}. \tag{29}$$

Moreover, if u_0 is continuous and bounded, there exists a positive constant $C > 0$ such that

$$\left\| u(\cdot\,,t) - \varphi \right\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{-\beta_1 t}. \tag{30}$$

Here β_1 is given by (27).

Proof. Let

$$H(t) = \frac{1}{2} \int_\Omega \big(u(x,t) - \varphi\big)^2 \,\mathrm{d}x.$$

Differentiating with respect to t and using (27) and the conservation of the total mass, we obtain:

$$H'(t) = -\frac{1}{2} \int_\Omega \int_\Omega J(x-y)\big(u(y,t) - u(x,t)\big)^2 \,\mathrm{d}y\,\mathrm{d}x \leqslant -\beta_1 \int_\Omega \big(u(x,t) - \varphi\big)^2 \,\mathrm{d}x.$$

Hence,

$$H'(t) \leqslant -2\beta_1 H(t).$$

Therefore, integrating we obtain:

$$H(t) \leqslant \mathrm{e}^{-2\beta_1 t} H(0), \tag{31}$$

and (29) follows.

In order to prove (30) let $w(x,t)$ denote the difference:

$$w(x,t) = u(x,t) - \varphi.$$

We seek for an exponential estimate in L^∞ of the decay of $w(x,t)$. The linearity of the equation implies that $w(x,t)$ is a solution of (8) and satisfies:

$$w(x,t) = \mathrm{e}^{-A(x)t} w_0(x) + \mathrm{e}^{-A(x)t} \int_0^t \mathrm{e}^{A(x)s} \int_\Omega J(x-y) w(y,s) \,\mathrm{d}y\,\mathrm{d}s.$$

Recall that $A(x) = \int_\Omega J(x-y)\,\mathrm{d}x$. By using (29) and the Holder inequality it follows that

$$\big|w(x,t)\big| \leqslant \mathrm{e}^{-A(x)t} w_0(x) + C\mathrm{e}^{-A(x)t} \int_0^t \mathrm{e}^{A(x)s - \beta_1 s} \,\mathrm{d}s.$$

Integrating this inequality, we obtain that the solution $w(x,t)$ decays to zero exponentially fast and moreover, it implies (30) thanks to Lemma 4.2. □

5. Final remarks on possible extensions

In this last section we briefly comment on some possible extensions of our results.

- First, concerning the Cauchy problem, one can study the behavior of the solutions when the asymptotic expansion of $\hat{J}$ near the origin is not of the form $\hat{J}(\xi) = 1 - A|\xi|^\alpha + \mathrm{o}(|\xi|^\alpha)$.

Let us just illustrate this topic by the following result concerning logarithmic perturbations (we thank Marc Peigné for showing this example to us). In dimension 1, we consider a function J such that $\hat{J}(x) \sim |x|^{-3}$ as $|x| \to +\infty$. Then we are just in the borderline case when the second moment of J is infinite. In fact, what happens for the Fourier transform is that

$$\hat{J}(\xi) \sim 1 + c\xi^2 \ln \xi \quad \text{as } \xi \to 0.$$

In this interesting case, the asymptotic behavior is a little bit different, it is still given by a solution of the heat equation, but with a different time velocity, as the following result says:

Theorem 5.1. *Let us assume that $\hat{J}$ has the following behavior near zero*:

$$\hat{J}(\xi) = 1 + c|\xi|^2 \ln(|\xi|) + o(|\xi|^2 \ln(|\xi|)),$$

and let $u_0 \in L^1(\mathbb{R}^N)$ such that $\hat{u}_0 \in L^1(\mathbb{R}^N)$. Now, if u is the solution of the Cauchy problem with initial data u_0 and v is the solution of the heat equation $v_t = (c/2)\Delta v$, with the same initial data $v(0) = u_0$, then

$$(t \ln t)^{N/2} \max_x |u(x,t) - v(x, t\ln t)| \to 0 \quad \text{as } t \to +\infty.$$

The proof is basically the same as for Theorem 1. Let us give a sketch. In Fourier variables we have to estimate the integral:

$$(t \ln t)^{N/2} \int |\hat{u}(\xi,t) - \hat{v}(\xi, t\ln t)| \, d\xi.$$

Writing $\hat{u}$ and $\hat{v}$ as exponentials we obtain that we have to deal with

$$(t \ln t)^{N/2} \int |e^{(\hat{J}(\xi)-1)t} - e^{c\xi^2 (t \ln t)/2}| \, d\xi.$$

Thus, for low frequencies $|\xi| < r(t)$, we first change variables:

$$(t \ln t)^{N/2} \int_{|\xi|<r(t)} |e^{(\hat{J}(\xi)-1)t} - e^{c\xi^2 (t \ln t)/2}| \, d\xi = \int_{|\eta|<r(t)(t\ln t)^{1/2}} |e^{(\hat{J}(\frac{\eta}{(t\ln t)^{1/2}})-1)t} - e^{c\eta^2/2}| \, d\eta.$$

We use the fact that in these variables,

$$\begin{aligned}\left(\hat{J}\left(\frac{\eta}{\sqrt{t\ln t}}\right) - 1\right)t &= c\frac{\eta^2}{\ln t} \ln\left(\frac{\eta}{\sqrt{t \ln t}}\right) + l.o.t. \\ &= c\frac{\eta^2}{\ln t}\left(\ln \eta - \frac{1}{2}\ln t - \frac{1}{2}\ln(\ln t)\right) + l.o.t. = \frac{c\eta^2}{2} + l.o.t.\end{aligned}$$

Here with $l.o.t.$ we denote lower order terms as $t \to +\infty$. Choosing a suitable $r(t) \to 0$, we recover the fact that for such low frequencies, the solution $u(x,t)$ is close to the solution of the heat equation, yet viewed at a later time $v(x, t\ln t)$.

Now as we did in the proof of Theorem 1, high frequencies $\{|\xi| > r(t)\}$ goes to zero fast as time goes by, both for u and v. Hence we obtain indeed the fact that

$$(t \ln t)^{N/2} \int_{\mathbb{R}^N} |\hat{u}(\xi,t) - \hat{v}(\xi, t\ln t)| \, d\xi \to 0 \quad \text{as } t \to +\infty.$$

Then the conclusion follows as in Theorem 1, taking the inverse Fourier transform. Details are left to the reader.

- An interesting problem to look at is to study diffusions given by kernels that depend on x and y and not only on $x - y$. That is, equations of the form $u_t(x,t) = \int_{\mathbb{R}^N} J(x,y)(u(y,t) - u(x,t))\, dy$. In this case our results do not apply since the use of the Fourier transform was the key of our arguments.
- Also, let us remark that our proofs strongly rely on hypothesis (H). It is interesting to known up to what extend (H) is necessary. To answer this one can consider a kernel J that is non-symmetric or verifies $\int_{\mathbb{R}^N} J(r)\, dr \neq 1$ (this fails out of the original model).
- Another interesting problem is to look at the Dirichlet or Neumann problems in unbounded domains, for example in a half-space. In this case it is not clear what the asymptotic behavior should be.
- Finally, one may try to analyze discrete in space versions of these problems (like the ones considered in [2]) and see if they behave as their continuous counterpart. We believe that this is an interesting issue in order to develop numerical approximations for these problems.

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