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Characterizations and recognition of circular-arc graphs and subclasses: A survey☆

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ABSTRACT

Circular graphs are intersection graphs of arcs on a circle. These graphs are reported to have been studied since 1964, and they have been receiving considerable attention since a series of papers by Tucker in the 1970s. Various subclasses of circular-arc graphs have also been studied. Among these are the proper circular-arc graphs, unit circular-arc graphs, Helly circular-arc graphs and co-bipartite circular-arc graphs. Several characterizations and recognition algorithms have been formulated for circular-arc graphs and its subclasses. In particular, it should be mentioned that linear time algorithms are known for all these classes of graphs. In the present paper, we survey these characterizations and recognition algorithms, with emphasis on the linear time algorithms.

1. Introduction

Circular-arc graphs are the intersection graphs of arcs on a circle. Early papers, considering this class have been written by Hadwiger, Debrunner and Klee in 1964 [22] and by Klee in 1969 [38]. Tucker wrote his Ph.D. thesis on this class in 1969 [71], and in the 1970s published a series of articles [72,73,75–77] containing some fundamental properties on circular-arc graphs and subclasses. These graphs have since received considerable attention for a few reasons, their beautiful structure being one of them. There are applications of circular-arc graphs in areas such as genetics, traffic control and many others. See [2,7,8,33,44,58,66–69,76] for applications.

Circular-arc graphs are a generalization of interval graphs, the intersection graphs of intervals on a real line. However, in general, problems for circular-arc graphs tend to be more difficult than for interval graphs. One of the reasons is that intervals of a real line satisfy the Helly property, while arcs of a circle do not necessarily satisfy it. This implies that the maximal cliques of an interval graph can be associated to chosen points of the line. The latter means that an interval graph can have no more maximal cliques than vertices. In contrast, circular-arc graphs may contain maximal cliques which do not correspond to points of the circle. In fact, circular-arc graphs can have an exponential number of maximal cliques. Complements of induced matchings are examples of such graphs.

In the present paper, we survey some structural characterizations and recognition algorithms for circular-arc graphs and subclasses of these graphs. The emphasis is on linear time recognition algorithms, both for this class and subclasses of it. It is worth mentioning that in the last year, at least five papers appeared in conference proceedings, describing linear time algorithms for circular-arc graphs and subclasses [34,36,37,41,42]. The books by Spinrad [65] and by Golumbic [19] contain comprehensive accounts on the class. See also the books [5,48] and the article [51].

☆ This work is dedicated to Pavol Hell, on his 60th birthday, for his many contributions on this subject.

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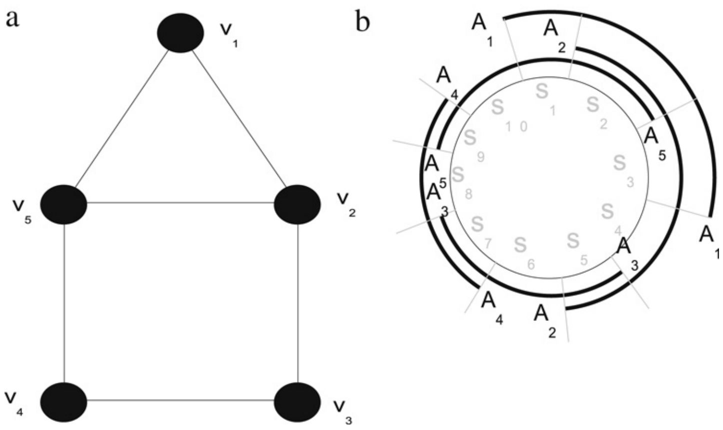

Fig. 1. A PCA graph and a normal model of it.

The following is the plan of the paper. In the next section, we describe the basic definitions and notation. Section 3 contains some characterizations and recognition algorithms for circular-arc graphs. The subclasses are examined in the following sections. That is, characterizations and recognition algorithms for the classes of proper circular-arc graphs, unit circular-arc graphs and Helly circular-arc graphs are presented in Sections 4–6 respectively. Co-bipartite circular-arc graphs and other subclasses are considered in Sections 7 and 8. Some open problem are formulated in the last section, which contains a summary of the results.

2. Preliminaries

We describe the basic notation and concepts needed. By G, we denote an undirected graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. For an edge $e \in E(G)$, write $e = v_iv_j$, where $v_i, v_j \in V(G)$ are the *extremes* of e. Say that v_i, v_j are *adjacent* vertices or *neighbors* and that e is *incident* to v_i, v_j. Write $N(v_i)$ to represent the set of neighbors of v_i and $N[v_i] = \{v_i\} \cup N(v_i)$. If $N[v_i] = V(G)$ then v_i is a *universal* vertex in G. On the other hand, if $N[v_i] = N[v_j]$ then v_i, v_j are *twins*. The *degree* of v_i is the value $d(v_i) = |N(v_i)|$. A *complete set* (*independent set*) of G is a set of pairwise adjacent (non-adjacent) vertices. A maximal complete set is called a *clique*. The *clique matrix* A of G is a 0,1-matrix where the rows correspond to cliques and the columns to vertices of G, such that $a_{ij} = 1$ when the ith clique contains the jth vertex of G.

Say that G is *connected* when there is path in G between any two of its vertices. A *bipartite* graph is one whose set of vertices can be partitioned into two independent sets. The *complement* of G is the graph $\overline{G}$ having the same vertex set as G, and where two vertices are adjacent in $\overline{G}$ precisely when they are not so in G. We employ the prefix *co* to refer to the complement of a graph. Therefore a graph G is *co-connected* when its complement is connected; G is *co-bipartite* when its complement is bipartite, and so on.

We use a similar notation for a directed graph D. For a directed edge $e = v_iv_j$, say that e *starts* at v_i and *ends* at v_j. Denote $N^+(v_i) = \{v_j \in V(D)/v_iv_j \in E(D)\}$, $N^-(v_i) = \{v_j \in V(D)/v_jv_i \in E(D)\}$, $N^+[v_i] = \{v_i\} \cup N^+(v_i)$ and $N^-[v_i] = \{v_i\} \cup N^-(v_i)$. Also $d^+(v_i) = |N^+(v_i)|$ and $d^-(v_i) = |N^-(v_i)|$ are the *outdegree* and *indegree* of v_i, respectively. Say that D is *eulerian* when $d^+(v_i) = d^-(v_i)$, for all v_i. Also, D is *connected* when its underlying undirected graph is, and D is *strongly connected* when it contains directed paths from v_i to v_j and from v_j to v_i, for all $v_i, v_j \in V(D)$.

A *circular-arc* (CA) model, or simply a *model*, $M = (C, \mathcal{A})$ is a circle C, together with a collection $\mathcal{A}$ of arcs of C. Unless otherwise stated, we always traverse C in the clockwise direction. Each arc $A_i \in \mathcal{A}$ is written as $A_i = (s_i, t_i)$, where $s_i, t_i \in C$ are the *extreme points* of A_i, with s_i the *start point* and t_i the *end point* of A_i, respectively, in the clockwise direction. The extremes of $\mathcal{A}$ are those of all arcs $A_i \in \mathcal{A}$. Denote by $|A_i|$ the length of arc A_i. The *complement* of arc $A_i = (s_i, t_i) \in \mathcal{A}$ is the arc $\overline{A_i} = (t_i, s_i)$. The complement model $\overline{M}$ of M is the model $(C, \overline{\mathcal{A}})$, where $\overline{\mathcal{A}} = \{\overline{A_i}/A_i \in \mathcal{A}\}$. As usual, we assume that no single arc of $\mathcal{A}$ covers C, that no two extremes of $\mathcal{A}$ coincide and that all arcs of $\mathcal{A}$ are open. When traversing C, we obtain a circular ordering of the extreme points. In particular, traversing C defines a circular ordering $s_1, \ldots, s_n$ of the start points, which implies a circular ordering $A_1, \ldots, A_n$ of the corresponding arcs. In general, when dealing with a sequence $x_1, \ldots, x_t$ of t objects circularly ordered, we assume that all additions and subtractions of the indices i are modulo t. A *segment* of M is the arc of C defined by two consecutive extreme points of $\mathcal{A}$, traversed in the circular ordering. Consequently, M has $2n$ segments, represented by $S_1, \ldots, S_{2n}$, where S_1 starts at s_1. Fig. 1(b) shows a CA model and its segments.

It is useful to classify ordered pairs of arcs of $\mathcal{A}$, according to their relative positions in C. There are five types of pairs of arcs $A_i, A_j \in \mathcal{A}$: *disjoint* (di), when $A_i \cap A_j = \emptyset$; *contained* (cd), when $A_i \subset A_j$; *contains* (cs), when $A_i \supset A_j$; *circle-cover* (cc), when $A_i \cup A_j = C$; *overlap* (ov), otherwise. See Fig. 2.

There are some special models $M = (C, \mathcal{A})$ of interest. When $\bigcup\{A_i \in \mathcal{A}\} \neq C$ then M is an *interval model*. More generally, when M does not contain circle-cover arcs then M is *normal*. On the other hand, if M has no contained arcs then M is a *proper*

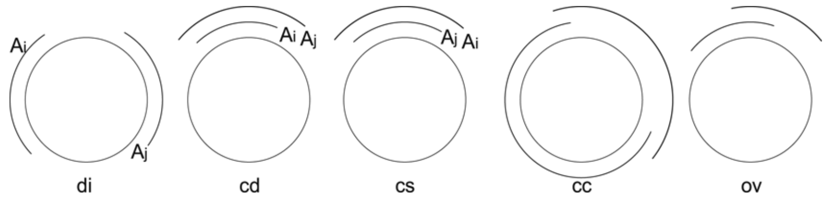

Fig. 2. Types of pairs of arcs.

circular-arc (PCA) model. If all arcs have the same length, M is called an *unit circular-arc* (UCA) model. Finally, if $\mathcal{A}$ is a Helly family of arcs then M is a *Helly circular-arc* (HCA) model. The arcs of $\mathcal{A}$ form a Helly family precisely when the arcs of every pairwise intersecting subfamily of $\mathcal{A}$ contains a common point of C.

A *CA graph* G is the intersection graph of the arcs of a CA model M. In this case, say that M is a model of G, and that G admits M, as a model. The above classes of models motivate the definitions of the following classes of CA graphs. A *PCA*, *UCA* and *HCA graph* is one admitting a PCA, UCA and HCA model, respectively. Similarly, an *interval graph* is one having an interval model. In particular, a *proper interval graph* is a graph admitting an interval model with no contained arcs. Observe that interval graphs are HCA, and that UCA graphs are PCA.

The following concept closely relates to circular-arc graphs. An *interval bigraph model* is an interval model whose set of arcs (intervals) are partitioned into two parts. An *interval bigraph* is a graph G whose vertices correspond to the intervals of an interval bigraph model, and such that $v_i, v_j \in V(E)$ is an edge of G if and only if the corresponding intervals intersect and belong to distinct parts of the bipartition. When the interval bigraph model is additionally a PCA model then G is a *proper interval bigraph*.

3. General circular-arc graphs

In this section, we discuss the recognition of general circular-arc graphs.

Circular-arc graphs have been characterized by Tucker in terms of circular orderings of its vertices, as follows.

Theorem 1 ([72]). *A graph G is a circular-arc graph if and only if there is a circular ordering $v_1, \ldots, v_n$ of its vertices, such that for $i < j$, if $v_iv_j \in E(G)$ then either*

$$v_{i+1}, \ldots, v_j \in N(v_i), \quad \textit{or}$$
$$v_{j+1}, \ldots, v_i \in N(v_j).$$

The question of recognizing circular-arc graphs has been first posed by Hadwiger, Debrunner and Klee, in 1964 [22]. Tucker described the first polynomial time algorithm [77] for it. The complexity of this algorithm is $O(n^3)$. The general outline of this method has also been somehow employed in the later algorithms. A key concept of this method can be described in terms of matrices, as follows.

Let G be a graph. For our purpose, there is no loss of generality to consider that G has no twins. The *intersection matrix* of G is a $n \times n$ matrix λ, where each entry λ_{ij} informs the type of intersection, if any, between $N[v_i]$ and $N[v_j]$, for $v_i, v_j \in V(E)$. That is,

$$\lambda_{ij} = \begin{cases} di, & \text{if } v_iv_j \notin E(G), \qquad\qquad\qquad\qquad\qquad \text{otherwise} \\ cd, & \text{if } N[v_i] \subset N[v_j], \qquad\qquad\qquad\qquad \text{otherwise} \\ cs, & \text{if } N[v_i] \supset N[v_j], \qquad\qquad\qquad\qquad \text{otherwise} \\ cc, & \text{if } N[v_i] \cup N[v_j] = V(G), \qquad\qquad\quad\;\, \text{and} \\ & \text{for each } v_k \in N[v_i] \setminus N[v_j], N[v_k] \subset N[v_i], \quad \text{and} \\ & \text{for each } v_k \in N[v_j] \setminus N[v_i], N[v_k] \subset N[v_j], \quad \text{otherwise} \\ ov. \end{cases}$$

As an example, Fig. 3(b) shows an intersection matrix of the graph of Fig. 3(a). The idea is to try to establish a similarity between neighborhood relations of the vertices of the graph G, and relations among the arcs that would correspond to the vertices of G, in the assumption that G is a circular-arc graph. We switch from a graph to a model.

Let $M = (C, \mathcal{A})$ be a CA model. The *intersection matrix* of M is an $n \times n$ matrix μ, where μ_{ij} expresses the type of relation between A_i and A_j, where $A_i, A_j \in \mathcal{A}$, as:

$$\mu_{ij} = \begin{cases} di, & \text{if } A_i, A_j \text{ are disjoint}, \quad \text{or} \\ cd, & \text{if } A_i \subset A_j, \qquad\qquad\quad\; \text{or} \\ cs, & \text{if } A_i \supset A_j, \qquad\qquad\quad\; \text{or} \\ cc, & \text{if } A_i \cup A_j = C, \qquad\quad\;\; \text{or} \\ ov, & \text{otherwise.} \end{cases}$$

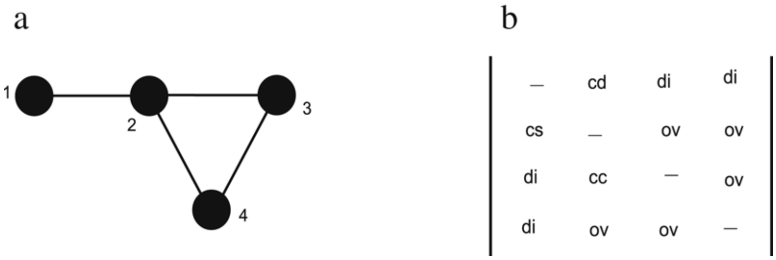

Fig. 3. A graph and its intersection matrix.

So, given a graph G, the basis of the methods [14,28,36,49,77] is to construct an intersection matrix λ of G, and use it for guiding the construction of a CA model of G, if one exists. That is, the model that is looked for is one whose intersection matrix is precisely λ. The following theorem asserts that this is always possible, whenever G is a CA graph.

Theorem 2 (*[77,28]*). *Let G be a graph with no universal vertex, nor twins. Then an intersection matrix of G is an intersection matrix of some CA model of G.*

Consequently, the initial step of a recognition algorithm for CA graphs would be to compute the intersection matrix of the input graph G.

The algorithm [77] considers two cases, namely G being co-bipartite, or not. Each of the cases is handled separately. When G is not co-bipartite, the method consists of looking for an independent set of G. Again there are two cases to be considered, namely when this independent set is of size 3, or greater than 3. In both situations, the basic principle is to allocate a set of pairwise disjoint arcs, to correspond to the independent set. Subsequently, other arcs are placed into the model, which is refined, step-by-step. The algorithm [77] performs these operations in $O(n^3)$ time. Eschen and Spinrad [14] described improved methods, which lowered the complexity to $O(n^2)$. See Eschen [15]. Nussbaum [54] pointed out that the above-mentioned method to allocate the pairwise disjoint arcs ought to be modified. In fact in his M.Sc. Thesis [54], he described the required correction. Recently, Kaplan and Nussbaum [36] improved the allocation of the disjoint arcs, so as to perform it in linear time. Furthermore, in [36] a sharper analysis of Eschen and Spinrad's algorithm is formulated, showing that in fact they run in linear time. Spinrad [64] described an elegant method for handling the case where G is co-bipartite. Let $V_1 \cup V_2 = V(G)$ be a co-bipartition of G. Define a directed graph D_G, where $V(D_G) = V(G)$ and having a directed edge from $v_i \in V(D_G)$ to $v_j \in V(D_G)$, $v_i \neq v_j$, whenever

$$
\begin{array}{ll}
\text{if } v_i, v_j \in V_1 \text{ and } N[v_i] \subset N[v_j], & \text{or} \\
\text{if } v_i, v_j \in V_2 \text{ and } N[v_i] \supset N[v_j], & \text{or} \\
\text{if } v_i \in V_1, v_j \in V_2, \text{ and } v_iv_j \notin E(G), & \text{or} \\
\text{if } v_i \in V_2, v_j \in V_1, \text{ and } N[v_i] \cup N[v_j] = V(D_G). &
\end{array}
$$

The following theorem characterizes co-bipartite CA graphs G, in terms of the digraphs D_G.

Theorem 3 (*[64]*). *Let G be a co-bipartite graph with no twins. Then G is a CA graph if and only if D_G is a partial order of dimension* 2.

The above theorem leads directly to an algorithm for recognizing co-bipartite CA graphs of complexity $O(n^3)$. Eschen and Spinrad [14] reduced the complexity to linear time, by describing an algorithm for computing intersection matrices in $O(n^2)$ time and employing a convenient chordal bipartite graph. Note that $m = \Omega(n^2)$, given that G is co-bipartite. See also [63].

Hsu [28] formalized the basis of the existing recognition algorithms for CA graphs, through Theorem 2. Hsu's recognition algorithm employs the idea of intersection matrices to construct a canonical model for the input graph G. Such a model is then transformed into a circle graph, that is, an intersection graph of chords of a circle. The problem of recognizing CA graphs is then reduced to the recognition of circle graphs. An algorithm for recognizing circle graphs is described in [28] and is employed in the CA graph recognition process. The method uses different kinds of decomposition techniques, such as modular decomposition and join decomposition. Besides describing recognition algorithms for circle graphs and CA graphs, [28] formulates isomorphism algorithms for both classes. The overall complexity is $O(nm)$.

The algorithm by McConnell [49] also employs the reduction to chordal bipartite graphs [14] for constructing the intersection matrix. However, [49] uses a sharper analysis, and shows that the computation can be done in linear time. The algorithm employs arc flippings, that is, replacing arcs with their complements. It also applies this notion to intersection matrices, as follows. Let μ be a $n \times n$ intersection matrix, and $1 \leq i \leq n$. The *complement of* μ, *relative to* i, denoted by $\overline{\mu_i}$ is the matrix obtained from μ by possibly changing each element of row i and column i, as follows:

BEFORE		*AFTER*
di	$\longrightarrow$	cs
cd	$\longrightarrow$	cc
cs	$\longrightarrow$	di
cc	$\longrightarrow$	cd
ov	$\longrightarrow$	ov.

Clearly, $\overline{(\overline{\mu_i})_i} = \mu$. Observe that the complement of μ, relative to i, reflects the changes in a possible CA model $M = (C, \mathcal{A})$ produced by complementing arc $A_i \in \mathcal{A}$. The notion of complement relative to a row is further generalized to subsets, as follows. Let $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$. The *complement of* μ, *relative to* I, denoted by $\overline{\mu_I}$ is the matrix obtained from μ by iterating the operations of taking complements, for each $j \in I$. That is, first consider the complement of μ, relative to i_j. Then take the complement of $\overline{\mu_{i_1}}$, relative to i_2, and so on, until i_k is considered. Again, $\overline{(\overline{\mu_I})_I} = \mu$.

Lemma 1 (*[49]*). *If μ is an intersection matrix of a model $(C, \mathcal{A})$ then $\overline{\mu_i}$ is the intersection matrix of the model $(C, (\mathcal{A} \setminus A_i) \cup \overline{A_i})$.*

In order to compute the complement $\overline{\mu_I}$ of an intersection matrix μ, relative to some subset I, we consider the set $\mathcal{M}$ of complement matrices, obtained by iteratively taking the required complements. Call $\mathcal{M}$ a *complement related family* of matrices. The theorem below describes two properties of complement related families. The term *interval matrix* refers to an intersection matrix of an interval graph or model.

Theorem 4 (*[49]*). *Let $\mathcal{M}$ be a complement related family of matrices. Then*

(i) *$\mathcal{M}$ is an equivalence class on the set of all intersection matrices.*
(ii) *If $\mathcal{M}$ contains an intersection matrix of a CA graph or model then $\mathcal{M}$ contains an interval matrix.*

Property (ii) of this theorem means that given any intersection matrix of a CA graph, it is always possible to find a subset of indices I, such that $\overline{\mu_I}$ is an interval matrix.

The outline of algorithm [49] is as follows: Let G be a CA graph with no universal vertices, nor twins.

(1) Construct an intersection matrix μ of G.
(2) Find the complement $\overline{\mu_I}$, relative to a chosen $I \subseteq \{1, \ldots, n\}$, so that $\overline{\mu_I}$ is an interval matrix.
(3) Construct a model M' for $\overline{\mu_I}$, such that M' is an interval model, whenever G is a CA graph.
(4) Construct the model M, from M', by complementing all arcs A_i, such that $i \in I$.
(5) Verify if M is a model of G. If yes, G is CA, otherwise it is not.

Note that the algorithm may fail to perform one of the above steps, in which case the graph is not CA. Conversely, if G is not a CA graph and the algorithm reaches Step 5, then it would report failure in this step. However, this could also be possibly detected in some previous step.

For computing Step 1, basically employ the method of evaluating neighborhoods of [14], which [49] shows can be performed in linear time.

The aim of Step 2 is to transform μ into an interval matrix $\overline{\mu_I}$. With this purpose, we have to properly choose the set I of indices. Keeping in mind that the aim is to obtain a linear time algorithm, I should be restricted to have $O(m/n)$ size. This ensures that complementing the members of I to obtain $\overline{\mu_I}$ from μ would take $O(m)$ time. The details of actually constructing such a set I can be found in [65].

In Steps 1 and 2, we still deal with intersection matrices of graphs. Step 3 represents the change to intersection matrices of models. If G is a CA graph, in Step 3 we obtain an interval model. The method is not simple and involves the application of tools as transitive orientations, modular decompositions and partitionings [47].

The computation of Step 4 is straightforward. Finally, Step 5 requires to verify if a given model M is a CA model of a given graph G. Also, there is no difficulty in computing it.

McConnell [49] showed that all the above steps can be implemented in $O(n + m)$ time.

4. Proper circular-arc graphs

In this section, we survey some of the characterizations and recognition algorithms for PCA graphs.

The first characterization of PCA graphs was described by Tucker [73], based on the adjacency matrix of the graph. A characterization of PCA graphs by forbidden subgraphs has been also described by Tucker [75], as below. For a graph G, denote by G^* the graph obtained by adding an isolated vertex to G.

Theorem 5 (*[75]*). *A graph G is a PCA graph if and only if it does not contain as induced subgraphs the graphs C_n^*, $n \geq 4$; $\overline{C_{2j}}$, $j \geq 3$; $\overline{C_{2j+1}^*}$, $j \geq 1$ and the graphs $\overline{H_1}$, $\overline{H_2}$, $\overline{H_3}$, $\overline{H_4}$, $\overline{H_5}$, and H_1^* (see Fig. 4).*

The following concepts are relevant. Let A be a $\{0, 1\}$-matrix. Say that A has the *consecutive 1's property on the columns*, when there is a permutation of rows of A, such that the 1 entries become consecutive in the columns [74]. More generally,

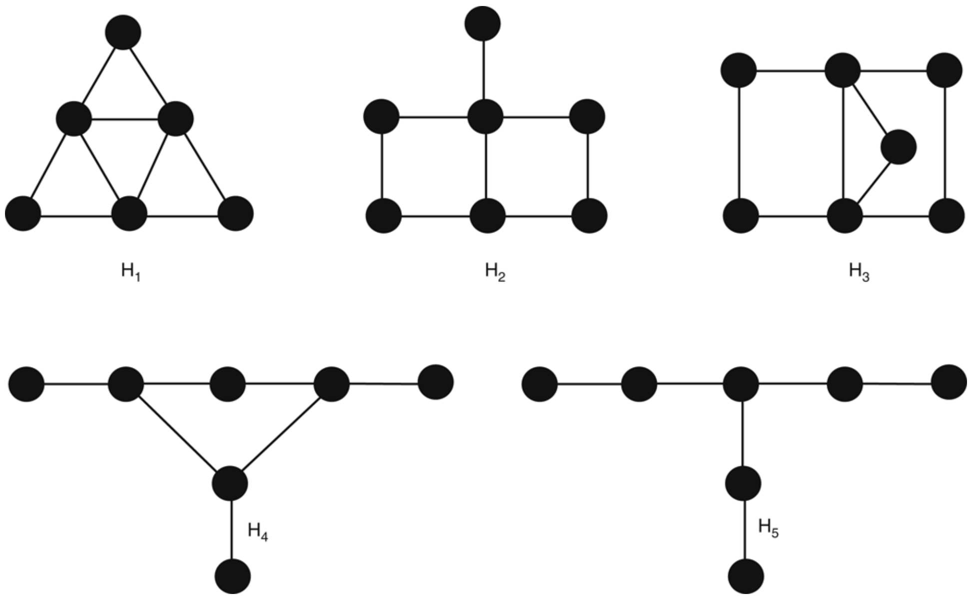

Fig. 4. Complements of forbidden subgraphs for PCA graphs.

A has the *circular consecutive 1's property on the columns*, when after a permutation of its rows the 1's or the 0's become consecutive in each column. Among other applications, the consecutive 1's property plays a key role in the recognition of interval graphs [4,29], while the circular consecutive 1's property is employed in a characterization of *PCA* graphs [73].

Some characterizations and recognition algorithms for PCA graphs consider separately the PCA graphs which are co-bipartite, and the PCA graphs which are not. The following concept is useful for characterizing the latter class.

The *augmented adjacency matrix* of a graph G is the one obtained from the adjacency matrix of G, by setting to 1 each entry of its main diagonal.

Tucker characterized non-co-bipartite PCA graphs, in terms of their augmented adjacency matrices.

Theorem 6 ([73]). *Let G be a graph, which is not co-bipartite. Then G is a PCA graph if and only if it admits an augmented adjacency matrix having the circular consecutive 1's property on the columns.*

On the other hand, Tucker proved a characterization by forbidden subgraphs for co-bipartite PCA graphs.

Theorem 7 ([73]). *Let G be a co-bipartite graph. Then G is a PCA graph if and only if it does not contain the following families as induced subgraphs: even induced cycles of length ≥ 6, and the complements of the graphs H_2, H_4 and H_5 of Fig. 4.*

Furthermore, Hell and Huang [26] proved that co-bipartite PCA graphs are closely related to proper interval bigraphs.

Theorem 8 ([26]). *Let G be a co-bipartite graph. Then G is a PCA graph if and only if $\overline{G}$ is a proper interval bigraph.*

The following are key concepts for a characterization of PCA graphs, introduced by Hell, Bang-Jensen and Huang [23], Huang [31,32] and Skrien [62]. They generalize *tournaments*, that is, orientations of a complete graph. A graph G is a *local tournament* when G admits an orientation, such that $N^-(v)$ and $N^+(v)$ induce tournaments in G, for all $v \in V(G)$. A *round enumeration* of a digraph D is a *circular ordering* $v_1, \ldots, v_n$ of its vertices, such that for each v_i there are integers r and s, such that $N^-(v_i) = \{v_{i-1}, v_{i-2}, \ldots, v_{i-r}, \}$ and $N^+(v_i) = \{v_{i+1}, v_{i+2}, \ldots, v_{i+s}, \}$. Finally, say that G is *round* when it admits an orientation having a round enumeration. In the special case that $v_1, \ldots, v_n$ is a linear ordering, instead of a circular ordering, then $v_1, \ldots, v_n$ is a *straight enumeration* and G is called *straight*. The latter concept has been employed in the following characterization of proper interval graphs.

Theorem 9 ([9,57]). *A graph G is a proper interval graph if and only if G is straight.*

The above theorem has been generalized to a characterization of PCA graphs, as follows.

Theorem 10 ([9,62]). *The following statements are equivalent for a connected graph G.*

(i) *G is a PCA graph.*
(ii) *G can be oriented as a local tournament.*
(iii) *G is round.*

As for the recognition problem, the first algorithm for recognizing PCA graphs is by Tucker [73], following his matrix characterization of this class. The algorithm has complexity $O(n^2)$. This complexity is achieved by applying a linear time algorithm for checking the circular consecutive 1's property of the augmented adjacency matrix of G, cf. [65]. Such a linear time algorithm can be obtained by employing PQ trees.

An algorithm for recognizing PCA graphs based on orientations has been described by Hell and Huang [24]. Its complexity is $O(\Delta m)$, where Δ is the maximum degree of the graph. The first linear time recognition algorithm is by Deng, Hell and Huang [9]. The following concepts are useful for describing this algorithm.

Let G be a graph. A *block* of G is a maximal set of pairwise twins. The *reduced graph* G_R of G is the graph obtained from G by identifying each subset of vertices formed by pairwise twins. A *mixed graph* is a graph in which possibly some edges are oriented and some are not. A *straight mixed orientation* of G is a mixed graph, in which the oriented edges form a straight orientation of the reduced graph G_R of G. Let S_1, S_2 be mixed orientations of subgraphs of G. Then S_1, S_2 are *consistent* when the edges of G which are directed in both S_1, S_2, are directed in the same way in these two orientations.

The following lemma is employed in the algorithm.

Lemma 2 (*[9]*). *Every connected non-trivial proper interval graph has exactly two mixed straight orientations, one the reserve of the other.*

The algorithm by Deng, Hell and Huang, constructs a round orientation of G (Theorem 9). With this purpose it finds an appropriate subgraph B of the input graph G, which is a proper interval graph, whenever G is a PCA graph. First, the algorithm operates with the reduced graph B_R of B, and computes a straight orientation of B_R. In what follows, the algorithm identifies three not necessarily disjoint subgraphs C, D, E, covering all the edges of $G-B$. Again, these subgraphs must be proper interval graphs, whenever G is PCA. Then the algorithm constructs straight mixed orientations of C, D, E, which are transformed into compatible orientations, by possibly changing directions of some edges. At this point, it possibly remains to orient only the edges in the blocks of B. Consequently, the orientation is completed by turning each block into a transitive tournament. The final orientation of G is then transformed into a PCA model.

The article [9] also describes an algorithm for recognizing proper interval graphs. Such an algorithm is based on finding a straight orientation of the graph (Theorem 9). The algorithm for constructing such an orientation is then used in different parts of this algorithm for recognizing PCA graphs. The outline of the PCA algorithm is as follows.

(1) Verify if G is a proper interval graph. If yes, stop: G is PCA. Otherwise,
(2) Let $v_{\min}$ be the minimum degree vertex of G, A be the subgraph of G induced by $N[v_{\min}]$, and $B = G - A$. If B is a clique then apply Tucker's algorithm [73] for recognizing if G is a PCA graph. Otherwise,
(3) If B is not a proper interval graph then stop: G is not PCA. Otherwise, orient B as a straight mixed graph. Let $B_1, \ldots, B_q$ be the straight enumeration of the reduced graph B_R of B.
(4) Define

$$\begin{aligned} L &= \text{ subset of vertices of } A, \text{ adjacent to some vertex of } B_q \\ R &= \text{ subset of vertices of } A, \text{ adjacent to some vertex of } B_1 \\ C &= G - L \\ D &= G - R \\ E &= A \cup B_q. \end{aligned}$$

If each C, D, E is not a proper interval graph then stop: G is not PCA. Otherwise, construct straight mixed orientations C', D', E' of C, D, E, respectively.
(5) If C', D' and E' are not consistent, try to make them consistent, by reversing the orientations of some edges. If this is not possible, stop: G is not PCA. Otherwise,
(6) Orient the edges of G, still undirected, so that each block B_i becomes a transitive tournament.
(7) Transform the orientation of G, into a PCA model.

See Fig. 5.

Besides Theorems 9 and 10, the correctness of the algorithm also relies on the following theorem.

Theorem 11 (*[9]*). *If G is a PCA graph then the subgraphs A, B, C, D, E are all proper interval graphs.*

As for the complexity, all the operations involved in the algorithm can be done in linear time, except possibly the application of Tucker's algorithm, in Step 2. As mentioned, the complexity of the latter algorithm is $O(n^2)$. However, Tucker's algorithm is applied only in the case where B is a complete graph. In this case, it is not difficult to conclude that $m = \theta(n^2)$, meaning that the overall complexity of the algorithm [9] is $O(n + m)$.

The following property is useful, and employed in some different algorithms for recognizing PCA graphs and UCA graphs. Recall that a normal model is a CA model in which no two arcs cover the circle. Then

Theorem 12 (*[19,73]*). *Every PCA graph admits a normal PCA model.*

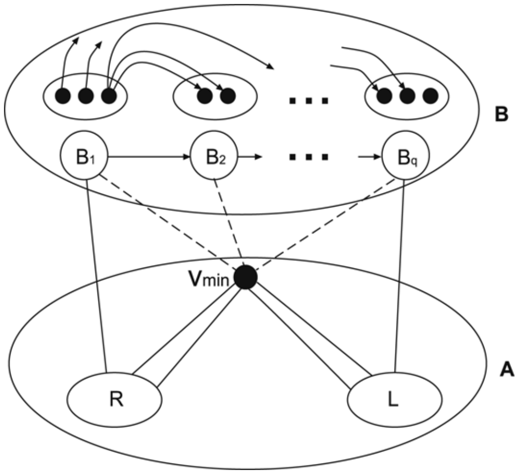

Fig. 5. The algorithm by Deng, Hell and Huang.

A conceptually simple linear time algorithm for recognizing PCA graphs has been described by Spinrad [65], as follows. Divide the graphs G into two kinds, those which are co-bipartite and those which are not. In the latter situation, apply Theorem 6. That is, deciding if G is PCA is equivalent to deciding if its augmented adjacency matrix has the circular consecutive 1's property. This can be verified in linear time, using PQ trees. For the case where G is co-bipartite, Spinrad [65] described another characterization for this PCA subclass, in relation to permutation graphs.

Theorem 13 (*[65]*). *Let G be a co-bipartite graph. Then G is a PCA graph if and only if $\overline{G}$ is a permutation graph.*

The algorithm [65] then recognizes co-bipartite PCA graphs G by deciding if $\overline{G}$ is a permutation graph. Since we know that $\overline{G}$ is bipartite, a simpler linear time algorithm can be applied to decide if $\overline{G}$ is indeed a permutation graph. Consequently, the overall complexity of this algorithm is $O(n + m)$.

Note that Theorems 8 and 13 imply that the classes of proper interval bigraphs and bipartite permutation graphs are equivalent. Moreover, these classes are also equivalent to bipartite asteroid-free graphs [5].

More recently, Kaplan and Nussbaum [37] described another linear time recognition algorithm for PCA graphs, aiming at producing certificates. Such an algorithm again considers two cases.

When G is not co-bipartite, Kaplan and Nussbaum also apply Tucker's characterization (Theorem 6). In this case, for obtaining a certificate for the circular consecutive 1's property, they employ the algorithm by McConnell [50] (see also [30]). The algorithm [50] produces a certificate for the augmented adjacency matrix not having the consecutive 1's property. However, in this situation we need a relation between the consecutive 1's property and the circular consecutive 1's property. The following theorem provides this relation.

Theorem 14 (*[73]*). *Let M_1 be a $\{0, 1\}$-matrix and M_2 be the matrix obtained from M_1 by complementing the entries of all rows having a 1 in some fixed column. Then M_1 has the circular consecutive 1's property if and only if M_2 has the consecutive ones property.*

When G is co-bipartite, the algorithm [37] applies the characterization by Hell and Huag (Theorem 8), which transforms the problem into deciding if $\overline{G}$ is a proper interval bigraph. The algorithm for recognizing proper interval bigraphs by Hell and Huang [27] is then applied. Such an algorithm constructs a PCA model for G, when $\overline{G}$ is indeed a proper interval bigraph. For producing a certificate for a negative answer, Kaplan and Nussbaum employ Theorem 7. The algorithm [37] then exhibits a convenient forbidden structure.

All the certificates of the algorithm [37] are obtained in linear time.

5. Unit circular-arc graphs

In this section, we survey the characterizations and recognition algorithms for UCA graphs.

Recall that a UCA graph is a CA graph admitting a model where all arcs are of the same size. Clearly, a UCA graph is PCA, similarly any unit interval graph is also a proper interval graph. However these two subclasses of interval graphs coincide, whereas PCA graphs properly contain UCA graphs. Fig. 6(b) illustrates a PCA model of the graph of Fig. 6(a). Any CA model for this graph would have its extreme points in the same circular ordering. By observing that the sum of the lengths of arcs A_0, A_1, A_2, A_3 must be greater than the sum of the lengths of A_4, A_5, A_6, A_7, we conclude that G cannot be a UCA graph.

Tucker [75] formulated a characterization of UCA graphs by forbidden subgraphs. First, we describe a family of graphs defined by Tucker, using the notation of [11]. For integers $n, k, n > k \geq 1$, denote by $CI(n, k)$ the graph admitting the

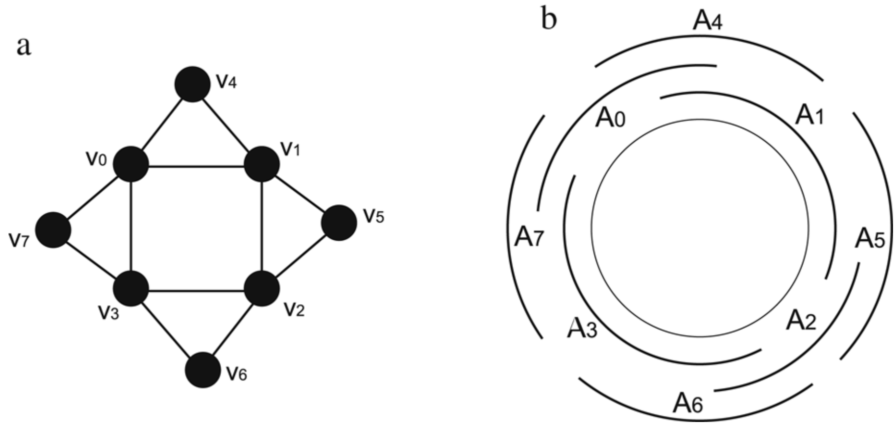

Fig. 6. A PCA graph which is not UCA.

following model. In a circle of radius 1, there are n arcs A_i and n arcs B_i, $0 \le i \le n-1$. Each arc A_i starts at point $\frac{2\pi i}{n}$ and has length $\frac{2\pi k}{n} + \epsilon$. And each arc B_i starts at $\frac{2\pi i+\pi}{n}$ and has length $\frac{2\pi k}{n} - \epsilon$. The graph of Fig. 6(a) is exactly $CI(4, 1)$.

These graphs were employed in Tucker's characterization for the PCA graphs which are UCA.

Theorem 15 ([75]). *Let G be a PCA graph. Then G is UCA if and only if G contains no $CI(n, k)$ induced subgraphs, for $n > 2k$ and n, k relatively prime.*

The following property has been used in the proof of the above theorem.

Theorem 16 ([75]). *Let G be a UCA graph and M a normal PCA model of it. Then G admits a UCA model whose extreme points are in the same circular ordering as those of M.*

The following concepts were used in a useful alternative formulation of Theorem 15.

Let G be a graph, $M = (C, \mathcal{A})$ with $|C| = 1$ a normal CA model of G, and n, k, m, l integers. An (n, k)-*circuit* of G is a subset of n vertices, whose corresponding arcs admit a circular ordering $A_1, \dots, A_n$ in M, such that consecutive arcs intersect, s_{i+1} is the start point which immediately precedes t_i in C and $k = \lfloor \sum_{i=1}^{n} |A_i| \rfloor$. Similarly, an (m, l)-*independent set* of G is a subset of m vertices, whose corresponding arcs admit a circular ordering $A_1, \dots, A_m$ in M, where consecutive arcs are disjoint, s_{i+1} is the start point which immediately succeeds t_i in C and $l = \lceil \sum_{i=1}^{m} |A_i| \rceil$. See the description by Durán and Lin [10].

An (n, k)-circuit is *minimal* if it contains no (n', k')-circuit where $n'/k' < n/k$, and $k, k' > 0$. Similarly, define a *maximal* (m, l)-independent set.

Theorem 17 ([75]). *A PCA graph G is not UCA if and only if it admits a minimal (n, k)-circuit and a maximal (n, k)-independent set, with respect to any PCA model, where n, k are relatively prime and $n > 2k$.*

Corollary 1 ([11,37,75]). *Let G be a PCA graph. Then G is not a UCA graph if and only if G has a minimal (n, k)-circuit and a maximal (m, l)-independent set satisfying $n/k = m/l$.*

Lin and Szwarcfiter [41] have described a characterization of UCA graphs, in terms of a generalization of eulerian digraphs. The following definitions are needed for such a characterization.

Let $M = (C, \mathcal{A})$ be a PCA model of a graph G, and $\mathcal{A} = \{A_1, \dots, A_n\}$. Let $S_1, \dots, S_n$ be the segments of M, and denote by l_j the length $|S_j|$ of S_j. Clearly, $|A_i| = \sum_{S_j \subseteq A_i} l_j$.

The idea of the characterization [41] is to assign variables to the lengths of the segments of a normal PCA model of the graph. The next step is to express that the lengths of the arcs must be equal, through a system of equations. With a solution to such a system, we adjust the model so as to reflect the segment lengths, maintaining the same circular ordering of the extreme points, and obtain a UCA model of the graph.

By Theorem 16, G is a UCA graph if and only if there is a solution formed by positive reals for the following system.

$$\begin{cases} |A_1| = |A_2| \\ |A_2| = |A_3| \\ \vdots \\ |A_{n-1}| = |A_n|. \end{cases}$$

The system can be written as:

$$\begin{cases} q_i : \sum\limits_{S_j \subseteq A_i} l_j = \sum\limits_{S_j \subseteq A_{i+1}} l_j, \quad 1 \le i \le n-1 \\ l_j > 0, \quad 1 \le j \le 2n. \end{cases}$$

$$\begin{array}{llllll}
 & \ldots = \ldots & & \ldots = \ldots & & \ldots = \ldots \\
q_i : & .l_j. = \ldots & q_k : & \ldots = .l_j. & q_i : & .l_j. = \ldots \\
 & \ldots = \ldots & & \ldots = \ldots & q_k : & \ldots = .l_j.
\end{array}$$

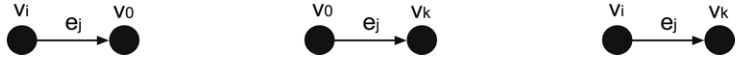

Fig. 7. The segment digraph.

These $n-1$ equations and $2n$ inequalities form the *full system* of M. The term $\sum_{S_j \subseteq A_i} l_j$ of the equation q_i is called the *left side* of q_i, and similarly we define the *right side* of it. By eliminating the common variables on both sides of q_i, we obtain the *reduced system* R. The following lemma describes a useful property of reduced systems.

Lemma 3 ([41]). *Let R be the reduced system of a normal PCA model M. Then*

(i) *Each segment length l_i of M appears at least once in R, unless S_j is contained in all arcs of M, or in none of them.*
(ii) *Furthermore, l_j appears at most twice in R. In the latter case, it appears in different sides and equations.*

In order to efficiently solve the system of equations, we describe a graph-theoretical model for R. Define the *segment digraph* D of R, having a vertex $v_i \in V(D)$, for each equation q_i of R, and an edge e_j for each variable l_j. Also, there is a distinguished vertex $v_0 \in V(D)$. That is, D has n vertices and no more than $2n$ edges. The directions of the edges e_j are defined as follows. If l_j is in the left (right) side of an equation q_i (q_k), e_j starts (ends) at v_i (v_k) otherwise e_j starts (ends) at v_0. See Fig. 7.

In order to formulate a characterization for UCA graphs, [41] employs the following generalization of eulerian digraphs. Let D be an arbitrary digraph and W a weighting of D, that is, an assignment of real positive weights w_j to the edges e_j of D, $1 \le j \le |E(D)|$. For vertex $v_i \in V(D)$, define its *in-weight* $w^-(v_i) = \sum_{e_j \in E^-(v_i)} w_j$, and its *out-weight* $w^+(v_i) = \sum_{e_j \in E^+(v_i)} w_j$. Say that W is *weakly eulerian* when $w^-(v_i) = w^+(v_i)$, for all vertices $v_i \in V(D)$. Finally, D is *weakly eulerian* when it admits a weakly eulerian weighting. Clearly, if D is eulerian then it is weakly eulerian.

Lin and Szwarcfiter have characterized UCA graphs, in terms of segment digraphs.

Theorem 18 ([41]). *A PCA graph is UCA if and only if its segment digraph is weakly eulerian.*

As a consequence, recognizing a UCA graph is equivalent to deciding if the corresponding segment digraphs are weakly eulerian. The following theorem characterizes general weakly eulerian digraphs.

Theorem 19 ([41]). *A digraph is weakly eulerian if and only if each of its connected components is strongly connected.*

The following corollary expresses a desired property for UCA models.

Corollary 2 ([41]). *Let D be a weakly eulerian digraph. Then D admits a weakly eulerian weighting in which all weights are integers.*

Next we refer to the recognition problem. Tucker's proof of the forbidden subgraph characterization for UCA graphs, in fact constructs a UCA model, whenever the graph does not contain any of the forbidden subgraphs. Therefore such a construction leads to an actual recognition algorithm for the class. However, as reported in [11,65], the process involves shrinking and lengthening of arcs, implying the manipulation of large integers. It is not known whether this process terminates in polynomial time.

The first polynomial time recognition algorithm for UCA graph has been formulated by Durán, Gravano, McConnell, Spinrad and Tucker [11]. It is based on Tucker method [75] and applies Corollary 1. Fixing the arcs to have unit length, the idea is to determine upper and lower bounds for the length of the circle. The upper bound is the smallest n/k value, among all (n, k)-circuits, whereas the lower bound is the largest m/l among all (m, l)-independent sets. Since $m/l \le n/k$ it follows that G is not a UCA if and only if $m/l = n/k$. The algorithm [11] constructs (n, k)-circuits by starting at every arc, and choosing as the next arc in the circuit, the neighbor arc which extends farthest in the model. For (m, l)-independent sets, the algorithm also starts at every possible arc, but simply chooses the next arc to be the first non-neighbor arc, in the clockwise direction. For performing these operations, the algorithm [11] requires $O(n^2)$ time, using union-find techniques.

A linear time recognition algorithm has been described by Lin and Szwarcfiter [41]. It is a direct consequence of the characterizations of Theorems 18 and 19. Given a PCA graph G by a normal PCA model of it, the algorithm [41] constructs the corresponding full system, reduced system and the segment digraph D. Then, it finds the connected components of D and verifies if each of them is strongly connected. In the affirmative case, G is UCA, while in the negative, it is not. All these operations can be performed in $O(n)$ time. Observe that for representing the full and reduced systems, we need just $O(n)$ indices, that is, $4n - 4$ indices for the full system and at most $4n$ for the reduced system.

When G is given by its sets of vertices and edges, both algorithms, [11,41], employ a recognition algorithm of PCA graphs for constructing a PCA model. Algorithms of complexity $O(n + m)$ with this purpose have been described in [9,36]. If the PCA model is not normal then we need an algorithm for transforming a given PCA model into a normal PCA model, while preserving the implied adjacencies. An algorithm with this purpose has been described in [11], with complexity $O(n^2)$. In [41] another algorithm has been formulated, with linear complexity $O(n)$. Consequently, the complexity of the entire recognition algorithm [41] is $O(n + m)$. This complexity reduces to $O(n)$, whenever the graph is given by a PCA model of it.

Kaplan and Nussbaum [37] described a recognition algorithm for UCA graphs, also based on Tucker's characterization [75]. Basically, it employs the same strategy as the algorithm [11], that is, to compute upper and lower bounds for the circle length of a possible UCA model. However, instead of considering (n, k)-circuits and (m, l)-independents set starting at every arc of the model, the algorithm [37], needs only to start from one of the vertices of these special subgraphs. In addition, [37] applies a generalization of (n, k)-circuits and (m, l)-independent sets, by allowing repetitions of vertices and employing paths, instead of cycles. These alterations allows [37] one to decrease the complexity of the algorithm [11] from $O(n^2)$ to $O(n + m)$. In addition, the algorithm [37] is simpler than [11]. Besides recognizing UCA graphs. [37] contains an algorithm for exhibiting a negative certificate, that is a certificate for the case that the graph is PCA, but not UCA. This certificate is described in terms of a minimum (n, k)-circuit and maximum (m, l)-independent set, and requires $O(n + m)$ time to be computed.

Finally, for constructing UCA models, so far the only known algorithm is by Lin and Szwarcfiter [41], described below.

The algorithm [41] again employs the characterization of UCA graphs in terms of weakly eulerian digraphs, and the characterization of the latter digraphs in terms of strongly connected components, respectively Theorems 18 and 19. Given a graph G, let M be a normal PCA model of it. First consider the reduced system of equations. Now the idea is to solve the system. The solution informs the required segment lengths that would lead to a UCA model. Then modify M accordingly, while preserving the circular ordering of its extreme points. The notation below is employed.

An *arborescence* is a orientation of a tree. An *out-arborescence* (*in-arborescence*) is a rooted arborescence where all edges are directed towards the leaves (root).

The algorithm for the model construction, first runs the recognition algorithm. Let D be the segment digraph of G. The aim is to determine a weakly eulerian weighting for G. Consider each connected component B of D. We know that it induces a strongly connected component. To start, assign a unit weight w_j to each of the edges e_j of B. We need to conveniently increase some of the weights of the edges, so as to achieve $w^-(v) = w^+(v)$, for each vertex $v \in B$. When $w^-(v) < w^+(v)$ say that v is *in-deficient*, while $w^+(v) < w^-(v)$ means v is *out-deficient*. All the possible in-deficiencies are corrected by conveniently increasing the weights of the edges of an arbitrary spanning out-arborescence of B. Similarly, the out-deficiencies are eliminated, using a spanning in-arborescence of B. An incremental technique is employed for the weights' assignments. The algorithm for the UCA model construction requires $O(n)$ time, given a PCA model of the graph.

All the weights constructed by the above algorithm are integers of size less than $4n$ [41]. This mean that the UCA model, obtained by the above construction, is such that its extreme points correspond to integers of size less than $4n$.

The results contained in [41] were extended and described in paper [43]. In fact, [43] reduces the problem of solving the reduced system to the problem of finding a feasible circulation in a network with non-negative lower capacities and unbounded upper capacities. A linear time solution for the latter problem is then formulated, which corresponds to a generalization of the above-described method for constructing UCA models.

6. Helly circular-arc graphs

We describe in this section results concerning the structure of HCA graphs. Recall that these graphs are exactly the CA graphs admitting a model whose arcs satisfy the Helly property. Such models are called HCA models.

Among all the CA graphs, the HCA graphs are those closest to interval graphs. This is precisely due to the Helly property. Intervals of a line satisfy the Helly property, as already mentioned. Therefore, as for interval graphs, cliques of HCA graphs correspond to selected points of the circle. A consequence is that, like interval graphs, HCA graphs have at most n cliques.

The first characterization of HCA has been given by Gavril [17]. It shows how HCA graphs arise from interval graphs, by extending the linear ordering of cliques to a circular ordering.

Theorem 20 (*[18]*). *A graph G is an interval graph if and only if it admits a clique matrix with the consecutive 1's property on its columns.*

The generalization to HCA graphs is as follows.

Theorem 21 (*[17]*). *A graph G is an HCA graph if and only if it admits a clique matrix with the circular consecutive 1's property on its columns.*

Other characterizations of HCA graphs have been recently formulated, by Joeris, McConnell and Spinrad [34] and by Lin and Szwarcfiter [42]. Such characterizations are essentially similar, and were obtained independently. These results are presented below, under the notation of [42].

Before considering HCA graphs, the first aim is to characterize HCA models. Let $M = (C, \mathcal{A})$ be a CA model. Recall that $\overline{M} = (C, \overline{\mathcal{A}})$ denotes the complement model of M.

The following theorem characterizes HCA models.

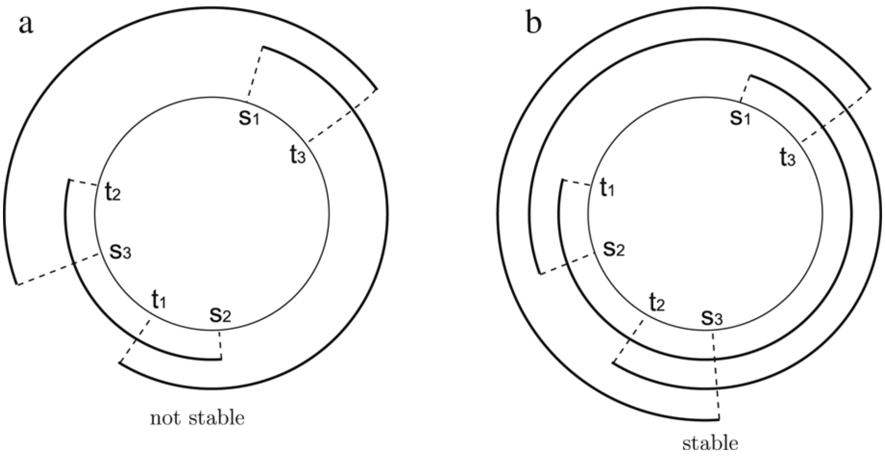

Fig. 8. Stable and not stable models.

Theorem 22 ([34,42]). *A CA model M is HCA if and only if*

(i) *whenever three arcs of $\mathcal{A}$ cover C, two of them also cover C, and*
(ii) *the intersection graph of $\overline{M}$ is chordal.*

Let $M = (C, \mathcal{A})$ be a model which is not Helly. Say that M is *minimally non-Helly* when removing any arc of $\mathcal{A}$ would turn the model Helly. Clearly, a minimally non-Helly model has at least 3 arcs. The following is a characterization for such models.

Corollary 3 ([34,42]). *A CA model $M = (C, \mathcal{A})$ is minimally non-Helly if and only if*

(i) *$\mathcal{A}$ is intersecting and covers C, and*
(ii) *two arcs of $\mathcal{A}$ cover C precisely when they are not consecutive in the circular ordering of $\mathcal{A}$.*

Next, we describe the characterization of HCA graphs. The following notation is employed.

Let $\mathcal{A}$ be a CA model. An *s-sequence* of M is a maximal consecutive sequence of start points, in the circular ordering. Similarly, define a *t-sequence*, for the end points. In general, an *extreme sequence* is an s-sequence or a t-sequence. Clearly, the extreme sequences are a partition of the extreme points, and are circularly ordered.

Let S be a extreme sequence. Denote by $NEXT(S)$ and $NEXT^{-1}(S)$, respectively the extreme sequences that immediately precede S and succeed S, in the ordering. Also, if p is an arbitrary extreme point of C then $SEQUENCE(p)$ denotes the unique extreme sequence that contains p. The following definitions are central.

Let s_i be a start point of M, and $S = SEQUENCE(s_i)$. Say that s_i is *stable* precisely when $i = j$ or $A_i \cap A_j = \emptyset$, for every $t_j \in NEXT^{-1}(S)$. More generally, the model M is *stable* when all starting points are so. Fig. 8 illustrates two CA models of a triangle. The first of them is not stable, but the second is so.

Next, we describe a family of graphs. It will be shown that these graphs form a forbidden family for HCA graphs. An *obstacle* is a graph H, containing a complete set $Q \subseteq V(H)$, $|Q| = q \geq 3$, admitting a circular ordering $v_1, \ldots, v_q$ of its vertices, where each edge v_i, v_{i+1} satisfies

(i) $N(w_i) \cap Q = Q \setminus \{v_i, v_{i+1}\}$, for some $w_i \in V(H) \setminus Q$, or
(ii) $N(u_i) \cap Q = Q \setminus \{v_i\}$ and $N(z_i) \cap Q = Q \setminus \{v_{i+1}\}$, for some adjacent vertices $u_i, z_i \in V(H) \setminus Q$.

In Figs. 9(a) and 9(b), the above cases (i) and (ii) are respectively satisfied, for edge v_5v_1. Figs. 10(a) and (b) depict two obstacles.

The following is a characterization of HCA graphs, in terms of stable models and obstacles.

Theorem 23 ([34,42]). *The following affirmatives are equivalent for a CA graph G.*

(i) *G is HCA.*
(ii) *G does not contain obstacles as induced subgraphs.*
(iii) *All stable models of G are HCA.*
(iv) *One stable model of G is HCA.*

Next, we consider the algorithms that arise from the described characterizations.

Gavril's characterization leads directly to a simple algorithm for recognizing HCA graphs. Let G be a graph. Construct the clique matrix of G and verify if its columns satisfy the circular consecutive 1's property. By Theorem 21, G is a HCA graph precisely when the answer is positive.

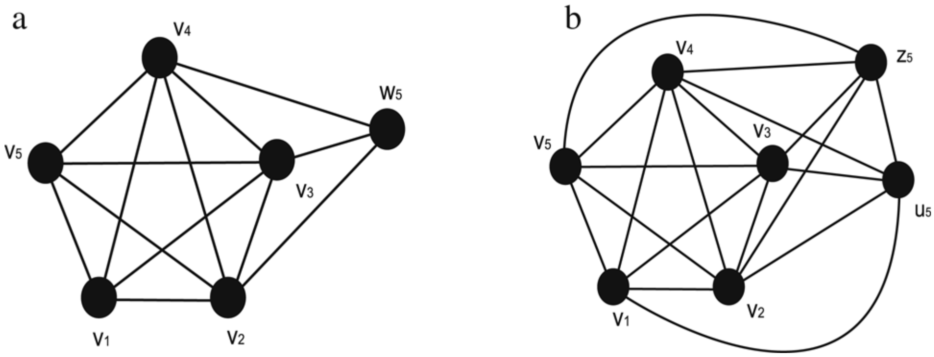

Fig. 9. Cases (i) and (ii).

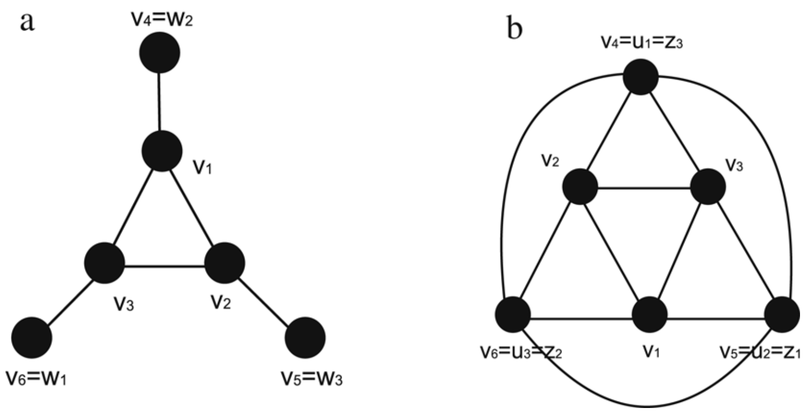

Fig. 10. Two obstacles.

An HCA graph has at most n cliques. We can therefore restrict to clique matrices having at most n cliques, otherwise G is not HCA. Using the algorithm by Paull and Unger [55], we can generate these $O(n)$ cliques in $O(n^3)$ time. Checking the circular consecutive 1's property can be done in linear time. Consequently, the complexity of algorithm [17] is $O(n^3)$.

Finally, consider the algorithms based on the characterizations of [34,42].

For recognizing minimally non-Helly models, apply Corollary 3. There is no difficulty in verifying conditions (i) and (ii) in $O(n)$ time, given the model.

For recognizing HCA models, the basis is Theorem 22. A direct implementation of it would consist of verifying whether the intersection graph G_c of $\overline{M}$ is chordal. This would lead to an algorithm of complexity $O(n^2)$, because G_c has $O(n^2)$ edges. However checking co-chordality of $\overline{G_c}$, instead of chordality of G_c, can be done in $O(m)$ time, because $\overline{G_c}$ has at most m edges, and verifying co-chordality can be done in linear time [21]. Consequently, HCA models can be recognized in $O(m)$ time. However, Joeris, McConnell and Spinrad [34] describe more elaborate implementations, employing union-find tools, which requires just $O(n)$ time.

The algorithm for recognizing HCA graphs [34,42] follows from Theorem 23. Its outline is as follows. Let G be a graph.

(1) Recognize whether G is CA. If affirmative, let M be a CA model of it. If negative, stop as G is not HCA.
(2) Transform M into a stable model.
(3) Recognize if M is HCA. If affirmative G is HCA. If negative it is not.

For Step 1, apply the algorithm by McConnell [49] or that by Kaplan and Nussbaum [36] which output a CA model M for G, or report that G is not CA. It should be observed that the algorithm [36] outputs a stable model of G, in case G is CA. However, to include the case where the graph might be given by an arbitrary CA model of it, we report on the algorithm for transforming any CA model into a stable one, corresponding to Step 2, as below described.

The basic idea of the method for transforming M into a stable model is to increase as much as possible the lengths of the arcs, by possibly extending both of its extremes, while maintaining the adjacencies of the corresponding graph. A possible strategy for accomplishing it is first to extend the end points t_j of the arcs of $\mathcal{A}$. That is, for each t_j, we find the closest start point s_i, in the clockwise direction, satisfying $i = j$ or $A_i \cap A_j = \emptyset$. Then we move t_j so as to become the extreme point just before s_i, clockwise. After considering all end points, we possibly extend the start points, as follows. For each s_i, determine the closest end point t_j, in the counterclockwise direction, satisfying $i = j$ or $A_i \cap A_j = \emptyset$. Let $T = SEQUENCE(t_j)$. Then move t_j towards T, counterclockwise, possibly reordering T and transforming this sequence into $T' s_i T''$, where T' is the set of end points $t_k \in T$, such that $i = k$ or $A_i \cap A_k = \emptyset$, while $T'' = T \setminus T'$. This algorithm transforms M into a stable model. A consequence of it is the following.

Corollary 4 ([34,42]). *Every graph admits a stable model.*

The stable model algorithm can be implemented in $O(m)$ time. Again, a more elaborate implementation by Joeris, McConnell and Spinrad [34] shows that the time bound is $O(n)$. Consequently, HCA graphs can be recognized in $O(n + m)$ time.

We remark that there is no difficulty in producing a negative certificate, for the case where G is a CA graph, but not HCA. In this situation, Step 3 of the algorithm would report that M is not an HCA model. On the other hand, Theorem 20 ensures that G would contain an obstacle as an induced subgraph. The algorithm for producing the negative certificate consists of obtaining the obstacle from the non-HCA stable model [34,42]. This operation can be implemented in linear time.

Finally, we mention that reference [35] contains a revised and extended version of the results of [34,42].

7. Co-bipartite circular-arc graphs

One of the main classes of CA graphs, in general, is that of co-bipartite CA graphs. There are some nice structural characterizations for this class. Besides, many of the algorithms formulated in Sections 3 and 4, for recognizing CA and PCA graphs, respectively, handle separately the case where the given graph is co-bipartite. The following property was employed in Tucker's algorithm for recognizing CA graphs, and afterwards in other algorithms for CA and PCA graphs.

Theorem 24 ([77]). *Let $M = (C, \mathcal{A})$ be a CA model of a co-bipartite graph. Then there are two point of C, such that every arc of $\mathcal{A}$ contains at least one of them.*

The above property has been generalized by Hell and Huang [25], to the entire class of CA graphs.

Theorem 25 ([25]). *Let $M = (C, \mathcal{A})$ be a CA model of some graph, whose vertices can be covered by k cliques. Then there are k points of C, such that every arc of $\mathcal{A}$ contains at least one of them.*

The first characterization of co-bipartite CA graphs is by Trotter and Moore [69], which also described a forbidden subgraph characterization for the class.

Theorem 26 ([69]). *Let G be a co-bipartite graph. Then G is a CA graph if and only if $\overline{G}$ can be oriented as a partial order of interval dimension* 2.

There are several characterizations for co-bipartite CA graphs. Recall that Spinrad has characterized such graphs in terms of partial order dimension (Theorem 3). The following characterization by Hell and Huang describes the class, in terms of a special edge coloring.

Theorem 27 ([25]). *Let G be a co-bipartite graph and $U \cup U' = V(G)$, a bipartition of its vertices into cliques U, U'. Then G is a CA graph if and only if the edges joining U, U' can be bi-colored, in such a way that the edges of any induced C_4 have distinct colors.*

Hell and Huang have formulated a further simple characterization of co-bipartite CA graphs. Let G be a graph. Denote by G^* the graph where $V(G^*) = E(G)$, and two vertices of G^* are adjacent when the end points of the corresponding edges of G induce a chordless cycle of length 4.

Theorem 28 ([25]). *A graph G is a co-bipartite CA graph if and only if both $\overline{G}, G^*$ are bipartite.*

The following concept has been employed in the characterizations of co-bipartite CA graphs. An *edge-asteroid* of a graph G is a set of edges e_i, admitting a circular ordering $e_0, e_1, \ldots, e_{2k}$, such that for any $i = 0, 1, \ldots, 2k$, there is a path containing both e_i, e_{i+1}, which avoids the neighbors of e_{i+k+1}. The following theorem is by Feder, Hell and Huang.

Theorem 29 ([16]). *A graph G is a co-bipartite CA graph if and only if $\overline{G}$ is chordal bipartite and contains no edge-asteroids.*

The following concept has been employed by Sen, Das and West, in a characterization of co-bipartite CA graphs. Let D be a digraph, and $\mathcal{I}, \mathcal{I}'$ two families of intervals of the line, $|\mathcal{I}| + |\mathcal{I}'| = n$. Say that D is an *interval containment* digraph when there is an one-to-one correspondence between vertices v_i of D and intervals I_i of $\mathcal{I} \cup \mathcal{I}'$, such that $v_iv_j \in E(G)$ if and only if $I_i \in \mathcal{I}, I_j \in \mathcal{I}'$ and $I_i \supset I_j$, for all $v_i, v_j \in V(G)$.

Theorem 30 ([60]). *A graph G is a co-bipartite CA graph if and only if G can be oriented as an interval containment digraph.*

Another approach for describing co-bipartite CA graphs, employs the following concept introduced by Gutman [20] and Riguet [56]. A *Ferrers* digraph is a digraph D, where $v_iv_j \in E(D)$ if and only if $N^+(v_i) \supset N^+(v_j)$. For an arbitrary digraph D, the *Ferrers dimension* of D is the least number of spanning Ferrers subdigraphs of D, whose intersection equals D. Digraphs of Ferrers dimension 2 are of special interest. In fact, Sen, Sanyal and West characterized co-bipartite CA graphs in terms of digraphs of Ferrers dimension 2.

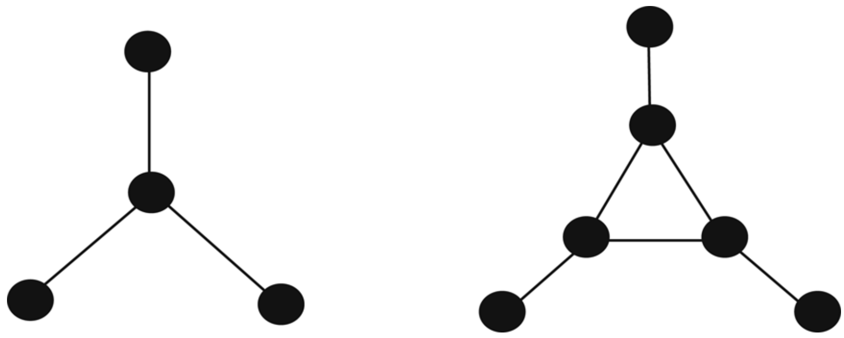

Fig. 11. The forbidden subgraphs for PCA chordal graphs.

Theorem 31 (*[61]*). *A graph G is a co-bipartite CA graph if and only if $\overline{G}$ can be oriented as a digraph of Ferrers dimension* 2*.*

Cogis [6] related the Ferrers dimension to partial order dimension. Ferrers dimension can also be directly related to interval dimension and to trapezoid graphs; see Spinrad [65], and Ma and Spinrad [46,45].

Co-bipartite CA graphs have also been employed in the problem of list homomorphisms to a fixed graph H. This problem can be solved in polynomial time when $\overline{H}$ is a co-bipartite CA graph, and it is NP-complete otherwise [16].

Hell and Huang have shown that a special class of co-bipartite CA graphs is closely related to interval bigraphs.

Theorem 32 (*[26]*). *A graph G is an interval bigraph if and only if $\overline{G}$ is a co-bipartite CA graph admitting a normal CA model.*

It should be noted that Müller [52], described a polynomial time algorithm for recognizing interval bigraphs. The complexity of it is $O(n^5m^6 \log n)$. However, no algorithm with lower complexity is known so far, for this recognition problem.

The class of co-bipartite proper CA graphs has been characterized by Hell and Huang, in terms of forbidden subgraphs.

Theorem 33 (*[26]*). *Let G be a co-bipartite graph. Then G is a proper CA graph if and only if $\overline{G}$ does not contain an induced cycle of length at least 6, or any of the graphs H_2, H_4 and H_5 of Fig. 4.*

8. Other classes

In this section, we consider some other classes of circular arc graphs.

It seems natural to consider the classes of CA graphs admitting a model which simultaneously satisfies two of the models PCA, UCA and HCA. Recall that the latter are the main models of CA graphs and are defined by imposing some restriction directly on the arcs of the model. Clearly, PCA graphs contain UCA graphs, so we only need to consider models simultaneously PCA and HCA, and UCA and HCA.

Lin, Soulignac and Szwarcfiter [40] have defined a *proper Helly circular-arc (PHCA)* graph, as a graph admitting a model which is simultaneously proper and Helly. Observe that a graph G may be simultaneously PCA and HCA, but not PHCA.

The following is a characterization of PHCA graphs, by forbidden induced subgraphs.

Theorem 34 (*[40]*). *Let G be a PCA graph. Then G is PHCA if and only it contains neither the Hajós graph nor the 4-wheel as an induced subgraph.*

Another characterization described in [40] leads to a linear time algorithm for recognizing PHCA graphs. The algorithm also produces certificates, both positive and negative, within the same bound.

Similar results were also described in the above paper, for the class of graphs admitting a model which is simultaneously UCA and HCA.

Subclasses of CA graphs can also be obtained by imposing restrictions on the actual graphs, instead of on the models. In this way, CA graphs which also belong to some other known class have been studied.

Chordal PCA graphs have been characterized by Bang-Jensen and Hell, through forbidden subgraphs.

Theorem 35 (*[1]*). *Let G be a chordal graph. Then G is a proper CA graph if and only if it does not contain the graphs of Fig. 11 as induced subgraphs.*

More recently, Bonomo, Durán, Grippo and Safe [3] characterized four subclasses of CA graphs, each of them formed by graphs simultaneously belonging to well-known classes. All the characterizations are in terms of minimal forbidden induced subgraphs.

The following is a characterization for a P_4-free graph to be a CA graph. Denote by G^0 and G^1 the graphs obtained from G, by adding an additional vertex adjacent to no vertex of G and to exactly one vertex of it, respectively.

Theorem 36 ([3]). *Let G be a graph with no P_4's. Then G is a CA graph if and only if it contains neither $K_{2,3}$ nor C_4^0 as induced subgraphs.*

In [3], characterizations have also been described for a $K_{1,3}$-free graph to be a CA graph, for a K_3^1-free graph to be a CA graph and for a $(K_4 - e)$-free graph to be a CA graph.

Some subclasses of CA graphs have been considered, in relation to specific problems. Trotter [70] has proved that a partial order of a CA comparability graph has dimension at most 4. Durán, Lin, Mera and Szwarcfiter [12,13] have described polynomial time algorithms for finding maximum clique-independent sets and minimum clique-transversals in $\overline{3K_2}$-free CA graphs. Lin, McConnell, Soulignac and Szwarcfiter [39] have studied the clique structure of HCA graphs.

Finally, we mention that some other classes of graphs are closely related to CA graphs. Among them we mention circular-arc containment graphs (Nirkhe [53], cf. [5]) and circular-arc digraphs (Sen, Das and West [59]).

9. Conclusions and open problems

We have described some structural characterizations and recognition algorithms for CA graphs and subclasses of it, such as PCA, UCA, HCA and co-bipartite CA graphs. The following table summarizes the considered recognition algorithms.

Recognition algorithms for circular-arc graphs and subclasses				
Class	Model construction	Negative certificate construction	Complexity	Reference
CA	Yes	No	$O(n^3)$	[77]
CA	Yes	No	$O(nm)$	[28]
CA	Yes	No	$O(n+m)$	[65]
CA	Yes	No	$O(n+m)$	[49]
CA	Yes	No	$O(n+m)$	[36]
PCA	Yes	No	$O(n^2)$	[73]
PCA	Yes	No	$O(\Delta m)$	[24]
PCA	Yes	No	$O(n+m)$	[9]
PCA	Yes	Yes	$O(n+m)$	[37]
UCA	No	No	$O(n^2)$	[11]
UCA	Yes	No	$O(n+m)$	[41]
UCA	No	Yes	$O(n+m)$	[37]
HCA	Yes	No	$O(n^3)$	[17]
HCA	Yes	Yes[a]	$O(n+m)$	[42]
HCA	Yes	Yes[a]	$O(n+m)$	[34]

[a] Certificate for the graph to be CA and not HCA.

We close this survey with the following list of open problems.

(1) Characterize CA graphs by forbidden induced subgraphs.
(2) Describe an algorithm for finding a certificate for a graph not to be a CA graph.
(3) Characterize CA graphs admitting a normal CA model. Describe a recognition algorithm for this class.
(4) Characterize CA graphs admitting a CA model where the arcs have at most two sizes.
(5) Find a UCA model for a UCA graph whose extreme points correspond to integers, and such that it minimizes the maximum segment length (respectively, minimizes the circle length).
(6) Characterize circular-arc bigraphs (that is, the generalization of interval bigraphs, where the intervals of the models are replaced by arcs of a circle).
(7) Characterize clique graphs of CA graphs.

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