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AN ANALYSIS OF UNIDIMENSIONAL SOLITON GAS MODELS OF MAGNETOHYDRODYNAMIC TURBULENCE IN THE SOLAR WIND

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ABSTRACT

We compare two statistical models of Alfvén solitons whose evolution is described by the one-dimensional derivative nonlinear Schrödinger (DNLS) equation, contrasting their predictions with solar wind observations. Both distribution functions give the same mean number of solitons. This confirms a previous calculation of Ponce Dawson and Ferro Fontán of the number of solitons which evolve from an arbitrary initial condition. One of the distribution functions follows an exponential law with soliton energy and the other follows a power law; the latter gives better results than the former. Within these models, we can explain the variation of the observed spectra (spectral index, outer scale, and maximum value) with the heliocentric distance. This variation is related to the radial dependence of the mean level of modulational instability in the medium. Concerning the spectral index, our calculation improves that of Ovenden, Shah, and Schwartz, because an average over the soliton phases is included.

Subject headings: hydromagnetics — Sun: solar wind

I. INTRODUCTION

A great amount of information about the structure of the fluctuations in the solar wind has been collected from spacecraft observations for a long time. The results are currently presented in the form of autocorrelation or cross-correlation functions of the magnitudes of interest (magnetic field, density enhancements, and velocities). In general, these correlation functions are Fourier transformed to produce power spectra which are functions of the frequency. There has been much interest in explaining these observed spectra.

The magnetic field fluctuations are generally measured *in situ*. Their spectra exhibit similar behavior (see, e.g., Fig. 3 of Jokipii 1971): they remain almost constant for low-frequency values (wavenumbers k and frequencies ν are related by $k = 2\pi\nu/v_w$, with $\boldsymbol{v}_w$ the solar wind velocity), follow a power law k^{-s} along approximately two orders of magnitude of k with $s \leq 2$, and afterward decrease more rapidly (see, e.g., Denskat and Neubauer 1983). The unidimensional power index increases with the heliocentric distance. According to the observations analyzed in Denskat and Neubauer (1983), the index ranges between 1.59 and 1.69 at 0.97 AU and between 0.87 and 1.15 at 0.24 AU. The k values at which the spectra change from a constant to a power law may be associated with the existence of an outer scale $L_0(k_0 \sim L_0^{-1})$. The value of k_0 decreases with increasing heliocentric distance (see, e.g., Denskat and Neubauer 1983).

The density fluctuations spectra, which are generally obtained indirectly by interplanetary scintillation (IPS) techniques, also show the same behavior. According to Scott, Ricket, and Armstrong (1983) and Scott, Coles, and Bourgois (1983), a broken power law is the best fit to the observations with a unidimensional power index $s \cong 1.0 \pm 0.5$ if $0.002\ \text{km}^{-1} \leq k \leq 0.016\ \text{km}^{-1}$ and $s \cong 1.7 \pm 0.3$ for $0.016\ \text{km}^{-1} \leq k \leq 0.05\ \text{km}^{-1}$ at heliocentric distances between 0.27 and 0.47 AU. The k value at which the power index changes, leading to a steeper spectrum, may be associated with the existence of an inner scale $L_i(k_i \sim L_i^{-1})$. The outer and inner scales define an interval (k_0, k_i) in wavenumber space over which the spectra follow a power law.

Another fact that has been known since the work of Belcher, Davis, and Smith (1969) is the presence of finite-amplitude Alfvén waves in the solar wind. The nonlinear evolution of Alfvén waves in the presence of sound waves has been studied extensively, and simplified evolution equations have been derived for different cases. In this work, we will be concerned with the so-called derivative nonlinear Schrödinger (DNLS) equation (Mjølhus 1976, Spangler and Sheerin 1982*a*, Sakai and Sonnerup 1983):

$$iq_t + iq_x + \frac{i}{4(1-\beta)}(q|q|^2)_x + q_{xx} = 0 , \tag{1}$$

where $q = (B_y - iB_z)/B_0 = (-v_{iy} + iv_{iz})/v_A$ (B_0 is the static magnetic field which is supposed to lie on the x-direction, $\boldsymbol{v}_i$ is the velocity of the ions, v_A is the Alfvén velocity $v_A^2 = B_0^2/(4\pi n_0 m_i)$, n_0 is the unperturbed density, and $\beta = c_s^2/v_A^2(c_s = [\gamma(T_e + T_i)/(m_i + m_e)]^{1/2}$ is the sound velocity with γ the ratio of specific heats, $\gamma = 5/3$ in the adiabatic case, and T_e, T_i, m_e, m_i are the temperatures and masses of electrons and ions). In equation (1), time is measured in units of $\alpha/\omega_{ci}(\alpha = \frac{1}{2}[1 - (m_e/m_i)] \cong \frac{1}{2}\omega_{ci} = eB_0/m_i c$ is the ion-cyclotron frequency) and lengths are measured in units of $d_i = \alpha v_A/\omega_{ci}$.

The DNLS equation describes the evolution of weakly nonlinear, weakly dispersive MHD waves propagating parallel to the ambient magnetic field $\boldsymbol{B}_0 = B_0 \hat{\boldsymbol{x}}$. Dispersion enters through finite ion-inertial effects (see, e.g., Roberts 1984), and the nonlinear term arises due to a coupling between transverse magnetic field (Alfvén waves) and plasma density perturbations.

Equation (1) may be obtained from the set of nondissipative MHD equations, where all the variables are supposed to depend only on the x-coordinate (see Sakai and Sonnerup 1983 or Spangler and Sheerin 1982*a*) by means of a reductive perturbation method (Taniuti and Wei 1968). Because of this geometric restriction, the model obtained is unidimensional and the equations governing the evolution of the x-component of the magnetic field are simply $\partial_x B_x = \partial_t B_x = 0$. This means that $B_x = B_0$ always, and therefore longitudinal perturbations on $\boldsymbol{B}$ cannot be described by equation (1). The expansions performed on the remaining variables $B \equiv B_y - iB_z$, $v \equiv v_{iy} - iv_{iz}$, $n(n \equiv n_i \cong n_e)$, and $u = v_{ix}$, and the stretching of the coordinates,

$$X = \delta(x - t), \qquad T = \delta^2 t , \tag{2}$$

are standard. Two small positive parameters δ and ϵ are introduced, and the following scaling is assumed:

$$\frac{\delta n}{n_0} \equiv \frac{n - n_0}{n_0} \sim \frac{\delta u}{v_A} \equiv \frac{u}{v_A} \sim O(\epsilon) , \tag{3a}$$

$$\frac{\delta B}{B_0} \equiv \frac{B}{B_0} \sim \frac{\delta v}{v_A} \equiv \frac{v}{v_A} \sim O(\epsilon^{1/2}) . \tag{3b}$$

$$\delta = \epsilon . \tag{4}$$

The quantities δn, δu, δB, and δv are perturbations of the variables around an equilibrium solution which, if we intend to apply the model to the solar wind, corresponds to a description in the wind reference frame.

It may be seen from equation (3) that while the perturbations of the Alfvénic variables δB and δv have nonvanishing terms of order $\epsilon^{1/2}$, the lowest order terms of the sound variables δn and δu are $O(\epsilon)$. It means that the magnetic field perturbation is $\epsilon^{-1/2}$ times greater than that of the density. This is the same assumption as the one considered by Hollweg (1971). It is valid if we restrict our attention to almost pure Alfvén waves (Spangler 1987) and would fail if the decay instability of an Alfvén wave into ion sound waves (Terasawa *et al.* 1986) had produced a high level of the latter ones (Spangler 1987). However, it is realistic for the study of fluctuations in the solar wind (see, for instance, Fig. 1 of Denskat and Neubauer 1983, where $|\delta B|/B_0 \cong 0.4$, while $|\delta n|/n_0 \cong 0.1$; according to this figure, ϵ would be of order 0.1). Besides, the scaling in equation (3) also means that, to lowest order in ϵ, the magnetic and thermal energy perturbations are of the same order. This differs with the situation described in Patel and Dasgupta (1987), where a KBM expansion method (Kakutani and Sugimoto 1974), instead of the reductive perturbation method (Taniuti and Wei 1968), is used, leading to the NLS equation for the Alfvén wave amplitude if a scaling $|\delta n|/n_0 \sim |\delta B|/B_0$ is assumed.

We must note then that expression (2) implies a change to a frame of reference, which we will call the Alfvén reference frame, that travels at the Alfvén velocity in the positive x-direction with respect to the wind. This imposes a further restriction, because only forward propagation of waves may be described in this way. Therefore, the DNLS equation cannot model the decay instability of Alfvén waves, because it is not applicable for a case where backward propagating waves are present (Terasawa *et al.* 1986).

The stretching of the x-coordinate in equation (2) is valid when long wavelengths predominate, with the quantity δ representing a measure of the characteristic wavenumber of the perturbation ($\delta \ll 1$). The characteristic length of the microscale L_m in the interplanetary medium satisfies $L_m \leq 0.01$ AU (Burlaga 1972). Considering the typical values $v_A \cong 50$ km s^{-1} and $\omega_{ci} \cong 0.5$ s^{-1} (which corresponds to a magnetic field $B_0 = 5\gamma$), L_m measured in our units ($d_i = 50$ km) results in $L_m \leq 3 \times 10^4 d_i$. Thus, we may see that the stretching in equation (2) is suitable for the study of a wide range of microscale phenomena in the solar wind. However, the predictions of the model may fail in the region of large wavenumbers.

The quantity T in equation (2) serves to describe the slow time variation of the variables involved in the Alfvén reference frame. This slow variation is related to a frequency shift $\Delta\omega$ (because ω does not exactly satisfy the dispersion relation $\omega = k$) and is due to the combined effect of dispersion and nonlinear interactions between sound and Alfvén waves. To lowest order in δ, dispersion produces a shift of order δ^2. In order for nonlinear effects to balance the effect of dispersion, it is necessary that they contribute with a shift of order δ^2 also, and therefore $\delta = \epsilon$ (eq. [4]). The stretching given in equation (2), while being reasonable for the quantities relevant to the Alfvén wave (v and B), may become invalid for the sound wave–associated quantities (u and n). It means that we are only describing slow time variations in the Alfvén reference frame (more rapid variations are "integrated away"). More quantitatively, the model describes only processes with time scales much greater than the time it takes for a sound wave to propagate across an Alfvén wave packet (Spangler 1985, 1986). This is called the static approximation and has been discussed in Sakai and Sonnerup (1983) and in Spangler (1987). The nonstatic case may be described by the set of equations

$$i\delta B_{t'} + iv_A \delta B_{x'} + \frac{i}{2}\left[(\delta u\, \delta B)_{x'} + \left(\delta u - v_A \frac{\delta n}{n_0}\right)\delta B_{x'}\right] + \delta B_{x'x'} d_i v_A = 0 , \tag{5a}$$

$$\delta n_{t't'} - \beta v_A^2 \delta n_{x'x'} = (|\delta B|^2)_{x'x'} \frac{n_0}{2B_0^2} v_A^2 , \tag{5b}$$

$$\delta n_{t'} + n_0 \delta u_{x'} = 0 \tag{5c}$$

(Sakai and Sonnerup 1983), where x' and t' are related to the dimensionless coordinates x and t of equation (1) by $x' = d_i x$ and $t' = \alpha t/\omega_{ci}$. A similar set of equations is obtained in Spangler (1987).

The static approximation is recovered by inserting equation (2) into equation (5) retaining the terms of lowest order in ϵ, and changing back to the wind reference frame. During this procedure, the following equations are found:

$$\frac{\delta n}{n_0} = \frac{\delta u}{v_A} , \tag{6a}$$

$$\frac{\delta B}{B_0} = -\frac{\delta v}{v_A}, \tag{6b}$$

$$\delta n = \frac{|\delta B|^2}{2(1-\beta)} \frac{n_0}{B_0^2}, \tag{6c}$$

A comparison of equations (5) and (6) shows that the density and magnetic field perturbations are no longer proportional in the nonstatic case. Instead, the evolution of the density perturbation is described by an inhomogeneous wave equation equal to the one derived by Hollweg (1971) with a ponderomotive force term proportional to the gradient of the Alfvén wave intensity. An analysis of the differences between the evolution of the density fluctuations as predicted by equations (5) and (6) has been made in the work by Spangler (1987). It is shown in this paper that while the evolution of δn in the presence of an Alfvén wave packet which does not steepen with time is perfectly describable by the static approximation, this is not the case for steepening wave packets. The departure from the static approximation is mainly determined by β and the ratio $l_0/(v_A \tau_{NL})$, where τ_{NL} is the characteristic steepening time for an initial wave packet of modulation scale l_0 (see eq. [59] below). In the nonstatic case, the position of the maximum density perturbation is displaced with respect to that of the Alfvén wave packet, and, for sufficiently large values of $l_0/(v_A \tau_{NL})$, the sign of δn can even change within one wave packet. However, the value of δn usually does not exceed the static estimate in equation (6) (Spangler 1987). Moreover, because $l_0/(v_A \tau_{NL}) \sim |\delta B|^2/(B_0^2|1-\beta|)$ (Spangler 1987), we can conclude that the static approximation is valid in the case of small-amplitude waves, and, as it is speculated in Spangler (1987), may still be valid for large-amplitude waves if damping processes are included. Moreover, because nonstatic features appear only during the phase of wave packet steepening, they will not be observable in the presence of Alfvén solitons, which, as we will show later, are special noncontracting traveling solutions of the DNLS equation. It may be also seen from equation (6) that the static approximation becomes invalid for $\beta = 1$, when the sound and Alfvén speeds are the same. This singularity appears in all treatments which use fluid equations (Spangler and Sheerin 1982a; Khanna and Rajaram 1982). In Mjølhus and Wyller (1988), a DNLS–like equation is obtained from a hybrid fluid and kinetic model adequate for a collisionless situation, using the reductive expansion method. This treatment provides an additional term which represents nonlinear Landau damping by resonant particles. When this term is not taken into account, the relevant equation reduces to equation (1), but with coefficients c_1 and c_3 in front of the nonlinear and dispersive terms, respectively. For $\beta = 0$, c_1 and c_3 reduce to those of equation (1). In the limit of cold ions, they still resemble the coefficients of equation (1), but become less similar as the temperature ratio T_i/T_e is increased (Mjølhus and Wyller 1986). The value of c_1 always remains finite (though large) in the vicinity of $v_A = c_s$ and only changes sign at $\beta = 1$ for extremely low values of T_i/T_e. For higher values of this ratio, the coefficient c_1 hardly differs from its cold-limit value. According to Mjølhus and Wyller (1986), the important effect of finite temperature is that of resonant particles. In Mjølhus and Wyller (1988), the treatment is extended further to cover collision-dominated cases obtaining a DNLS equation with an additional diffusive term and coefficients c_1 and c_3 similar to those of equation (1). These results would mean that equation (1) is only valid for low β values. However, for high β values (neglecting resonant effects that will be discussed later), only the coefficients in front of the nonlinear and dispersive terms should be modified, and these factors can be absorbed by rescaling q, x, and t to obtain equation (1).

Let us analyze now the relationship between solar wind observations and equation (6). Studies of the correlation between δn and δu give a phase which clusters around zero for frequencies $\nu \geq 10^{-5}$ s^{-1} (see Fig. 4 in Goldstein and Siscoe 1972), and this agrees with equation (6a). It may also be seen in Denskat and Neubauer (1983; Fig. 3) and in Goldstein and Siscoe (1972; Fig. 9) that the phase of the correlation between δv and δB is π for almost the entire frequency range, which agrees with equation (6b). The correlation between δn and $|\delta B|^2$ also gives a phase equal to π (see Fig. 10 of Goldstein and Siscoe 1972) for $\nu \geq 10^{-5}$ s^{-1}. This would mean that $\beta > 1$ if equation (6c) is supposed to be valid. In any case, even if the static approximation is not valid, as for typical solar wind parameters $|\delta B|^2/(B_0^2|1-\beta|) \cong 0.25$ (according to the results of Spangler 1987), a sign for $\delta n/\delta|B|^2$ different from the one in equation (6c) would hardly be seen. Although typical values of β in the solar wind satisfy $\beta \leq 1$ (Sakai and Sonnerup 1983), evidence exists of the occurrence of $\beta > 1$ values during highly perturbed periods (Burlaga 1972; Spangler and Sheerin 1982*b*). However, we must remember that new effects should probably be taken into account if β is too high.

With all these limitations of the DNLS equation in mind, let us review some of its properties. A circularly polarized Alfvén wave of finite amplitude

$$q = \bar{b} \exp(ikx - i\omega t), \qquad \omega = k + k\frac{\bar{b}^2}{4(1-\beta)} + k^2, \tag{7}$$

is an exact solution of equation (1). Such a wave is unstable against amplitude modulations The conditions for modulational stability or instability are

$$8(1-\beta)k + \bar{b}^2 > 0 \qquad \text{(stable)}, \tag{8a}$$

$$8(1-\beta)k + \bar{b}^2 < 0 \qquad \text{(unstable)}, \tag{8b}$$

This means that, in a $\beta < 1$ case, while the right-hand polarization is stable, the left-hand polarization may be unstable. This picture is reversed for $\beta > 1$. The evolution of modulated wave packets has been numerically studied in Spangler, Sheerin, and Payne (1985), Spangler (1985), and in Ponce Dawson and Ferro Fontán (1988*a*). In the last two papers, it is shown that the modulational instability stops leading to the formation of a soliton train.

Solitons are special solutions of certain nonlinear equations which travel without deformation owing to a balance between nonlinear and dispersive effects. The DNLS equation has soliton solutions of the form (Kaup and Newell 1978)

$$q_s = \phi[x - (1+v)t] \exp [\![i\{\varphi[x - (1+v)t] + 4\Delta^4 t\}]\!], \tag{9}$$

where

$$\phi^2(z) = \frac{32\Delta^2 \sin^2 \gamma |1-\beta|}{\cosh [4\Delta^2 \sin \gamma (z - x_0)] + \cos \gamma}, \tag{10a}$$

$$\varphi_s(z) = \frac{v}{2}(z - x_0) - \frac{3}{16(1-\beta)} \int_{-\infty}^{z} dz' \phi^2 + \varphi_0 , \tag{10b}$$

$$v = -4s\Delta^2 \cos \gamma , \tag{10c}$$

$$s = \text{sgn}(1-\beta) , \tag{10d}$$

The term x_0 is the position of the peak of the soliton, and φ_0 is an arbitrary phase.

The form of these solitons is characterized by two parameters. Following Kaup and Newell (1978), we have chosen Δ^2 ($\Delta^2 > 0$) and γ ($0 < \gamma < \pi$), but the soliton peak amplitude

$$\phi_{\max}(z) = \left(\frac{32\Delta^2 \sin^2 \gamma |1-\beta|}{1 + \cos \gamma}\right)^{1/2}, \tag{11}$$

the soliton velocity in the Alfvén reference frame v (eq. [10c]), or the pair of parameters

$$k_0 = -s2\Delta^2 \cos \gamma , \tag{12a}$$

$$\omega_0 = 4\Delta^4(\sin^2 \gamma - \cos^2 \gamma) , \tag{12b}$$

that are used in Mjølhus and Wyller (1986) could have been chosen also. Owing to the existence of these two parameters, there is not a unique relationship between $\phi_{\max}$ and v (see eq. [10]). Besides, there are two characteristic length scales: that of the soliton amplitude

$$l_\phi \sim (4\Delta^2 \sin \gamma)^{-1} , \tag{13}$$

which we may call the modulation scale or soliton width, and that of the phase l_φ, which may be referred to as the soliton wavelength. Because the soliton phase is related nonlinearly to the amplitude (see eq. [10]), this wavelength and its inverse, the soliton wavenumber

$$k_s = -2s\Delta^2 \cos \gamma - \frac{3}{16(1-\beta)} \phi^2 \tag{14}$$

$l_\varphi \sim |k_s|^{-1}$, are amplitude dependent. The behavior of k_s and of the ratio l_ϕ/l_φ is strongly dependent on the value of the soliton parameter γ (which is proportional to its energy). This may be seen in Figure 1, where the amplitude $\phi/(4|1-\beta|)^{1/2}$ and the real part $(\phi \cos \varphi_s)/(4|1-\beta|)^{1/2}$ are plotted against x for $\Delta^2 = 0.25$ and for different values of γ. For low-γ values, the nonlinear term in equation (14) may be neglected, and therefore

$$l_\varphi \sim |2\Delta^2 \cos \gamma|^{-1} . \tag{15}$$

Moreover, $\phi_{\max}$ is very small, while the soliton modulation scale is very large and contains several wavelengths ($l_\phi/l_\varphi \gg 1$). If the value of Δ^2 is kept constant, $\phi_{\max}$ increases, while l_ϕ decreases with γ. The ratio l_ϕ/l_φ decreases with γ in the interval $(0, \pi/2)$ and increases for $\gamma \in (\pi/2, \pi)$. In the latter case, the nonlinear term in equation (14) becomes very important and k_s changes its sign, and therefore the soliton changes its sense of polarization within one wave packet. For this reason, solitons with $\gamma \in (0, \pi/2)$ are called normal, and those with $\gamma \in (\pi/2, \pi)$ are called anomalous in Mjølhus (1978).

The DNLS equation has the property of being completely integrable by the inverse scattering transform (IST) method (Kaup and Newell 1978). For this reason, an arbitrary initial condition $q(x, 0)$ decays into the superposition of a soliton train (which may be absent) and a radiation residue which disperses away with time (Ablowitz *et al.* 1974). The soliton number and the parameters which define its form remain invariant during the evolution. Besides, it has an infinity of conserved quantities. Three of them are

$$C_{-1} = \frac{i}{4|1-\beta|} \int_{-\infty}^{\infty} dx\, q \int_{-\infty}^{x} dy\, q^*(y) , \tag{16}$$

which have been found in a previous work (Ponce Dawson and Ferro Fontán 1988*b*) and is proportional to the helicity $\int dx\, \boldsymbol{A} \cdot \boldsymbol{B}$, with $\boldsymbol{A}$ the vector potential in the Coulomb gauge,

$$C_0 = \int_{-\infty}^{\infty} dx \frac{|q_0|^2}{8|1-\beta|}, \tag{17}$$

which may be called the energy of the system, and

$$C_1 = \frac{1}{32|1-\beta|^2} \int_{-\infty}^{\infty} dx \left[i4(1-\beta) q_x^* q + \frac{|q|^4}{2} \right] . \tag{18}$$

This last constant seems to play a fundamental role in the formation of solitons. Ponce Dawson and Ferro Fontán (1988*a*) observed

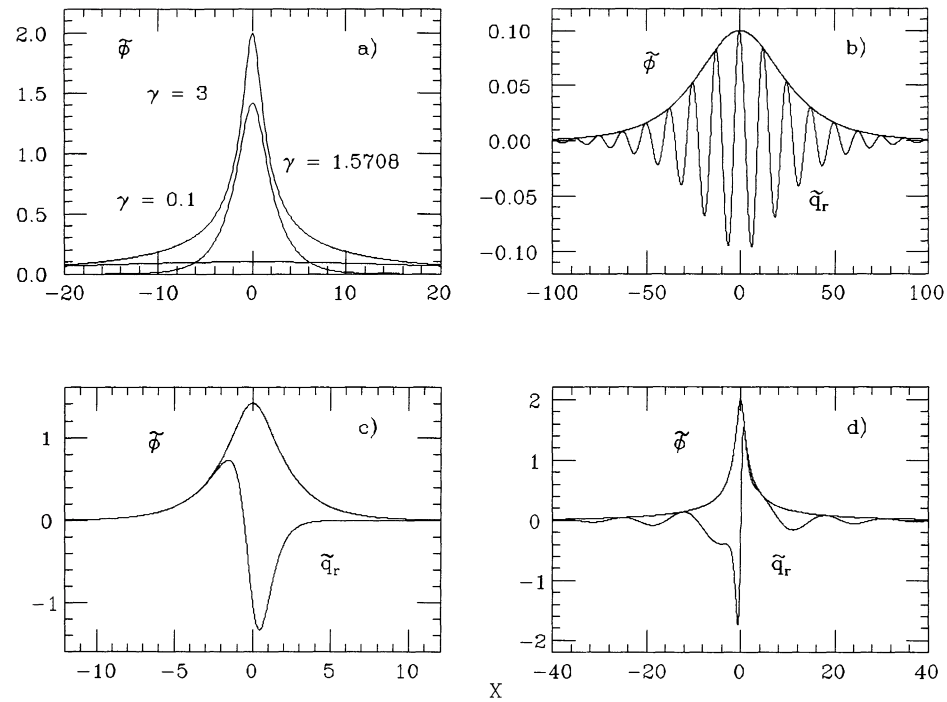

FIG. 1.—(*a*) Soliton amplitude $\tilde{\phi} = \phi/(4|1-\beta|)$ with ϕ defined in eq. (10a) is plotted against x for $\Delta^2 = 0.25$ and $\gamma = 3$, 1.5708, 0.1. It may be observed how the soliton peak amplitude decreases with γ while the modulation scale is increased. (*b*), (*c*), (*d*) Soliton amplitude $\tilde{\phi}$ and the real part $\tilde{q}_r = \tilde{\phi} \cos \varphi_s$ are plotted against x for $s = -1$ and (*b*) $\gamma = 0.1$, (*c*) $\gamma = 1.5708$, and (*d*) $\gamma = 3$. It may be observed how the ratio l_ϕ/l_φ decreases with increasing γ for $0 < \gamma \leq \pi/2$ and afterward increases for $\pi/2 < \gamma < \pi$. The particular behavior of $\tilde{q}_r$ for $\gamma = 3$ reflects the fact that anomalous solitons ($\pi/2 < \gamma < \pi$) change their sense of polarization within the soliton wavepacket.

that solitons appear whenever $C_1 < 0$; this last condition is an integrated version of Mjølhus (1976) criterion (eq. [8]). This may be seen readily if we write the initial condition as

$$q_0 \equiv q(x, 0) = |q_0(x)| \exp [i\varphi(x)] , \tag{19}$$

because the condition of modulation instability may be written as

$$8(1-\beta)\varphi_x + |q_0|^2 < 0 , \tag{20}$$

and $C_1 = \int dx |q_0|^2/(64|1-\beta|^2)(8[1-\beta]\varphi_x + |q_0|^2)$.

The values of the constants (16)–(18) for a one-soliton solution are

$$C_{-1} = \frac{2 \sin \gamma}{\Delta^2} ; \qquad C_0 = 2\gamma ; \qquad C_1 = -4\Delta^2 \sin \gamma . \tag{21}$$

It may be seen from equation (21) that the parameter γ is proportional to the soliton energy and that $C_1 < 0$ for each soliton.

Solitons play a fundamental role in nonlinear physics, and not only in the case of integrable equations. They appear also during the evolution of nonintegrable equations, such as the Zakharov set (Zakharov 1972). Actually, these are not solitons but solitary waves, because they do not remain invariant after nonlinear interactions. Nevertheless, a turbulent state can be envisaged as a collection of these nonlinear entities. Solitons are the building blocks of the treatments which have been used in Kingsep, Rudakov, and Sudan (1973), Matsuno (1977), and Meiss and Horton (1982), among others.

We are interested in obtaining a statistical description of DNLS solitons, contrasting the results derived from the model with observations in the solar wind. There are two previous works which attempted a similar approach: Spangler and Sheerin (1982*b*), which also used DNLS solitons, and Ovenden, Shah, and Schwartz (1983). In the latter, a set of equations is derived, which written in our variables reads

$$ib_{t'} + iv_A(1 + 2k'_A d_i)b_{x'} + v_A k'_A \frac{\delta n}{2n_0} b - \delta u\, b\, k'_A + d_i v_A b_{x'x'} = 0 , \tag{22a}$$

$$\delta n_{t't'} - \beta v_A^2 \delta n_{x'x'} = k'_A |b|^2 \frac{n_0}{2B_0^2} v_A^2 , \tag{22b}$$

$$\delta n_{t'} + n_0 \delta u_{x'} = 0 , \tag{22c}$$

instead of equation (5), where b is the amplitude of a circularly polarized Alfvén wave of the form in equation (7a) with $b = B_0 \bar{b}$,

$k'_A = k_A/d_i$, and $\omega'_A = v_A k'_A + k'^2_A v_A d_i$. In the static approximation (when a scaling similar to eq. [2] may be introduced), equations (22a)–(22c) reduce to the relationships (6a) and (6c), and the nonlinear Schrödinger (NLS) equation which, written in dimensionless variables, reads

$$i\bar{b}_t + 2ik_A \bar{b}_x - k_A \frac{|\bar{b}|^2 \bar{b}}{4(1-\beta)} + \bar{b}_{xx} = 0 \tag{23}$$

(see, e.g., Roberts 1984; Mjølhus and Wyller 1986). We want to stress that the solitons considered in the paper of Ovenden, Shah, and Schwartz (1983) are solutions of equation (23) and not of equations (22a)–(22c).

It is necessary to mention that solitons do not apply only to solar wind observations. More recently, the presence of solitons in Earth's magnetosphere has been reported (Patel and Dasgupta 1987). Also, in cometary plasmas, the relation between the perturbations in the magnetic field and in the density shows that the description using a gas of solitons may be suitable (Lakhina 1987). Moreover, in Lakhina and Buti (1988), the interaction of Alfvén solitons and plasma waves is invoked to explain some features of the observed cometary kilometric radiation in the vicinity of comet Halley. It is also mentioned in this work that coherent structures, probably Alfvén solitons, have been seen in the spectra of hydromagnetic turbulence by *Vega 2* (Galeev 1986).

The organization of the paper is as follows. In § II, the statistical description of the DNLS equation is developed, comparing different models which use solitons as their building blocks. In § III, the limitations of the DNLS equation and of the soliton gas model are discussed. In § IV, the theoretical correlation functions are obtained and compared with observations. Finally, the conclusions are summarized in § V.

II. STATISTICAL DESCRIPTION OF ALFVÉN SOLITONS

We will now analyze the application of different statistical models to the DNLS equation. First, we will study the method of Matsuno (1977), which applies to integrable equations such as the DNLS one. The basic assumption involved is that the radiation component may be neglected and that the asymptotic solution of the dynamical equation (in this case, the DNLS) may be written as a superposition of solitons

$$q = \sum_{i=1}^{N} q_s(x, t; \Delta_i^2, \gamma_i, x_{0i}, \varphi_{0i}) , \tag{24}$$

where q_s represents a soliton solution (9)–(10) of parameters Δ_i^2, γ_i, initial peak position x_{0i} and initial phase φ_{0i}. The particular set of solitons that enters into the sum depends on the initial condition. Although each initial condition determines unambiguously the whole evolution $q(x, t)$, the interaction among solitons can produce phase shifts (leaving the parameters Δ_i^2 and γ_i invariant). Therefore, for a statistical description, it is usual to assume an uniform distribution of x_{0i} and φ_{0i} over their ranges of variation, retaining the influence of the initial condition only on the set $\{\Delta_i^2, \gamma_i\}$.

If the number of solitons is large enough, the discrete summation in equation (24) may be replaced by an integral of the form

$$q = \int F(a, b, [q_0]) q_s \, da \, db , \tag{25}$$

where we have chosen as the soliton parameters $a = s\Delta^2 \cos \gamma$ and $b = \Delta^2 \sin \gamma$, and where $F(a, b, [q_0]) \, da \, db$, which is a functional of the initial condition q_0, is the number of solitons with a between a and $a + da$ and b between b and $b + db$. The quantity b is proportional to the inverse of the soliton modulation scale l_ϕ (see eq. [13]), and a is proportional to the soliton wavenumber (14) when $\gamma \ll 1$.

Because expression (25) is valid only for cases with a large number of solitons and a negligible radiation component, it is necessary to determine under which conditions this assumption holds. We have seen during previous numerical simulations of the DNLS equation (Ponce Dawson and Ferro Fontán 1988*a*) that the asymptotic solution of equation (1) is of the form of equation (24) when the constant $C_1 < 0$. As we have already stated, this means that the initial condition is modulationally unstable in an integrated sense. We have also found in a recent work (Ponce Dawson and Ferro Fontán 1989) that solitons develop from regions of the initial condition which are modulationally unstable (those that satisfy eq. [20]) and that their number is large while the radiation component remains negligible in a strongly unstable case (Mjølhus 1978),

$$-8(1 - \beta)\varphi_x \gg |q_0|^2 . \tag{26}$$

In equation (26), the initial condition has been written the same way as in equation (19). In such a case, once the modulational instability is saturated, the complete solution of equation (1) may be written the same way as in equation (25).

Extending the treatment of Karpman and Sokolov (1967), which applies to the Korteweg–de Vries (KdV) equation, we have found that, for initial conditions of the form (19) that satisfy

$$l_0 \equiv |q|/|q|_x \gg 1/k_0 \tag{27}$$

where $k_0 \equiv -s\varphi_x$, the soliton distribution function is

$$F(a, b) = \frac{1}{\pi} \int_{R > b^2} dx \, b\delta\left[a + \frac{1}{2}\left(\varphi_x + \frac{|q_0|^2}{4(1-\beta)}\right)\right](R - b^2)^{-1/2} , \tag{28}$$

where $R \equiv [-8\varphi_x/(1 - \beta) + |q_0|^2/(1 - \beta)^2](|q_0|^2/64)$ (Ponce Dawson and Ferro Fontán 1989). A comparison of equation (28) and our previous numerical findings (Ponce Dawson and Ferro Fontán 1989) showed good agreement even for a case with only one soliton.

The function (28) gives the total number of solitons

$$N = \frac{1}{\pi} \int_{R>0} dx \left\{ \left[\frac{-8\varphi_x}{(1-\beta)} + \frac{|q_0|^2}{(1-\beta)^2} \right] \frac{|q_0|^2}{64} \right\}^{1/2} . \tag{29}$$

It may be seen from equations (28) and (29) that if the initial condition has no modulationally unstable regions (i.e., regions where eq. [20] is fulfilled), solitons are not formed. Expression (28) gives the correct value of the constant C_1 for any initial condition, whenever the radiation component is negligible. Because this constant is related to the first moment of the distribution function, we suppose that expression (29), which is the zeroth order moment of equation (28), is essentially correct for any initial condition, though the actual distribution for initial conditions which do not satisfy equation (27) may be different from equation (28). This conclusion is based on the fact that higher order moments, which are related to higher order constants of the DNLS equation, mostly affect the behavior of the distribution function in the tails.

Expressions (28) and (29) reduce to

$$F(a, b) = \delta\left(a - \frac{sk_0}{2}\right) \int_{8b^2|1-\beta| < |q_0(x)|^2 k_0} dx \, \frac{b}{\pi} \left[\frac{k_0 |q_0(x)|^2}{8|1-\beta|} - b^2 \right]^{-1/2} , \tag{30}$$

$$N = \frac{1}{\pi} \int_{k_0 > 0} dx \left(\frac{|q_0|^2}{8|1-\beta|} k_0 \right)^{1/2} , \tag{31}$$

in a strongly unstable case. In such a case, it may be seen from equations (30) and (31) that

$$a \cong -\frac{1}{2} \varphi_x ; \qquad 0 < b^2 \le \frac{|q_0|^2_{\max}}{8(1-\beta)} (-\varphi_x) . \tag{32}$$

Since the inequality (26) is also satisfied, equation (32) implies that

$$|b/a| = |\tan \gamma| \ll 1 ; \qquad \Delta^2 \cong sa \cong -s\varphi_x/2 . \tag{33}$$

This means that, at least in this case, the system prefers to share the available energy E among a large number of solitons of low energy and equal velocities $v = -4a$ (see eq. [10]) rather than to concentrate it in a few of them (remember that the energy content of one soliton is 2γ, and if the solitons are sufficiently separated, it may be assumed that $E[\text{sol}] = \sum_i 2\gamma_i$). This behavior is also observed in our previous numerical simulations. The situations plotted in Figures 4–6 of Ponce Dawson and Ferro Fontán (1988*a*) correspond to initial conditions with the same energy ($E = C_0 = 2.195$) and different values of C_1. It may be seen there that as the "level of the modulation instability" contained in the initial condition (measured by $-C_1$) is increased, the number of solitons is also increased, and, but of necessity, the energy content of each soliton decreases.

Since for $\gamma \ll 1$ the soliton wavenumber $k_s \cong -2s\Delta^2$, it may be seen from equation (33) that, in a strongly unstable case, solitons with the same wavenumber as that of the initial condition are formed ($k_s = \varphi_x$). Moreover, if the initial wavenumber $k_0 = -s\varphi_x \ll 1$, then Δ^2 will also satisfy $\Delta^2 \ll 1$. Therefore, solitons with large modulation scale l_ϕ and wavelength l_φ, but with a high ratio l_ϕ/l_φ, appear from strongly unstable initial conditions of short wavenumber. This property renders the model suitable for describing low-frequency waves upstream from Earth's bow shock as well, since, as stated in Spangler (1985), they remain coherent over several cycles.

It may also be shown that, for initial conditions of the form (19) with $-s\varphi_x = k_0$ that satisfy relation (27) (wave packet limit), the DNLS equation reduces in the strongly unstable case to the NLS equation (23) (Mjølhus and Wyller 1986). Therefore, under this assumption, the results using equations (1) or (23) should be similar.

Up to now, we have introduced no statistics. When the initial conditions are unknown, but a probability distribution in function space $P[q_0]$ may be given, a statistical description is possible. This is what occurs in many cases, particularly in the solar wind. Quantities such as the mean value of q- or n-point correlation functions must be used to describe the system, instead of $q(x, t)$. Mean values which are functions of x and t are calculated through a functional integration over the initial conditions

$$\langle \ldots \rangle = \int Dq_0 \, d\varphi_0 \, dx_0 \, P(x_0) Q(\varphi_0) \mathscr{P}[q_0] \ldots , \tag{34}$$

where Dq_0 stands for the integration measure in function space. Expression (34) involves an average over the set of initial conditions which are distributed according to the probability density in the space of functions $\mathscr{P}[q_0]$. An average over the initial peak position x_0 and phase φ_0 of the solitons is also included in equation (34) through the uniform distribution probabilities $P(x_0)$ and $Q(\varphi_0)$.

In this way, an averaged distribution function of solitons may be defined through

$$f(a, b) = \langle F((a, b), [q_0]) \rangle , \tag{35}$$

which is useful whenever one works with the continuous formulation (25). The function $f(a, b)$ gives the density of solitons in the parameter space (a, b) for an ensemble of initial conditions distributed according to $\mathscr{P}[q_0]$. Choosing different probability distributions $\mathscr{P}[q_0]$ for the initial conditions, different distribution functions $f(a, b)$ may be obtained. Particularly, functions which follow an exponential (canonical) or power law with soliton energy may appear.

Because the application of the method of Matsuno (1977) is rather complicated and the actual distribution $F(a, b)$ may differ from equation (28), we will follow the simplified version of Meiss and Horton (1983; hereafter MH). This method uses the value obtained in equation (29) for the total number of solitons, which is essentially correct, and it considers an averaged distribution function which follows an exponential law with soliton energy. Afterward, we will compare it with the method of Kingsep, Rudakov, and Sudan (1973; hereafter KRS), which naturally gives an averaged distribution function f which follows a power law instead. Both methods reproduce the feature of a high concentration of solitons in the low-energy region of the parameter space.

The method of Meiss and Horton (1983) was developed for the case of drift-wave turbulence. In that work, the authors use only the expression for the total number of solitons as a functional of the initial conditions, they calculate its mean value $\langle N \rangle$ with a canonical distribution of initial conditions, and they assume a canonical averaged distribution function of solitons f. The temperatures of both canonical distributions are related, equating the energy content of the soliton gas to the energy of the initial conditions.

Let us now apply this method to the DNLS equation. Let us thus consider a canonical distribution of initial conditions

$$\mathscr{P}[q_0] = \frac{1}{Z} \exp\left[\frac{-\int dx |q_0|^2}{(2T)}\right], \tag{36}$$

with $Z = \int Dq_0 \exp(-\int dx |q_0|^2/[2T])$. With this distribution, it is possible to calculate the mean number of solitons $\langle N \rangle$ by means of relation (34). Using the strongly unstable approximation (31) $\langle N \rangle$ results,

$$\langle N \rangle_{\rm I} = \frac{1}{6}\left(\frac{mL}{|1-\beta|}\langle |q_0|^2 \rangle\right)^{1/2}, \tag{37}$$

where $\langle |q_0|^2 \rangle$ is related to the mean energy content WL of the perturbations by $WL = L\langle |q_0|^2 \rangle/(8|1-\beta|)$, and the subscript I is used to identify the MH procedure. The functional integration which leads to equation (37) has been made by discretizing the variables over the distance L and by assuming $-s[\varphi(L) - \varphi(0)] = 2m\pi$. The integer number m measures the number of wavelengths of the initial condition that the system, of length L, contains. Since the distribution of the initial conditions does not depend on the phase φ, the functional integration over $\varphi \in (0, 2m\pi)$ is equivalent to having supposed a uniform distribution of initial wavenumbers $k_0 = -s\varphi_x$ on the interval $(0, 2m\pi/L)$. We suppose then that the $\langle N \rangle$ solitons are distributed canonically in energy. Since the soliton energy is 2γ and each soliton is characterized by Δ^2 and γ, this method gives no information about the Δ^2 distribution. We assume then that all the solitons have the same value of Δ^2. In this way, the distribution function results in

$$f_{\rm I} = \frac{\langle N \rangle_{\rm I}}{Z} \exp(-\Psi\gamma)\delta(\Delta^2 - \Delta_0^2), \tag{38}$$

with $Z = \int d\gamma \exp(-\Psi\gamma)$. The value of Ψ may be calculated by equating the mean energy content of the soliton gas to the initial state energy

$$\langle N \rangle_{\rm I}\langle 2\gamma \rangle = L\langle |q_0|^2 \rangle/(8|1-\beta|). \tag{39}$$

This calculation gives, in a strongly unstable case,

$$\Psi = \frac{8}{3}\left(\frac{m|1-\beta|}{L\langle |q_0|^2 \rangle}\right)^{1/2}. \tag{40}$$

Since $-s|q|^2\varphi_x \sim \langle |q|^2 \rangle m/L$ and $|q|^4 \sim \langle |q|^2 \rangle^2$, the assumption of a strongly unstable case means that $\langle |q_0|^2 \rangle L/|1-\beta| \ll m$. Then $\Psi \gg 1$.

The value of Δ_0^2 may be obtained from another constant of motion; for instance, C_{-1} or C_1. Supposing $C_{-1} \sim \langle |q|^2 \rangle L^2/(4m|1-\beta|)$, $-C_1 \sim \langle |q|^2 \rangle m/(4|1-\beta|)$ (see their expressions [16] and [18]), both constants give the same result:

$$\Delta_0^2 = \frac{m}{2L}. \tag{41}$$

The computation of f is now completed. It reads

$$f_{\rm I} = \frac{4m}{9} \exp\left[-\frac{8}{3}\left(\frac{m|1-\beta|}{L\langle |q_0|^2 \rangle}\right)^{1/2}\gamma\right]\delta\left(\frac{\Delta^2 - m}{2L}\right). \tag{42}$$

Another possibility for obtaining the Δ^2 distribution is to follow Matsuno's method. According to equation (33) (which holds in a strongly unstable case), if $-s\varphi$ is distributed uniformly in the interval $(0, 2m\pi)$ and the system is of length L, a natural choice for Δ^2 may be that of a uniform distribution over the interval $(0, m\pi/L)$. In such a case, the mean value of Δ^2 would be $\Delta^2 = m\pi/2L$, the same that could have been obtained following the argument which led us to equation (41) if we had taken $C_1 \sim -m\pi\langle |q|^2 \rangle/(3|1-\beta|)$.

The second approach we will discuss is the method introduced by Kingsep, Rudakov, and Sudan (1973) for unidimensional Langmuir turbulence as described by the Zakharov equations (Zakharov 1972). In this treatment, it is supposed that all the available states of the system are made up of N identical solutions with randomly distributed initial peak positions and phases. Since the governing equations are nonintegrable, the number of solitons can vary, due mainly to the process of soliton fusion. In this way, each state is characterized by the number of solitons N. The soliton parameters of an N-state are determined from the value of the energy available in the initial conditions.

It is then assumed that all states with $1 \leq N \leq N_{\rm max}$ are equiprobable. In order to determine the value $N_{\rm max}$, the following argument is used. The soliton width l_ϕ depends on N through the soliton parameters. Thus, the maximum number of solitons that may be supported by a system of length L may be calculated from

$$l_\phi(N_{\max})N_{\max} \sim L . \tag{43}$$

The treatment of Kingsep, Rudakov, and Sudan (1973) is used by Ovenden, Shah, and Schwartz (1983) to exploit the similarity between equations (22a)–(22c) and the Zakharov set. It must be mentioned that these authors considered only the absolute value of the soliton and not its phase. Therefore, the average over the phases did not play any role in their description.

The KRS method cannot be applied directly to the DNLS equation, because it is completely integrable and its solitons cannot be destroyed. However, it may be applied to equations (5a)–(5c). This set of equations has soliton solutions which, written in dimensionless variables x and t, are of the form

$$\delta n = \frac{|\delta B|^2}{2[(1+v)^2 - \beta]} \frac{n_0}{B_0^2} , \tag{44a}$$

$$\delta u = \frac{(1+v)|\delta B|^2}{2[(1+v)^2 - \beta]} \frac{v_A}{B_0^2} , \tag{44b}$$

$$\delta B = B_0 |\delta \bar{B}[x - (1+v)t]| \exp [\![i\{\varphi[x-(1+v)t] + 4\Delta^4 t\}]\!] , \tag{44c}$$

with

$$|\delta \bar{B}(z)|^2 = \frac{32\Delta^2 \sin^2 \gamma [(1+v)^2 - \beta]}{\alpha \cosh [4\Delta^2 \sin \gamma (z - x_0)] + (1+2v)\cos\gamma} , \tag{45a}$$

$$\varphi(z) = \frac{v}{2} z - \frac{3+4v}{16[(1+v)^2 - \beta]} \int_{-\infty}^{z} dz' |\delta \bar{B}(z')|^2 + \varphi_0 , \tag{45b}$$

where $\alpha = [(1+2v)^2 + (4v/3)(1+v)\sin^2\gamma]^{1/2}$ and v is given by equation (10c) (with $s = \operatorname{sgn}[(1+v)^2 - \beta]$). These solitons are very similar to those of the DNLS equation. Particularly when $\gamma = \pi/2$ ($v = 0$), equations (45) and (9)–(10) are exactly equal, and, if $v \ll 1$, the solitons of the DNLS equation are approximate solutions of the set (5). If we limit ourselves to the study of solitons with long wavelengths (small wavenumbers), which is one of the assumptions that led to the DNLS equation and to equation (5a), according to equation (33) the soliton velocities $v = -4a$ would satisfy $|v| \ll 1$. In such a case, we can consider that the DNLS solitons with δn and δu given by equation (6) are solutions of equations (5a)–(5c).

Let us apply the KRS method with these solitons. Since each state is made up of N identical solitons, the parameters $\Delta^2(N)$ and $\gamma(N)$ may be obtained from the constants of motion as

$$\frac{2N \sin\gamma}{\Delta^2} \cong C_{-1} , \tag{46}$$

$$2N\gamma \cong C_0 , \tag{47}$$

$$-N 4\Delta^2 \sin\gamma \cong C_1 . \tag{48}$$

It is clear from equations (46)–(48) that, if $C_{-1} \cong (L^2\langle|q|^2\rangle/4m|1-\beta|)$, $C_0 \cong (L\langle|q|^2\rangle/8|1-\beta|)$, $C_1 \cong -(m\langle|q|^2\rangle/8|1-\beta|)$ and $\gamma = \langle|q|^2\rangle L/4N \ll 1$ (which is equivalent to the assumption $\Psi \gg 1$ in the first approach), equations (46) and (48) both give the same result:

$$\gamma(N) = \frac{\langle|q|^2\rangle L}{16|1-\beta|N} , \tag{49}$$

$$\Delta^2(N) = \frac{m}{2L} . \tag{50}$$

If we take the soliton width $l_\phi = A/(4\Delta^2 \sin\gamma)$ (where A is a scaling parameter introduced for later convenience, $A = 1$ both in Kingsep, Rudakov, and Sudan 1973 and in Ovenden, Shah, and Schwartz 1983), then, when $\gamma \ll 1$, it reads

$$l(N) = \frac{8AN|1-\beta|}{m\langle|q|^2\rangle} , \tag{51}$$

and the maximum number of solitons

$$N_{\max} = \left(\frac{mL\langle|q|^2\rangle}{8A|1-\beta|}\right)^{1/2} . \tag{52}$$

Since in this model all the states of the system with $1 \le N \le N_{\max}$ are equiprobable, the distribution function $f(\Delta^2, \gamma)$ may be calculated as

$$f(\Delta^2, \gamma) = \int_1^{N_{\max}} dN \frac{N}{N_{\max}} \delta[\Delta^2 - \Delta^2(N)]\delta[\gamma - \gamma(N)] . \tag{53}$$

It results in

$$f_{\rm II} = \frac{2m}{A}\left(\frac{\gamma_m}{\gamma}\right)^3 \delta\left(\Delta^2 - \frac{m}{2L}\right) H(\gamma - \gamma_m)\,, \tag{54}$$

where $H(x)$ is the Heavyside function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0\,, \end{cases} \tag{55}$$

and γ_m is the minimum value of γ, $\gamma_m = (AL\langle|q|^2\rangle/32m|1-\beta|)^{1/2} = \sqrt{2A}/(3\Psi)$. The mean number of solitons for this distribution is

$$\langle N\rangle_{\rm II} = \left(\frac{mL\langle|q|^2\rangle}{32A|1-\beta|}\right)^{1/2}, \tag{56}$$

which for $A = 9/8$ is exactly equal to the one obtained through the first approach.

Actually, the lower limit of integral (53) is 1, and this would give an upper limit γ_M to the allowed values of γ. Moreover, γ_M should also satisfy $NA/(4\Delta^2 \sin\gamma_M) \le L$. However, if the contributions from $\gamma \ge \gamma_M$ to the correlation functions are negligible, expression (54) is essentially correct.

Let us now compare $f_{\rm I}$ and $f_{\rm II}$. Actually, the results of the first approach should be modified, because the system cannot support solitons of width greater than L. Taking this into account, $f_{\rm I}$ results in

$$f_{\rm I} = \frac{4m}{7}\exp\left[-\frac{24}{7}\left(\frac{m|1-\beta|}{L\langle|q_0|^2\rangle}\right)^{1/2}(\gamma - \gamma_m)\right]\delta\left(\frac{\Delta^2 - m}{2L}\right)H(\gamma - \gamma_m)\,. \tag{57}$$

Although $f_{\rm I}$ and $f_{\rm II}$ give a similar mean number of solitons, the distribution among the possible values of γ is quite different. This can be seen in Figure 2, where both functions are plotted against $\bar\gamma = (3\Psi/4)\gamma$ with $A = 9/8$. As we have already stated, both distributions show that almost all the solitons have small values of γ and that the concentration around γ_m is sharper for larger values of Ψ. The function $f_{\rm II}$ gives more low-energy solitons than $f_{\rm I}$ does. For $\bar\gamma > 3$, the values of f are almost negligible, proving that it is not necessary to take into account the upper limit γ_M.

Finally, we will briefly mention the model described by Rudakov and Tsytovich (1978) for Langmuir turbulence in the presence of a large number of ion-sound waves, that has also been used by Spangler and Sheerin (1982*b*) in the case of Alfvén turbulence. Considering the processes of soliton fusion and the breakup of solitons by ion-sound waves, the authors found a dependence of the distribution function with the soliton amplitude of the form

$$f \sim \phi_{\rm max}^{-1.84}\,. \tag{58}$$

This treatment is used by Spangler and Sheerin (1982*b*) for the DNLS equation. Although it is clear that it cannot be applied to an integrable equation such as the DNLS equation, it may be suitable for the nonintegrable equations (5a)–(5c), of which the DNLS equation is the static limit. We may also justify the application of this method or the KRS method with the argument that the dynamical equation is actually nonintegrable owing to the presence of other effects such as damping, resonant interaction with particles, etc., which have been neglected so far.

III. LIMITATIONS OF THE SOLITON GAS MODEL

As we have already mentioned, the model developed so far does not include damping or wave growth effects. The picture of an initial condition transforming into a soliton train plus a radiation residue may be changed in a dissipative case. The inclusion of additional damping and growth terms has been analyzed in a collection of papers (Spangler 1986; Ghosh and Papadopoulos 1987;

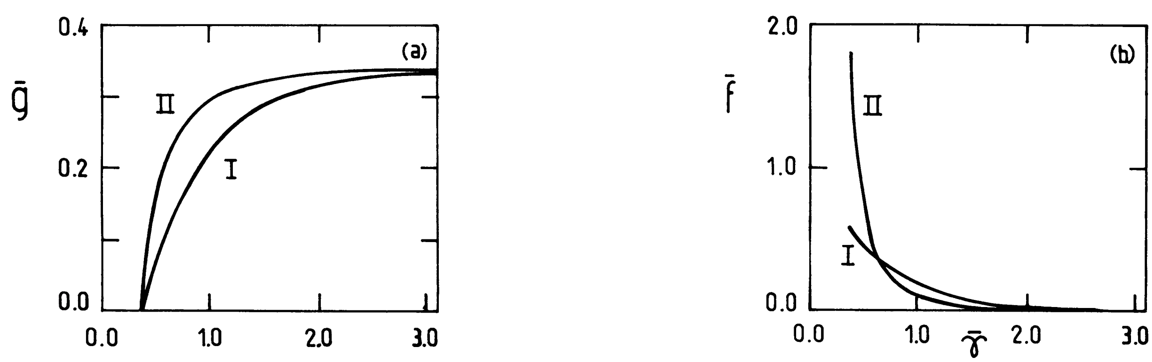

FIG. 2.—Functions (*a*) $\bar g(\gamma) = \int_0^{\gamma} d\gamma' \bar f(\gamma')$ and (*b*) $\bar f(\gamma) = f(\gamma, \Delta^2)/[m\delta(\Delta^2 - \Delta_0^2)]$, where f is the soliton distribution function, are plotted against $\bar\gamma = 3\Psi\gamma/4$, a quantity proportional to the soliton energy. Distribution function f is calculated following (I) the MH method (eq. [58]) and (II) the KRS method (eq. [55]). It may be seen in Fig. 1*a* that both functions give the same number of solitons, and, as is shown in (*b*), both distributions cluster around the minimum value γ_m. This effect turns out to be more important as the value of the mean level of modulational instability Ψ is increased. However, the γ distributions of $f_{\rm I}$ and $f_{\rm II}$ are different: $f_{\rm II}$ gives more low-energy solitons than $f_{\rm I}$.

Mjølhus and Wyller 1986, 1988; Flå, Mjølhus, and Wyller 1988*a*, *b*). Among the possible damping processes, we may cite decay instability into daughter Alfvén waves and a sound wave (Terasawa *et al.* 1986), ion-cyclotron damping (Spangler 1986), and nonlinear damping by resonant particles (Mjølhus and Wyller 1986, 1988).

The first two processes cannot be included consistently in a derivation similar to the one that led to the DNLS equation (1). As we have already discussed, the first process is precluded by the restriction of forward propagation. The decay instability converts the initial Alfvén wave in two Alfvén waves (forward- and backward-propagating) and one forward-propagating soundlike wave. The energy of the parent wave mainly goes to the sound and backward-propagating Alfvén waves (Goldstein 1978; Terasawa *et al.* 1986). The absence of such backward-propagating waves in the solar wind at 1 AU is explained in Terasawa *et al.* (1986), invoking the large characteristic time of the process for the long wavelengths observed therein. This problem may also arise for the development of the modulational instability and the subsequent formation of solitons. We will discuss this problem later.

The ion-cyclotron damping is of order $\exp(-1/\epsilon)$ (Flå, Mjølhus, and Wyller 1988*b*) and can only enter in a fully nonlinear model ($\epsilon \sim 1$). However, in Spangler (1986) the addition of such a linear damping term and a source term (amplification by particle beams) to equation (1) is analyzed. The growth rate mostly affects the region of low wavenumbers, while dissipation affects that of $k \geq \omega_{ci}/v_A$ (a region over which the DNLS equation may not be valid). The value of k at which dissipation becomes important decreases with increasing β. A picture is obtained in which energy is injected in the low-wavenumber region and then transferred to higher wavenumbers by the modulational instability, where it can be damped efficiently. Solitons may appear if their characteristic wavenumbers correspond to regions of low dissipation. Therefore, only solitons with sufficiently large characteristic wavelengths will survive.

In Ghosh and Papadopoulos (1987), equation (1) with an additional linear damping term is solved numerically. The system is driven by one unstable discrete Fourier mode. The following results are found. The asymptotic time behavior is independent of the initial conditions. For sufficiently low values of the unstable mode strength, the formation of solitons is observed. When this strength is increased, a transition from this ordered state (characterized by the presence of " solitons " whose amplitude may oscillate with time) to turbulence is obtained.

Finally, as we have already mentioned, a DNLS–like equation which takes into account the effect of resonant particles (nonlinear Landau damping and amplification) has been obtained in Mjølhus and Wyller (1988). The resonant term becomes important for $\beta \geq 1$ (Mjølhus and Wyller 1986), In Flå, Mjølhus, and Wyller (1988*b*) it is found that Alfvén waves may be excited by particle beams traveling at the Alfvén velocity. The effect of this new term on the evolution of a circularly polarized Alfvén wave is also analyzed therein. It is observed that the growth rate of the modulational instability is decreased when the damping coefficient is increased. However, for $\beta \leq 1$, the reduction does not exceed 35% of the value for $\beta = 0$. It is also found that a new modulational instability is introduced: while in the nondissipative nonresonant case condition (8b) must be fulfilled for the wave to be modulationally unstable, with this additional term waves with any wavenumber or polarization may be unstable. However, it has not been investigated yet whether this instability could lead to the formation of solitons. A transfer of energy toward the low-wavenumber region of the spectrum is also observed. The wavenumber k decreases proportionally to the energy owing to the conservation of the action $|E_k|^2/k$.

In Flå, Mjølhus, and Wyller (1988*a*), the effect of the resonant term on an Alfvén soliton is analyzed both analytically (using perturbation theory) and numerically when this new term is sufficiently low, obtaining the following results. Normal solitons are slightly perturbed, while anomalous ones develop an important nonsolitonic feature. Damping is most effective in the transition region between normal and anomalous solitons. In the damped case, all solitons eventually end up in the normal regime, because the energy dissipation tends toward zero with γ.

This discussion allows us to conclude that solitons could also appear even if damping and growth terms are added to the original equation, provided the growth rate is sufficiently low. We can also conclude that solitons with long characteristic length scales will survive, because damping effects are rather unimportant in the short-wavenumber region. Moreover, according to Flå, Mjølhus, and Wyller (1988*a*), low-energy ($\sim\gamma$) solitons will appear, which, as we have already seen, are characterized by large modulation scales. Therefore, the model of a gas of low-energy solitons similar to that presented in the preceding section may be suitable even if some of these new effects are included. We may also suppose that the distribution function $f_{\rm II}$ obtained with the KRS method could give better results than $f_{\rm I}$, because it is characterized by a larger number of low-energy solitons than $f_{\rm I}$ is.

Besides the limitations of the DNLS equation, the approach of a gas of solitons as a model of MHD turbulence pursues its own restrictions. First, of all, one may ask if there is enough time for the solitons to be formed under solar wind conditions. In Spangler (1985), a characteristic nonlinear time

$$\tau_{\rm NL}^{(1)} = \frac{|1-\beta| l_0}{v_A}\left(\frac{B_0}{b}\right)^2 , \tag{59}$$

for the development of the modulational instability from an initial Alfvén wave of modulation scale l_0 and amplitude b is obtained. The author observes that, for almost all the numerical simulations performed, the packet reaches its maximum peak amplitude for $0.7\tau_{\rm NL} \leq t \leq 2.4\tau_{\rm NL}$.

Another nonlinear time may be deduced from the maximum growth rate of modulational instability of an initial circularly polarized Alfvén wave of amplitude b and wavenumber k_0 obtained in Longtin and Sonnerup (1986):

$$\tau_{\rm NL}^{(2)} = \frac{4|1-\beta|}{v_A k_0}\left(\frac{B_0}{b}\right)^2 . \tag{60}$$

Equation (60) is equal to equation (59) with $4/k_0$ instead of l_0. For typical solar wind conditions ($|1-\beta| \cong 0.4$, $v_A = 50$ km s^{-1}, $b/B_0 = 0.4$), $\tau_{\rm NL}^{(1)} = 0.05 l_0$ s km^{-1} and $\tau_{\rm NL}^{(2)} = 0.2/k_0$ s km^{-1}. During this time, an initial perturbation is convected by the solar wind to an heliocentric distance $D \sim \tau_{NL} v_W$. Supposing $v_W \sim 500$ km s^{-1}, we obtain that, according to equation (59), the initial perturbation should travel a distance $D^{(1)} = 25 l_0$ and according to equation (60), $D^{(2)} = 100/k_0$ before solitons were formed. These distances must be compared with the heliocentric distances D_H at which the power spectra are obtained. This condition imposes a limit on the values of the characteristic scales the soliton model can describe at each heliocentric distance. For example, for $D_H = 0.3$ AU, we obtain from $D^{(2)}$, $k_0 \geq 2 \times 10^{-11}$ cm^{-1} and, from $D^{(1)}$, $l_0 \leq 2 \times 10^{11}$ cm. Since the model developed in the preceding section requires $k_0 l_0 \gg 1$, both conditions give a similar range of wavenumbers the soliton gas model can describe: 2×10^{-11} cm$^{-1} \leq k_0 \ll d_i^{-1} = 2 \times 10^{-7}$ cm^{-1} or, equivalently, the frequency range 1.6×10^{-4} s$^{-1} \leq \varphi \ll 1.6$ s^{-1} (the upper limit comes from the basic assumptions that led to the DNLS equation). This range covers a wide region of the observed power spectra (see, e.g., Denskat and Neubauer 1983). At larger heliocentric distances, the model applies for lower wavenumbers.

The characteristic times derived from equations (59) and (60) may be altered if other effects are taken into account. For example, in Spangler (1986), it is argued that the inclusion of a wave growth term in the low-wavenumber region may cause the modulational instability to develop in times rather shorter than equations (59) or (60). In this way, the application of the soliton gas model is enhanced.

However, even if solitons are not actually formed, they may be thought of as a nonlinear basis for solutions of the DNLS equation. In this way, soliton gas models are better than models which use plane waves as a basis, since even with a finite number of solitons the main nonlinear features of the dynamical equation can be reflected, while an infinite number of plane waves is necessary to reproduce them. The observation of density perturbations is evidence of the existence of some nonlinear processes which couple the density and the magnetic field. This coupling may not be describable by the model we analyze in this paper. For this reason, it is necessary to take into account all these limitations if one is interested in a quantitative comparison with the observations.

Finally, we must also mention that a full three-dimensional treatment could also lead to the appearence of a whole set of new effects. In the case of Zakharov equations, unidimensional solitons are not stable under transverse perturbations. We do not know what happens with these DNLS solitons; the presence of the static magnetic field could possibly render DNLS solitons stable. Nevertheless, this problem could only be studied within the frame of an extended model, such as the one described by the three-dimensional equation derived by Mjølhus and Wyller (1988).

IV. OBSERVATIONS AND THE SOLITON GAS MODEL

We have already said that relevant quantities for the study of the magnetic field and plasma parameter fluctuations in the interplanetary medium are the correlation functions

$$R_{HG}(X, T) = \langle \delta H(x, t) \cdot \delta G(x + X, t + T) \rangle \,, \tag{61}$$

and their Fourier transforms, the power spectra

$$P_{HG}(k, \nu) = \int dX_3\, R_{HG}(0, 0, X_3, 0) \exp\left(i\,\frac{2\pi\nu X_3}{v_W}\right), \tag{62}$$

where R'_{HG} is the correlation function between G and H (two quantities of interest, such as magnetic fields or density enhancements) in the wind frame, X_3 is the coordinate along the direction of v_W, and P_{HG} is the spectrum in the spacecraft frame (i.e., the observed frequency power spectrum).

If we want to apply the soliton gas model to the solar wind, the assumption which led to equation (62) is satisfied, because our solitons travel with a speed very close to the Alfvén velocity and $v_A \cong 40$ km s^{-1} while $v_W \cong 400$ km s^{-1}. Since all the relevant variables are related according to equation (6), the independent autocorrelations of the model are

$$P_{B_y}(\nu) = \int dT \exp(-2\pi i\nu T)\langle \delta\bar{B}_y(x, t + T)\delta\bar{B}_y(x, t)\rangle \,, \tag{63}$$

$$P_B(\nu) = \int dT \exp(-2\pi i\nu T)\langle |\delta\bar{B}(x, t + T)||\delta\bar{B}(x, t)|\rangle \,, \tag{64}$$

$$P_n(\nu) = \int dT \exp(-2\pi i\nu T)\langle \delta\bar{n}(x, t + T)\delta\bar{n}(x, t)\rangle \,, \tag{65}$$

where $\delta\bar{B}_y = \delta B_y/B_0$, $|\delta\bar{B}| = |\delta B|/B_0$, and $\delta\bar{n} = \delta n/n_0$ correspond to dimensionless variables in the spacecraft reference frame. Particularly, using relation (6c) and approximating $\delta|\boldsymbol{B}| = (1 + B_y^2 + B_z^2)^{1/2} - 1 \cong \frac{1}{2}(B_y^2 + B_z^2) \equiv \frac{1}{2}|B|^2$, it may be seen that

$$P_{|B|}(\nu) = (1 - \beta)^2 P_n \,, \tag{66}$$

where $P_{|B|}$ is the Fourier transform of $\langle \delta|B|\delta|B'|\rangle$, the power spectrum of fluctuations of the magnetic field amplitude. The correlation functions (63)–(65) can be calculated analytically leading to (see Appendix)

$$P_{B_y} = \int d\Delta^2\, d\gamma\, f(\Delta^2, \gamma)\, \frac{\pi|1 - \beta|}{2\Delta^2 v_{W_x} L}\left[p\left(\frac{k}{K} + k', \gamma\right) + p\left(\frac{k}{K} - k', \gamma\right)\right], \tag{67}$$

$$P_B = \int d\Delta^2\, d\gamma\, f(\Delta^2, \gamma)\, \frac{2\pi|1 - \beta|}{\Delta^2 v_{W_x} L}\, p\left(\frac{k}{K}, \gamma\right), \tag{68}$$

$$P_n = \int d\Delta^2\, d\gamma\, f(\Delta^2, \gamma)\, \frac{32\pi\Delta^2}{v_{W_x} L}\, \frac{\sinh^2[\gamma(k/K)]}{\sinh^2[\pi(k/K)]}\,, \tag{69}$$

where $k = 2\pi v/v_{W_x}$, $\bar{k} = (s/2) \cot \gamma$, $K = 4\Delta^2 \sin \gamma$, $p(\xi, \gamma) = [\text{sech}\,(\pi\xi)P_{-1/2+i\xi}(\cos \gamma)]^2$, and $P_{-1/2+i\xi}(\cos \gamma)$ is the Legendre function of the first kind, also called the conical function (Erdélyi 1953, p. 120). It is interesting that equation (67) contains a sech $(\pi k/4\Delta^2 \sin \gamma)$ term, because for a gas of equal NLS solitons, such as the one described by Ovenden, Shah, and Schwartz (1983), each soliton gives a contribution proportional to sech (ηk), with η a real constant. The effect of the $P_\nu(\cos \gamma)$ term in equation (67) is to slow down the falling off of the hyperbolic secant with k. This effect is more important as γ increases. Once again, this means that in a strongly unstable case, as $\gamma \ll 1$, the NLS and DNLS pictures will give similar results whenever one considers only one initial wavenumber.

Let us now use the expressions of f_{I} and f_{II} that were obtained in § II in order to calculate P_{B_y}, P_B, and P_n. In Figure 3, P_{B_y}, P_B, and P_n are plotted versus $\bar{k}$ for $\Psi = 7$, and, in Figure 4, for $\Psi = 50$. All functions behave qualitatively like the observed correlations.

Concerning the spectral indices, we may see that for P_{B_y} and P_B, values obtained with the KRS method (II) are better than those obtained with the MH method (I). In the case of P_n, both methods give similar values of s which do not depend on Ψ. The fact that the theoretical P_n does not follow a power law with k may be an indication that the density perturbations are not described correctly by the static approximation. The same problem arises for $P_{|B|}$, which according to the model is proportional to P_n. In this case, it is probably the unidimensionality of the DNLS equation that makes it impossible to describe longitudinal perturbations of the magnetic field.

The differences between the model and the observations is that the first shows a more rapid decreasing of the spectra with k than observed. However, for $\Psi = 50$, we find that both P_{B_y} and P_B, computed with f_{II}, show a k^{-s} dependence along two orders of magnitude with $s = 2$ for P_B and $s = 1.88$ for P_{B_y}. The behavior of P_B is thus similar to that of the correlation function calculated by Ovenden, Shah, and Schwartz (1983). Actually, if the lower limit $N = 1$ is maintained in the integral (35) of Ovenden, Shah, and Schwartz (1983), it may be seen that their result $P(k) \sim k^{-2}$ is valid for a finite k interval. The s value obtained decreases when one computes the correlation the way that we do for P_{B_y}. The s value thus obtained is closer to the observed value. This is due to the fact that an average over the soliton phases is taken into account in this calculation, while the phases are not considered either in P_B or in Ovenden, Shah, and Schwartz (1983).

For $\Psi = 7$, P_{B_y} and P_B also follow a power law, but only along one decade. In this case, it is more difficult to estimate the power index. It ranges between 1.8 and 2 for P_B and is approximately equal to 1.79 for P_{B_y}. The fact that with $\Psi = 50$, one obtains better

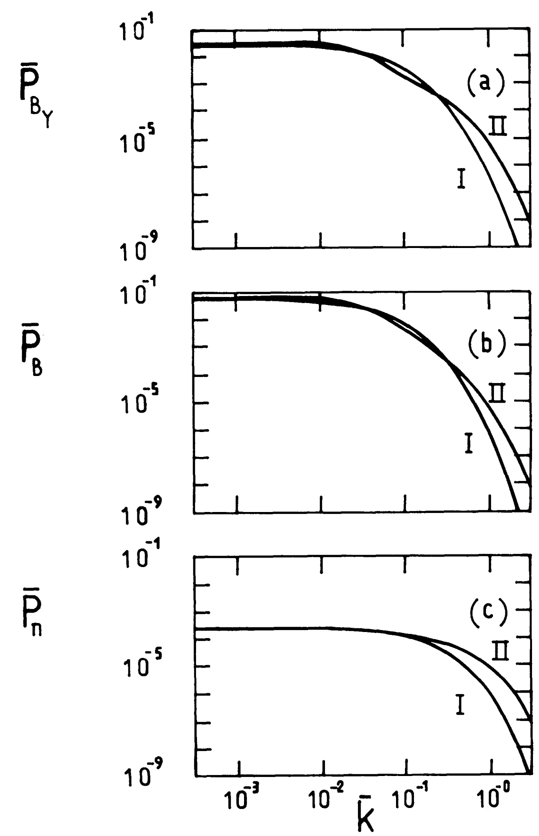

FIG. 3

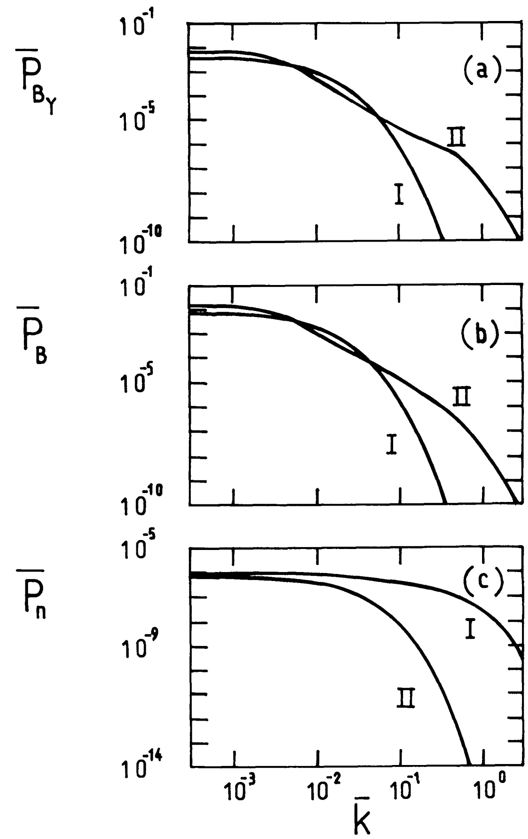

FIG. 4

FIG. 3.—Spectra (*a*) $\bar{P}_{B_y} = v_{W_x} L'P_{B_y}/(L\pi|1-\beta|)$, (*b*) $\bar{P}_B = v_{W_x} L'P_B/(4\pi L|1-\beta|)$, and (*c*) $\bar{P}_n = v_{W_x} L'LP_n/(16\pi m^2)$ are plotted against $\bar{k} = Lk/2m$ for $\Psi = 7$. In all cases, the spectra were calculated using the distribution functions f_{I} (eq. [58]) and f_{II} (eq. [55]). The spectra are qualitatively similar to the observed ones. However, they predict a faster decrease with the wavenumber than observed.

FIG. 4.—Same as Fig. 3, but for $\Psi = 50$. The results in this case are better than those of Fig. 3. This may be because the mean level of modulational instability Ψ is larger and the model developed applies to strongly unstable cases.

results than with $\Psi = 7$ may be due to the fact that the model has been developed for a modulationally strongly unstable case, and Ψ is a measure of such instability.

Concerning the outer and inner scales, we can only obtain those corresponding to P_{B_y} and P_B. Moreover, it is impossible to obtain the value of $\bar{k}_i$ using the MH method (see Figs. 3 and 4). From Figures 3 and 4, it may be deduced that $\bar{k}_0(\Psi = 7) > \bar{k}_0(\Psi = 50)$, where the actual wavenumbers k_0 and $\bar{k}_0$ are related by $\bar{k}_0 = k_0 L/2m$.

Let us now analyze the radial variations of the different quantities that enter the soliton model comparing them with the observations described in Denskat and Neubauer (1983). We are concerned mainly with the maximum level $P_{B_y}(k = 0)$ and with the wavenumber k_0 related to the outer scale. Actually, the soliton gas model cannot describe the $k \cong 0$ region of the spectrum. The term $P_{B_y}(k = 0)$ must be understood as $P_{B_y}(k_{\min})$, $k_{\min}$ the lowest k value the soliton gas model can describe.

The model we have developed so far assumes constant values of ambient magnetic field B_0 and mean density n_0 against which all quantities are dedimensionalized. In order to analyze what happens when B_0 and n_0 vary, we can divide the space into regions of length $\bar{L}$ over which B_0 and n_0 are almost constant. Assuming a power-law dependence with the heliocentric distance $r(B_0 \sim r^{-\alpha}$, $n_0 \sim r^{-\vartheta})$ the length $\bar{L}$ would scale according to $\bar{L} \sim r$, the Alfvén velocity $v_A \sim r^{-\alpha+\vartheta/2}$, the proton gyrofrequency $\omega_{ci} \sim r^{-\alpha}$, and our unit of length $d_i \sim r^{\vartheta/2}$. Therefore, the length L of the system used in the preceding section, which is equal to $\bar{L}$ measured in units of d_i, will satisfy $L \sim r^{1-\vartheta/2}$.

Each arbitrary pulse which enters a region of length L (in units of d_i) is finally transformed into a soliton train, whenever the evolution may be described with the DNLS equation. In a strongly unstable case, according to equation (33), the solitons will have the same wavenumber as the initial pulse. Let us consider two neighboring regions 1 and 2, with region 1 closer to the Sun than region 2. Let us suppose that all initial pulses entering region 1 have the same wavenumber k_1 (an assumption that according to eq. [33] is consistent with the $\delta[\Delta^2 - m/2L]$ function in f_I and f_{II}). All the solitons in region 1 will have the wavenumber $2\Delta^2 \cos\gamma \cong 2\Delta^2 = k_1 = m_1/2L_1$. These solitons travel without deformation until they arrive at region 2, their expression being equations (9)–(10) with $x_1 = x$ and $t_1 = t$ coordinates dedimensionalized with the units that correspond to region 1. If we write the expression of a region 1 soliton in terms of the coordinates of region 2 (x_2 and t_2), we find

$$q = |q_s| \exp\left\{ik_1 \frac{d_{i2}}{d_{i1}}\left[x_2 - \frac{v_{A1}}{v_{A2}}(1 + v_1)t_2\right]\right\}, \tag{70}$$

where the subscripts 1 and 2 identify the regions and where we have simplified expression (10b) supposing $|q|^2 \ll k_1 \cong 2\Delta^2 \ll 1$. We see from equation (70) that this soliton acts as an initial pulse of $k_2 = k_1(d_{i2}/d_{i1})$ for region 2, and therefore all the solitons of region 2 will have this wavenumber. We can conclude that the wavenumber $k(k = m/[2L])$, measured in d_i^{-1} units, evolves according to

$$\frac{m}{2L} = k \sim d_i \sim \frac{1}{\sqrt{n}} \sim r^{\vartheta/2}. \tag{71}$$

It is mentioned in Denskat and Neubauer (1983) that their measurements are consistent with saturated fluctuating amplitudes of the magnetic field. This means that the quantity $\langle|q|\rangle = \langle|\delta\bar{B}|\rangle$ does not depend on the heliocentric distance. If we suppose that $1 - \beta$ is also independent of r (as it is in Ovenden, Shah, and Schwartz 1983), we can deduce that

$$\Psi \sim \left(\frac{m}{L\langle|q|^2\rangle}\right)^{1/2} \sim r^{\vartheta/4}, \tag{72}$$

so that the mean level of modulational instability increases with r. If this is so, Figure 3 would correspond to a region which is closer to the Sun than that of Figure 4. The conclusion that Ψ should increase with r is also drawn if we compare the power index s in Figures 3a and 4a with observations at different heliocentric distances: s decreases with r, and $s(\Psi = 7) < s(\Psi = 50)$. However, the model cannot describe the variation of the power index for the $P_{|B|}$ case. This may be due to the fact that fluctuations of the magnetic field component parallel to B_0 are not taken into account.

We can further compute the radial variation of $P_{B_y}(k = 0)$ and k_0. If we make the estimates

$$P_{B_y}(k = 0) \cong P_B(k = 0) = \frac{4\pi L|1 - \beta|}{9\Psi L' v_{W_x}}, \tag{73}$$

$$k_0 \cong \langle 4\Delta^2 \sin\gamma\rangle \cong \frac{m}{L}\langle\gamma\rangle = \frac{m}{9L\Psi}, \tag{74}$$

we obtain

$$P_{B_y} \sim \frac{L v_A}{L'\Psi} \sim \frac{L r^{-\alpha+\vartheta/4}}{L'}, \qquad k_0 \sim r^{\vartheta/4}, \tag{75}$$

where we have assumed that the solar wind velocity is independent of the heliocentric distance and thus, when measured in units of v_A (as it is in expressions [63]–[65]) it varies according to $v_W \sim v_A^{-1}$.

In order to contrast these results with the observed ones, we must remember that P_{B_y} and k_0 are dedimensionalized quantities: P_{B_y} is measured in units of $\alpha B_0^2/\omega_{ci}(\alpha \cong \frac{1}{2})$, and k_0 is measured in units of $1/d_i$. Taking this into account, we can deduce from Figure 4 of Denskat and Neubauer (1983) that

$$P_{B_y} \sim r^{\alpha-2.6}, \tag{76}$$

$$k_0 \sim r^{\vartheta/2-0.6}. \tag{77}$$

If we use the observational scaling $B_0 \sim r^{-1.6}$, we obtain $P_{B_y} \sim r^{-1}$. In that case, equations (76) and (77) agree with equation (75) if

$$\vartheta = 2.4 , \tag{78}$$

$$\frac{L}{L'} = \text{const} . \tag{79}$$

The relation (78) implies that the mean density $n_0 \sim r^{-2.4}$, so it decreases slightly faster than with a r^{-2} law. This behavior agrees with that observed by *Helios* (Schwenn 1983). Had we assumed an r^{-2} law for n_0, the result from equation (74) would have given a dependence for k_0 similar to that obtained by Ovenden, Shah, and Schwartz (1983) (remember that their $k_{\text{br}} = k_0/d_i$), but is slightly different from the observed one (see, e.g., eq. [77]).

Let us now discuss the meaning of the quantity L' and that of the relation (78). In general, $L = L'$. However, if we want to extract from the observations the model values of Ψ, m/L, and $\langle |q|^2 \rangle$, we may see that it is impossible to adjust P_{B_y} and $P_{|B|}$ simultaneously with $L = L'$. This may be due to the fact that the model does not describe the spectrum of magnitude fluctuations $P_{|B|}$ correctly. Nevertheless, we can make the assumption that in a strongly unstable case the number of solitons is so large that it is not a good approximation to consider them as uncorrelated entities. Instead of trying a new model of correlated solitons, we can simply modify the existing model by introducing the quantity L', which represents the distance over which we can choose the peak position of each correlated soliton. Since $\langle N \rangle \langle l_\phi \rangle \sim$ const, where l_ϕ is defined in equation (13), it seems correct that L/L' does not depend on the heliocentric distance, as equation (79) asserts.

V. CONCLUSIONS

We have compared two statistical models of Alfvén solitons with solar wind observations. An analysis of the characteristic time that is necessary for the solitons to be formed shows that soliton gas models cannot explain spectra in a range of wavenumbers $k \leq k_{\min}$. The quantity $k_{\min}$ decreases when the heliocentric distance is increased. It may be reduced further if wave amplification is taken into account (Spangler 1986). However, it cannot be reduced indefinitely, because very low wavenumbers are modulationally unstable only for extremely low magnetic field perturbations. Although the number of solitons could be large even in this case, producing a large value of $P_{B_y}(k = 0)$, the processes described by the model in the low-wavenumber region could be masked by other processes related to higher perturbations of the magnetic field. However, we must remember that the inclusion of resonant effects produces a new kind of modulational instability (Flå, Mjølhus, and Wyller 1988*b*), and this may cause the soliton gas model to be valid even for $k \cong 0$.

The models we analyze in this paper give a similar mean number of solitons, and this fact reinforces the idea that the expression (29) found in a recent paper (Ponce Dawson and Ferro Fontán 1989) is essentially correct. The main difference between both models is the behavior of the distribution functions with the energy of the solitons. The first model (MH) is an extension of the one developed by Meiss and Horton (1982) for drift-wave turbulence and gives a distribution which follows an exponential law with energy. The second model (KRS) is an extension of the work of Kingsep, Rudakov, and Sudan (1973), which applies to strong Langmuir turbulence and uses a distribution function which follows a power law instead. It may be applied to the DNLS case, Keeping in mind that damping or wavegrowth effects render the equation nonintegrable. According to the results of § III, the consequences that produces the inclusion of these terms could be better described by the KRS model than by the MH model. This agrees with the fact that the former gives results which better resemble the observations than the latter.

We have also found that the results are improved when the quantity Ψ, which measures the mean level of modulational instability, increases. This is due to the fact that the model developed in this paper is valid for strongly unstable situations. In such cases, the DNLS equation reduces to the NLS if the IC is of the form of a plane wave with a given wavenumber. It is then meaningful to compare these results to those of Ovenden, Shah, and Schwartz (1983). We have found a qualitatively good agreement between the theoretical predictions and the observations. However, the model gives power spectra which decrease with the wavenumber faster than observed. This aspect could probably be improved if dissipative and amplification effects are consistently included in the model.

Concerning the spectral index, our results seem better than those of Ovenden, Shah, and Schwartz (1983), because we have considered an average over the phases that was not taken into account in that paper. Regarding the outer and inner scales, we have found that they vary with the heliocentric distance r, as observed, if the mean level of modulational instabilities Ψ increases with r and if the density perturbations decrease slightly faster than according to a r^{-2} law. This last result agrees with *Helios* observations (Schwenn 1983). Within this model, we could explain the radial variation of $P_{B_y}(k = 0)$, k_0, and s simultaneously (in Ovenden, Shah, and Schwartz 1983, only the radial dependence of k_0 was analyzed).

These results, which have been obtained with a unique initial wavenumber, remain almost unchanged if a uniform distribution of wavenumbers over a given range is considered.

Since the model is unidimensional, it cannot explain the perturbations of the magnetic field component parallel to the ambient field. Probably for this reason, it is impossible to adjust all the observed power spectra (P_{B_y} and $P_{|B|}$) simultaneously. However, the existence of some kind of correlation between the solitons may also explain this fact. This may be taken into account through the quantity L', the distance over which the initial peak position x_0 may be chosen. Moreover, if we try to adjust the model to the observations quantitatively, it is necessary that $L' \neq L$ if $\Psi \gg 1$. It would be interesting to extend these models to cover weakly unstable cases in order to decide which is the one that gives the best results. In a weakly unstable case, as solitons with $\pi/2 < \gamma < \pi$ would appear (Ponce Dawson and Ferro Fontán 1989), the power index would decrease due to the factor $P_{1/2+i\nu}$ in P_{B_y} (see eq. [67]). Besides, since the solitons would be narrower, no kind of correlation among them would be necessary. However, the maximum level $P_{B_y}(k = 0)$ would also decrease due to a diminishing of the number of solitons. It must be stressed that the weakly unstable case could only be analyzed within the frame of the DNLS equation, which does not reduce in such a case to the NLS one.

Finally, it must be noted that both kinds of distribution functions compared in this paper could have also been obtained within the frame of Matsuno's (1977) method choosing different probability densities for the initial conditions. We may then conclude that, in order to obtain spectra following a power law with wavenumber, it is necessary that the distribution of the initial conditions and that of the solitons follow a power law with the energy.

We are indebted to S. R. Spangler for his fruitful comments and suggestions.

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APPENDIX

We want to calculate

$$P_{B_y}(v) = \int dT \exp(-2\pi i v T)\langle \delta\bar{B}_y(x, t+T)\delta\bar{B}_y(x, t)\rangle , \tag{A1}$$

where $\delta\bar{B}_y$ is the dedimensionalized component of the magnetic field perturbation in the spacecraft reference frame, and the average must be calculated the same way as in equation (34). We assume that $\delta\bar{B}_y$ may be written as a superposition of solitons

$$\delta B_y = \sum_n \left\{ \frac{k_n(2|1-\beta|)^{1/2}}{\Delta_n[\cosh(k_n x_n) + \cos\gamma_n]^{1/2}} \cos\left[s \cot\gamma_n \frac{k_n x_n}{2} - \frac{3}{4} s \int_{-\infty}^{x_n} dz\, \phi_n^2(z) + 4\Delta^4 t + \varphi_{0n} \right] \right\} , \tag{A2}$$

where $k_n = 4\Delta_n^2 \sin\gamma_n$; $x_n = x - (v_{W_x} + 1 - sk_n \cot\gamma_n)t - x_{0n}$, $v_{W_x} = \boldsymbol{v}_W \cdot \hat{\boldsymbol{x}}$ is the solar wind velocity in the direction of the ambient magnetic field $\boldsymbol{B}_0$, and ϕ_n is the absolute value defined in equation (10a) with $\Delta^2 = \Delta_n^2$ and $\gamma = \gamma_n$. The quantities k_n and γ_n are functionals of the initial conditions, and uniform distributions are assumed for $\varphi_{0n} \in (0, 2\pi)$ and $x_{0n} \in (0, L_n)$.

Granting a sufficiently dilute gas of solitons, products of different soliton contributions in equation (A1) will be negligible. Moreover, for the case of a gas with predominantly low γ solitons and $k \ll 1$, the terms $4\Delta^4 t$ and $(3/4)s\int_{-\infty}^{x_n} dz\, \phi_n^2(z)$ (upper bounded by 3γ) can be neglected in front of $(\cot\gamma_n)(k_n x_n/2)$. If we also suppose $v_{W_x} \gg 1$ (remember that v_W is measured in units of v_A), equation (A1) reduces to

$$P_{B_y}(v) = \sum_n \int dz\, dp_n\, dq_n\, P(x_{0n})Q(\varphi_{0n}) \exp\left(-2\pi i v \frac{z}{v_{W_x}}\right)\left[\frac{2k_n^2|1-\beta|\cos(q_n)\cos(q_n - sk_n\cot\gamma_n z/2)}{\Delta_n^2 v_{W_x}[\cosh(k_n p_n) + \cos\gamma_n]^{1/2}\{\cosh[k_n(p_n - z)] + \cos\gamma_n\}^{1/2}}\right] , \tag{A3}$$

where we have defined $p_n = x - x_{0n} - v_{W_x} t$, $q_n = (s\cot\gamma_n)(k_n/2)(x - v_{W_x} t) + \varphi_{0n}$, and $z = Tv_{W_x}$. The integral over q_n can be performed immediately, leading to

$$P_{B_y}(v) = \sum_n \int dz\, dp_n\, P(x_{0n}) \exp\left(-2\pi i v \frac{z}{v_{W_x}}\right)\left[\frac{k_n^2|1-\beta|\cos(k_n\cot\gamma_n z/2)}{\Delta_n^2 v_{W_x}[\cosh(k_n p_n) + \cos\gamma_n]^{1/2}\{\cosh[k_n(p_n - z)] + \cos\gamma_n]^{1/2}}\right] . \tag{A4}$$

Writing the cosine term as a sum of exponentials, if $P(x_{0n})$ is a uniform distribution over the long distance L_n, equation (A4) can be thought of as the sum of two Fourier transforms of a convolution. This gives

$$P_{B_y} = \sum_n \frac{\pi|1-\beta|}{2\Delta_n^2 v_{W_x} L'}\left[\left[\left\{\operatorname{sech}\left[\pi\left(\frac{k}{k_n} + k'_n\right)\right] P_{-1/2+i(k'+k/k_n)}(\cos\gamma)\right\}^2 \left\{\operatorname{sech}\left[\pi\left(\frac{k}{k_n} - k'_n\right)\right] P_{-1/2+i(-k'+k/k_n)}(\cos\gamma)\right\}^2\right]\right] , \tag{A5}$$

where $k = 2\pi v/v_{W_x}$, $k'_n = (s/2)\cot\gamma_n$, and $L' = L_n$ is independent of n. In general, L' is taken to be equal to L. We will postpone the discussion of the value of L' until the comparison with the observations is made. In equation (A5), $P_{-1/2+iv}(\cos\gamma)$ is the Legendre function of the first kind, which is proportional to Gauss's hypergeometric series (Erdélyi 1953, p. 120):

$$P_{-1/2+iv}(\cos\gamma) = 1 + \frac{4v^2+1}{2^2}\sin^2\left(\frac{\gamma}{2}\right) + \frac{(4v^2+1)(4v^2+3^2)}{2^2 4^2}\sin^4\left(\frac{\gamma}{2}\right) + \ldots . \tag{A6}$$

Expression (A5) may be reobtained in the case $v_{W_x} \gg 1$ from the wind reference frame correlation function by using relation (61).

When the number of solitons is large enough, a continuous description may be applied. In such a case equation (A5) reduces to

$$P_{B_y} = \int d\Delta^2\, d\gamma\, f(\Delta^2, \gamma)\frac{\pi|1-\beta|}{2\Delta^2 v_{W_x} L'}\left[\left[\left\{\operatorname{sech}\left[\pi\left(\frac{k}{K} + k'\right)\right] P_{-1/2+i(k'+k/K)}(\cos\gamma)\right\}^2 + \left\{\operatorname{sech}\left[\pi\left(\frac{k}{K} - k'\right)\right] P_{-1/2+i(-k'+k/K)}(\cos\gamma)\right\}^2\right]\right] , \tag{A7}$$

where $K = 4\Delta^2 \sin\gamma$ and $k' = s/2 \cot g\gamma$.

The power spectra P_B and P_n can be calculated in a similar way (in these cases, there is no phase dependence, and some of the assumptions that led to eq. [A7] are not necessary). Their continuous versions read

$$P_B = \int d\Delta^2 \, d\gamma \, f(\Delta^2, \gamma) \, \frac{2\pi|1-\beta|}{\Delta^2 v_{W_x} L} \left[\operatorname{sech}\left(\pi \frac{k}{K}\right) P_{-1/2+i(k/K)}(\cos\gamma) \right]^2 , \tag{A8}$$

$$P_n = \int d\Delta^2 \, d\gamma \, f(\Delta^2, \gamma) \, \frac{32\pi\Delta^2}{v_{W_x} L} \, \frac{\sinh^2 [\gamma(k/K)]}{\sinh^2 [\pi(k/K)]} . \tag{A9}$$

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