COMPLETE INTERSECTIONS IN TORIC IDEALS

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ABSTRACT. We present examples that show that in dimension higher than one or codimension higher than two, there exist toric ideals I_A such that no binomial ideal contained in I_A and of the same dimension is a complete intersection. This result has important implications in sparse elimination theory and in the study of the Horn system of partial differential equations.

1. INTRODUCTION

Given a configuration $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^m$ of integral points generating $\mathbb{Z}^m$, the (toric) ideal $I_A \subset \mathbb{C}[x_1, \ldots, x_n]$ is generated by all binomials

$$x^u - x^v,$$

whose exponents $u, v \in \mathbb{N}^n$ satisfy $A \cdot u = A \cdot v$. Here we are also denoting by A the $m \times n$ matrix whose j-th column is a_j. Note that I_A is weighted-homogeneous for every weight w in the row-span of A. We shall assume that $(1, \ldots, 1)$ is in the row-span of A and, consequently, that I_A is a (standard) homogeneous ideal.

The configuration A and its associated toric ideal I_A are the central characters in several areas of very active research in commutative algebra, algebraic geometry, and differential algebra. The associated projective variety X_A has a natural action of the algebraic torus $(\mathbb{C}^*)^m$ making it into a projective toric variety of dimension $d := m - 1$. We call d the dimension of A.

Let $\mathcal{L}_A \subset \mathbb{R}^n$ be the lattice

$$\mathcal{L}_A = \{v \in \mathbb{Z}^n : A \cdot v = 0\}$$

and let $r := n - m$ be the codimension of A. Let $B = \{v_1, \ldots, v_r\} \subset \mathcal{L}_A$ be a maximal set of vectors linearly independent over $\mathbb{Q}$. We also denote by B the $n \times r$ matrix whose j-th column is v_j and consider the binomial ideal J_B generated by the binomials $x^{v_j^+} - x^{v_j^-}$, where $v_j = v_j^+ - v_j^+$ is the decomposition in positive and negative components.

It has been proven in [8] that $J_B \cdot \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is always a complete intersection in the Laurent polynomial ring, and that it coincides with $I_A \cdot \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ if and only if the greatest common divisor g of the maximal minors of B satisfies $g = 1$. This positive integer g is precisely the index with respect to $\mathcal{L}_A$ of the lattice spanned by B. We will refer to J_B as a *basis ideal*. Following [14] we will reserve

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the term lattice basis ideal for the case when B is a $\mathbb{Z}$-basis of $\mathcal{L}_A$. Clearly $J_B \subset I_A$ and, in general, this containment is proper.

In this note we study a question that arises naturally in the study of A-discriminants [7] and of Horn systems of differential equations [6]: does every toric ideal contain a complete intersection basis ideal (i.e., a complete intersection binomial ideal of the same dimension)? This is indeed the case if X_A is a monomial curve ($d = 1$) or in the codimension two case ($r = 2$). The purpose of this note is to show that in any dimension higher than one and any codimension higher than two, there exist toric ideals I_A such that no basis ideal contained in I_A is a complete intersection.

We recall that the dual variety X_A^* of X_A is the Zarisky closure of the locus of hyperplanes tangent to X_A at a smooth point. When X_A^* is a hypersurface, its defining equation is the A-discriminant. This notion, introduced by Gel′fand, Kapranov, and Zelevinsky [12], generalizes the classical notion of the discriminant of a univariate polynomial. Dickenstein and Sturmfels [7] have shown how to compute A-discriminants for codimension two (i.e., $n - m = 2$) configurations. A key ingredient of this work is the fact that every basis ideal in a codimension two toric ideal is a complete intersection. Our results show that a different approach is needed to describe A-discriminants in higher codimensions.

The work of Gel′fand et al. on sparse elimination was a step toward the study of A-hypergeometric (or GKZ) systems. Consider the "quantized" version of the ideal I_A, that is, the left ideal $H_A(\beta)$ in the Weyl algebra

$$D_n = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

generated by the toric operators $\partial^u - \partial^v$, $A \cdot u = A \cdot v$ together with the Euler operators associated with the $(\mathbb{C}^*)^m$ action:

$$a_{j1} x_1 \partial_1 + \cdots + a_{jn} x_n \partial_n - \beta_j \,,$$

where $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{C}^m$. An A-hypergeometric function of degree β is a locally defined (multivalued) holomorphic function $\mathbb{C}^n$ annihilated by $H_A(\beta)$. This notion of hypergeometric functions encompasses most of the classical univariate and multivariate hypergeometric functions. GKZ systems are holonomic for all choices of parameters β, and so in particular, the corresponding spaces of A-hypergeometric functions are finite dimensional. Another classical multivariable generalization of hypergeometric differential equations is given by the Horn systems, which are closely related to the GKZ systems. A Horn system consists of the Euler operators and only those toric operators coming from a basis ideal J_B contained in I_A. Dickenstein, Matusevich, and Sadykov [6] have shown that, in codimension two, the behavior of an A-hypergeometric system and that of any of its associated Horn systems is not very different. This relies on the fact that such basis ideals are complete intersections. However, if the basis ideal is not a complete intersection, the Horn system has an infinite dimensional local solution space for all choices of parameters and, thus, is never holonomic. Our examples show that in the general multivariate case, the A-hypergeometric and Horn systems have essentially distinct behavior.

2. Preliminaries

The study of binomial ideals is intimately connected with the study of (affine) semigroup algebras. It is from this perspective, and beginning with the work of Herzog [13] and Delorme [5], that the question of classifying complete intersection

binomial ideals has been extensively studied by many authors [1, 2, 3, 9, 10, 11, 14, 15, 16, 17, 18, 20, 21, 22]. A combinatorial characterization of these ideals is given in [11] in terms of a choice of B and the notion of mixed matrices; that is, matrices such that every column contains a strictly positive and a strictly negative entry. Note that since the columns of the matrix B add up to zero, B is automatically mixed. The following result follows from [8, Theorem 2.1] and [11, Theorem 2.3] (see also [17, Theorem 2.7]).

Theorem 2.1. *The ideal J_B is a complete intersection if and only if for every mixed $n' \times r'$-submatrix $B' \subset B$ we have $n' \geq r'$.*

Since a mixed submatrix must contain at least two rows it follows that:

Corollary 2.2. *If $r \leq 2$, every basis ideal J_B is a complete intersection.*

It is also easy to prove that if $m = 2$, i.e., for X_A a monomial projective curve, there exists B such that J_B is a complete intersection. Indeed, let

$$(2.1) \qquad A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix},$$

where $a_1 \leq \cdots \leq a_n$ are coprime. Performing a row operation that does not change I_A, we may assume without loss of generality that $a_1 = 0$, and therefore all $a_j \geq 0$. Consider now the following choice of B:

$$(2.2) \qquad B = \begin{pmatrix} a_3 - a_2 & a_4 - a_3 & \cdots & a_n - a_{n-1} \\ -a_3 & 0 & \cdots & 0 \\ a_2 & -a_4 & \cdots & 0 \\ 0 & a_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & -a_n \\ 0 & 0 & \cdots & a_{n-1} \end{pmatrix}.$$

Since $a_{i+1} - a_i \geq 0$, it follows that every mixed submatrix of B must contain more rows than columns and therefore J_B is a complete intersection.

Note that already in the simplest case of the twisted cubic; i.e., the curve X_A associated with

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix},$$

the toric ideal I_A is not a complete intersection, but it does contain a complete intersection basis ideal.

Given a vector $v \in \mathbb{Z}^n$, we define the support of v:

$$\mathrm{supp}(v) := \{i \in \{1, \ldots, n\} : v_i \neq 0\}.$$

Similarly we set $\mathrm{supp}^+(v) := \{i : v_i > 0\}$, $\mathrm{supp}^-(v) := \{i : v_i < 0\}$. If $u, v \in \mathbb{Z}^n$, we say that u is *conformal* to v if $\mathrm{supp}^+(u) \subset \mathrm{supp}^+(v)$ and $\mathrm{supp}^-(u) \subset \mathrm{supp}^-(v)$.

Given a configuration $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^m$, a vector $v \in \mathcal{L}_A$ is called a *circuit* if its support is minimal among all elements in $\mathcal{L}_A$ relative to inclusion. As shown in the proof of [19, Lemma 4.9], if u is a circuit in $\mathcal{L}_A$ of maximal dimension m with $\mathrm{supp}(u) = \{i_1, \ldots, i_{m+1}\}$, then up to a multiple:

$$(2.3) \qquad u = \sum_{j=1}^{m+1} (-1)^j \det \left(a_{i_i}, \ldots, a_{i_{j-1}}, a_{i_{j+1}}, \ldots, a_{i_{m+1}} \right) e_{i_j}.$$

The following is Lemma 4.10 in [19].

Lemma 2.3. *Every vector $v \in \mathcal{L}_A$ may be written as a non-negative rational linear combination of $n - m$ circuits each of which is conformal to v.*

Definition 2.4. Given a configuration $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^m$, we say that a basis ideal J_B is generated by circuits if and only if each of the column vectors of B is a circuit in $\mathcal{L}_A$.

Proposition 2.5. *A toric ideal I_A contains a complete intersection basis ideal if and only if it contains a complete intersection basis ideal generated by circuits.*

Proof. Let $\{v_1, \ldots, v_r\} \subset \mathcal{L}_A$ be a $\mathbb{Q}$-linearly independent set defining a complete intersection basis ideal. Let $i \in \{1, \ldots, r\}$ be the smallest index such that v_i is not a circuit. Using Lemma 2.3 write v_i as a non-negative rational combination of circuits $w_1^i, \ldots, w_r^i$:

$$v_i = q_1 w_1^i + \cdots + q_r w_r^i .$$

Clearly, some w_j^i must be linearly independent from $\{v_k, k \neq i\}$, hence we may replace v_i by the circuit w_j^i to obtain a new $\mathbb{Q}$-linearly independent set. Continuing in this manner we obtain a linearly independent set $\{w_1, \ldots, w_r\}$ consisting of circuits and such that w_i is conformal to v_i.

Let B (respectively C) denote the matrix whose columns are the vectors v_i (respectively w_i). Since w_i is conformal to v_i, it follows that if C' is a mixed submatrix of C, then the corresponding submatrix B' of B is also mixed. Hence, by Theorem 2.1, if J_B is a complete intersection, so is J_C. $\square$

3. Toric ideals of dimension at least two

In this section we will exhibit examples of configurations in any dimension $d = m - 1 \geq 2$, which do not contain any complete intersection basis ideal. By assumption we may suppose that $A \subset \{1\} \times \mathbb{Z}^d$. By abuse of notation we will identify A with its projection onto $\mathbb{Z}^d$.

Theorem 3.1. *Let A be the vertex set of a lattice polytope in $\mathbb{R}^d$ and $n = |A|$. There exists $N = N(d)$ such that, for $n \geq N$, I_A does not contain any complete intersection basis ideal.*

Proof. Let $B \subset \mathbb{Z}^{n \times (n-m)}$ be such that its columns are a $\mathbb{Q}$-basis of the kernel lattice $\mathcal{L}_A$. Note that every column of B must contain at least two strictly positive and two strictly negative entries since, otherwise, one of the points in A would be in the convex hull of some of the other points in A and that is impossible by assumption.

Suppose J_B is a complete intersection. It then follows from Theorem 2.1 that every $(n-m-1) \times (n-m)$ minor of B must be non-mixed. In other words, for any subset $J \subset \{1, \ldots, n\}$, $|J| = m + 1$, there exists a column v_i of B such that either the positive support $\mathrm{supp}^+(v_i)$ or the negative support $\mathrm{supp}^-(v_i)$ is contained in J. Clearly, there are $\binom{n}{d+2}$ index sets J of cardinality $m + 1 = d + 2$.

Now, for a given v_i, its positive support contains at least two indices and therefore it may be contained in at most $\binom{n-2}{d}$ distinct index sets J of cardinality $m + 1$. Thus, since B has $n - d - 1$ columns and taking into account the positive and negative supports, the condition may be satisfied for at most

$$2 (n - d - 1) \binom{n - 2}{d}$$

index sets J. But for n sufficiently large

$$(3.1) \qquad \binom{n}{d+2} > 2\,(n-d-1)\binom{n-2}{d}$$

since the left-hand side is a polyomial in n of degree $d+2$ with positive leading term, while the right-hand side is a polynomial in n of degree $d+1$. $\qquad\square$

Remark 3.2. In the planar case $d=2$, the inequality (3.1) is satisfied for $n \geq 22$. However, it is clear that the estimates above are very rough and that one should expect Theorem 3.1 to hold for n considerably smaller than 22.

Example 3.3. Consider a configuration $\{a_1, \ldots, a_{10}\}$ of ten points in $\mathbb{Z}^2$ that are the vertices of a polygon. We may assume them to be ordered counterclockwise. Given four indices $1 \leq i < j < k < \ell \leq 10$, there exists a relation

$$\lambda_i a_i - \lambda_j a_j + \lambda_k a_k - \lambda_\ell a_\ell = 0,$$

where $\lambda_i, \lambda_j, \lambda_k, \lambda_\ell$ are positive integers. Such a relation defines a circuit in the lattice kernel $\mathcal{L}_A$. Using the computer algebra system CoCoA [4], we searched for sets of seven such relations satisfying the condition in Theorem 2.1. The following is such an example. We have only indicated the sign of the coefficients since that is all that matters in Theorem 2.1 and, for generic coefficients, the matrix B will be of maximal rank.

$$B = \begin{pmatrix} + & + & + & 0 & + & 0 & 0 \\ - & - & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & - & + & 0 \\ - & + & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & - & 0 \\ 0 & 0 & 0 & - & + & + & + \\ 0 & 0 & 0 & + & 0 & 0 & - \\ 0 & 0 & + & 0 & - & 0 & + \\ 0 & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & - & 0 & - & - \end{pmatrix}.$$

We were unable to obtain similar examples with $n = 11$. We suspect that Theorem 3.1 holds for polygons in the plane with at least 11 vertices.

4. Toric ideals of codimension at least three

It follows from Remark 3.2 that Theorem 3.1 furnishes examples of toric ideals that do not contain any complete intersection basis ideal in codimension greater than twenty-one. In this section we will describe a different class of examples that show that in every codimension greater than two there exist toric ideals with that same property.

Let $n = m + r$, and consider a configuration consisting of the vertices of a cyclic polytope (we refer to [23] for other properties of this important class of polytopes):

$$(4.1) \qquad A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ t_1^2 & t_2^2 & \cdots & t_n^2 \\ \vdots & \vdots & & \vdots \\ t_1^{m-1} & t_2^{m-1} & \cdots & t_n^{m-1} \end{pmatrix},$$

where $0 < t_1 < t_2 < \cdots < t_n$ are integers. For appropriate choices of $t_1, \ldots, t_n$, the columns of A span $\mathbb{Z}^m$.

Theorem 4.1. *Let $r \geq 3$. For $n \geq 2(r^2 - r + 1)$ the toric ideal I_A associated with the matrix* (4.1) *does not contain any complete intersection basis ideal.*

Proof. According to Proposition 2.5, it suffices to show that I_A does not contain any complete intersection basis ideal generated by circuits that, up to constant, are given by the expression (2.3). On the other hand, note that all maximal minors of A are non-zero and the determinant

$$\det \left(a_{i_i}, \ldots, a_{i_{j-1}}, a_{i_{j+1}}, \ldots, a_{i_{m+1}} \right)$$

is strictly positive since it is the Vandermonde determinant for

$$t_{i_i} < \cdots < t_{i_{j-1}} < t_{i_{j+1}} < \cdots < t_{i_{m+1}}.$$

Therefore, if $v \in \mathcal{L}_A$ is a circuit, then it will have exactly $r - 1$ zero entries while the remaining entries will alternate in sign.

This means that if B is an $n \times r$ matrix whose columns are a circuit basis of $\mathcal{L}_A$, then each column of B contains exactly $r - 1$ entries that are zero and B contains a total of $r(r-1)$ zero entries. This implies that if $n \geq 2(r^2 - r + 1)$, B will have to contain two consecutive rows all of whose entries are non-zero. Let B' denote the 2×3 submatrix consisting of those two rows. Since the signs of the columns are alternating, B' is mixed and, by Theorem 2.1, J_B is not a complete intersection. In fact, if B' consists of the j-th and k-th rows of B, the codimension two ideal $\langle x_j, x_k \rangle$ is an associated prime ideal of J_B. $\square$

Remark 4.2. For $r = 3$ the lower bound in Theorem 4.1 is 14. One can show that for $n \leq 11$ every toric ideal I_A admits a complete intersection basis ideal. We do not know if there are examples of configurations with $n = 12$ or $n = 13$ that do not admit any such basis ideals.

Acknowledgments

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