CODIMENSION THEOREMS
FOR COMPLETE TORIC VARIETIES

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ABSTRACT. Let X be a complete toric variety with homogeneous coordinate ring S. In this article, we compute upper and lower bounds for the codimension in the critical degree of ideals of S generated by $\dim(X) + 1$ homogeneous polynomials that do not vanish simultaneously on X.

INTRODUCTION

In this paper, X will denote a complete toric variety of dimension n. We will work over $\mathbb{C}$, so that the torus of X is $(\mathbb{C}^*)^n$. The dual lattices will be denoted by M and N as usual, and the minimal edge generators of the fan of X will be denoted by $n_1, \ldots, n_r \in N$. Corresponding to each n_i we have the irreducible torus-invariant divisor D_i and the variable x_i in the homogeneous coordinate ring $S = \mathbb{C}[x_1, \ldots, x_r]$, which is graded by the Chow group $A_{n-1}(X)$.

Consider a torus-invariant Cartier divisor $D = \sum_i a_i D_i$ such that $\mathcal{O}_X(D)$ is generated by global sections. In Section 1, we recall a vanishing theorem of Batyrev and Borisov [2] that describes $H^i(X, \mathcal{O}_X(-D))$ in terms of the polytope

$$\Delta_D = \{m \in M_\mathbb{R} \mid \langle m, n_i \rangle \geq -a_i\}.$$

By Serre duality, we get a description of

$$H^i(X, \mathcal{O}_X(D + K)),$$

where K is the canonical class.

In Section 2, we consider homogeneous polynomials $f_0, \ldots, f_n \in S$ of degrees $\alpha_0, \ldots, \alpha_n \in A_{n-1}(X)$ which satisfy the following two properties: first, the f_i do not vanish simultaneously on X, and second, each α_i lies in $\mathrm{Pic}(X)$ and the corresponding line bundle $\mathcal{O}_X(\alpha_i)$ is generated by global sections. By abuse of terminology, we will say that α_i is *globally generated* in this situation.

Following [4], we define the *critical degree* to be

$$(0.1) \qquad \rho = \sum_{i=0}^{n} \alpha_i - \beta_0$$

where $\beta_0 = \sum_{j=1}^{r} \deg(x_j)$, and we let

$$I = \langle f_0, \ldots, f_n \rangle \subset S$$

be the homogeneous ideal generated by the f_i. Then the *Codimension Question* asks what is the dimension of $(S/I)_\rho$, i.e., what is the codimension of I_ρ in S_ρ?

The main result of Section 2 computes upper and lower bounds for this codimension, when the associated family of polytopes is *essential*. In particular, it is always nonzero. The assumption of having an essential family is natural since it is necessary for the existence of a nontrivial sparse resultant whose vanishing is equivalent to the condition that the f_i do not have common zeros on X (see [11]). We also give further geometric conditions on the polytopes under which the codimension at the critical degree attains the upper bound that we present.

When the divisors involved are big and nef, our results imply that the codimension is 1. The first toric codimension-one theorem, proved in [7] using ideas from [1], assumed that the α_i were all equal to the class of a single ample divisor. This was extended in [5] to the case when the α_i were positive integer multiples of a single ample class. In the general case when the α_i are all ample, a codimension-one theorem was proved in [4] provided that X was simplicial. All of these results now follow from part (1) of Corollary 2.5.

The codimension in the critical degree is important because of its relation to the theory of toric residues [3, 4, 5, 7, 8]. Toric residues are rational functions of the coefficients of the given polynomials $f_0, \ldots, f_n$; moreover, they are rational hypergeometric functions determined by the lattice points of the associated family of polytopes, with poles at the resultant locus [6]. As described in [7], a homogeneous polynomial H of critical degree gives rise to a rational n-form on X. Since the f_i do not vanish simultaneously on X, this n-form represents an element of $H^n(X, \widehat{\Omega}_X^n) \simeq H^n(X, \mathcal{O}_X(-\beta_0))$. The toric residue of H is defined to be the trace of this cohomology class. If the codimension is one and we have an explicit element J of critical degree with known residue, then the computation of the toric residue of H is reduced to writing $H = cJ$ modulo I. When all degrees are equal and ample, a choice of J is the toric Jacobian [7] associated to $f_0, \ldots, f_n$. In case all degrees are ample, explicit elements with residue equal to ± 1 are known [4, 8], but it is still an open problem to find such elements in the big and nef but not ample case.

1. The vanishing theorem

Now let D be a torus-invariant Cartier divisor on a complete toric variety X. We define the polytope Δ_D as in the introduction and we use $\text{int}(\Delta_D)$ to denote the relative interior of Δ_D. Here is the vanishing theorem from [2, Thm. 2.5].

Theorem 1.1. *Let D be a torus-invariant Cartier divisor on a complete toric variety X and let Δ_D be the polytope defined above. If $\mathcal{O}_X(D)$ is generated by global sections, then:*

(1) $H^i(X, \mathcal{O}_X(-D)) = 0$ *for* $i \neq \dim(\Delta_D)$.

(2) *There is an isomorphism*

$$H^{\dim(\Delta_D)}(X, \mathcal{O}_X(-D)) \simeq \bigoplus_{m \in M \cap \mathrm{int}(\Delta_D)} \mathbb{C} \cdot \chi^{-m}$$

which is equivariant with respect to the natural $(\mathbb{C}^*)^n$ *action on each side.*

Remark 1.2. We should note that for a complete toric variety X, the sheaf $\mathcal{O}_X(D)$ is generated by its global sections if and only if D is a nef divisor (this observation appears in [9] and [10]). Also, in the terminology of [9], such a divisor D is called ℓ-semiample, where $\ell = \dim(\Delta_D)$.

Using Serre Duality, we get the following corollary of Theorem 1.1. As usual, $K = K_X$ denotes the canonical divisor of X.

Corollary 1.3. *Under the same hypotheses as Theorem* 1.1*, we have:*

(1) $H^i(X, \mathcal{O}_X(D+K)) = 0$ *for* $i \neq n - \dim(\Delta_D)$.
(2) *There is an isomorphism*

$$H^{n-\dim(\Delta_D)}(X, \mathcal{O}_X(D+K)) \simeq \bigoplus_{m \in M \cap \mathrm{int}(\Delta_D)} \mathbb{C} \cdot \chi^m$$

which is equivariant with respect to the natural $(\mathbb{C}^*)^n$ *action on each side.*

Proof. We know that K is given by the Weil divisor $-\sum_{j=1}^r D_i$. By Serre Duality,

$$H^i(X, \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(K)) \simeq H^{n-i}(X, \mathcal{O}_X(-D))^*.$$

However, given any Weil divisors E and F on X, the natural map

$$\mathcal{O}_X(E) \otimes_{\mathcal{O}_X} \mathcal{O}(F) \to \mathcal{O}_X(E+F)$$

is easily seen to be an isomorphism when E or F is Cartier. Since D is Cartier by assumption, the above duality may be written

$$H^i(X, \mathcal{O}_X(D+K)) \simeq H^{n-i}(X, \mathcal{O}_X(-D))^*.$$

By functoriality, this is compatible with the $(\mathbb{C}^*)^n$ action on everything. From here, the corollary follows immediately from Theorem 1.1. $\square$

Remark 1.4. When D is big and nef, we have $\dim(\Delta_D) = n$. Then Corollary 1.3 implies that

$$H^i(X, \mathcal{O}_X(D+K)) = 0, \quad i > 0.$$

This is the Kawamata-Viehweg vanishing theorem from [10].

2. CODIMENSION IN THE CRITICAL DEGREE

As in the introduction, fix $\alpha_0, \ldots, \alpha_n \in \mathrm{Pic}(X) \subset A_{n-1}(X)$ such that each α_i is globally generated. This implies that each α_i determines a lattice polytope Δ_i which is well-defined up to translation by an element of M.

Given polynomials $f_i \in S_{\alpha_i}$ (so that $\deg(f_i) = \alpha_i$), we obtain the homogeneous ideal $I = \langle f_0, \ldots, f_n \rangle \subset S$. We want to study the codimension of I_ρ in S_ρ, where $\rho = \sum_{i=0}^n \alpha_i - \beta_0$ is the critical degree (0.1).

Before stating our main result we need a definition.

Definition 2.1. Let Δ_i, $i \in I$, be polytopes in $\mathbb{R}^n$. Given $J \subset I$, set

$$\Delta_J = \sum_{j \in J} \Delta_j.$$

The family $\{\Delta_i, i \in I\}$ is called **essential** if for every $J \subset I$ with $|J| \leq n$, we have

$$\dim(\Delta_J) \geq |J|.$$

We will also use the standard notation

$$(2.1) \qquad\qquad l^*(\Delta) = \#(M \cap \mathrm{int}(\Delta))$$

to denote the number of lattice points of a polytope Δ which lie in the relative interior of Δ.

Here is our codimension theorem.

Theorem 2.2. *Suppose that X is a complete toric variety of dimension n. Let $\alpha_i \in \mathrm{Pic}(X)$, $0 \leq i \leq n$, be globally generated and assume that the corresponding family of polytopes $\{\Delta_i, 0 \leq i \leq n\}$ is essential. If $f_i \in S_{\alpha_i}$, $0 \leq i \leq n$, do not vanish simultaneously on X, then:*

(1) *The codimension of I_ρ in S_ρ satisfies the inequalities*

$$1 + \sum_{\dim(\Delta_\ell)=1} l^*(\Delta_\ell) \leq \dim((S/I)_\rho) \leq 1 + \sum_{k=1}^{n-1} \sum_{\dim(\Delta_J)=|J|=k} l^*(\Delta_J),$$

where in the right-most sum we always assume that $J \subset \{0, \ldots, n\}$.

(2) *The codimension of I_ρ in S_ρ is given by*

$$\dim((S/I)_\rho) = 1 + \sum_{k=1}^{n-1} \sum_{\dim(\Delta_J)=|J|=k} l^*(\Delta_J)$$

if one of the following is satisfied for every J with $1 \leq |J| \leq n-2$:

(a) $\dim(\Delta_J) \neq |J| + 1$;

(b) $\dim(\Delta_J) = |J| + 1$ *but Δ_J has no interior lattice points;*

(c) $\dim(\Delta_J) = |J| + 1$ *but $\dim(\Delta_{J \cup I}) > |J| + |I|$ for nonempty subsets $I \subset \{0, \ldots, n\}$ such that $I \cap J = \emptyset$ and $|J| + |I| < n$.*

Proof. Since f_i is a global section of $\mathcal{O}_X(\alpha_i)$, we get a Koszul complex

$$0 \to \mathcal{O}_X(-\textstyle\sum_i \alpha_i) \to \cdots \to \bigoplus_i \mathcal{O}_X(-\alpha_i) \to \mathcal{O}_X \to 0,$$

which is exact since each $\mathcal{O}_X(\alpha_i)$ is locally free and the f_i have no common zeros.

By hypothesis, each sheaf in the Koszul complex is locally free. Since the sheaf $\underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ vanishes whenever $i > 0$ and $\mathcal{E}$ or $\mathcal{F}$ is locally free, it follows that the Koszul sequence remains exact after tensoring with $\mathcal{O}(\rho) = \mathcal{O}(\alpha_0 + \cdots + \alpha_n - \beta_0)$ (which need not be locally free). This gives the exact sequence

$$0 \to \mathcal{O}_X(-\textstyle\sum_i \alpha_i) \otimes_{\mathcal{O}_X} \mathcal{O}_X(\rho) \to \cdots \to \bigoplus_i \mathcal{O}_X(-\alpha_i) \otimes_{\mathcal{O}_X} \mathcal{O}_X(\rho) \to \mathcal{O}_X(\rho) \to 0.$$

Since the α_i all come from Cartier divisors, the reasoning used in the proof of Corollary 1.3 implies that we can write this exact sequence as

$$0 \to \mathcal{O}_X(-\beta_0) \to \bigoplus_i \mathcal{O}_X(\alpha_i - \beta_0) \to \cdots \to \mathcal{O}_X(\alpha_0 + \cdots + \alpha_n - \beta_0) \to 0.$$
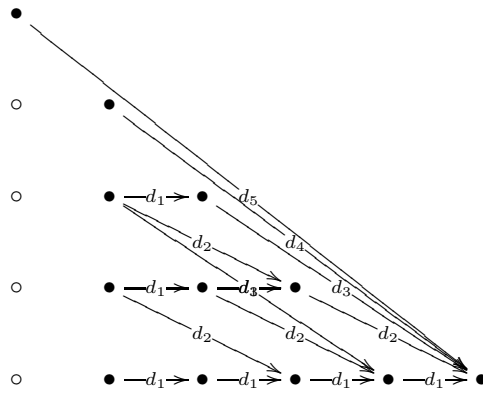
FIGURE 1. Possible nonzero differentials when $n = 4$

Using $\rho = \alpha_0 + \cdots + \alpha_n - \beta_0$, this becomes

$$(2.2) \quad 0 \to \mathcal{O}_X(-\beta_0) \to \underbrace{\bigoplus_i \mathcal{O}_X(\alpha_i - \beta_0)}_{} \to \cdots \to \underbrace{\bigoplus_k \mathcal{O}_X(\rho - \alpha_k)}_{} \to \underbrace{\mathcal{O}_X(\rho)}_{} \to 0.$$

$$\underbrace{\phantom{\mathcal{O}_X(-\beta_0)}}_{\mathcal{F}^0} \quad \underbrace{}_{\mathcal{F}^1} \quad \underbrace{}_{\mathcal{F}^n} \quad \underbrace{\phantom{\mathcal{O}_X}}_{\mathcal{F}^{n+1}}$$

We will now study the hypercohomology of the complex $\mathcal{F}^\bullet$.

Since (2.2) is exact, the hypercohomology $\mathbb{H}^*(X, \mathcal{F}^\bullet)$ vanishes identically. Hence we get a spectral sequence

$$E_1^{p,q} = H^q(X, \mathcal{F}^p) \Rightarrow 0$$

where the differential $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ is induced by the map $\mathcal{F}^p \to \mathcal{F}^{p+1}$ of the Koszul complex (2.2).

We compute $E_1^{p,q}$ as follows. When $p = 0$,

$$(2.3) \qquad \dim(E_1^{0,q}) = \dim\left(H^q(X, \mathcal{O}_X(-\beta_0))\right) = \begin{cases} 0, & q < n, \\ 1, & q = n, \end{cases}$$

where $q = n$ uses the isomorphism $H^n(X, \mathcal{O}_X(-\beta_0)) = H^n(X, \mathcal{O}_X(K)) \simeq \mathbb{C}$ given by the trace map.

When $p > 0$, we have

$$\mathcal{F}^p = \bigoplus_{|J|=p} \mathcal{O}_X(\alpha_J - \beta_0), \quad \alpha_J = \sum_{j \in J} \alpha_j.$$

Then Corollary 1.3 and (2.1) imply that

$$(2.4) \quad \dim(E_1^{p,q}) = \sum_{|J|=p} \dim\left(H^q(X, \mathcal{O}_X(\alpha_J - \beta_0))\right) = \sum_{|J|=p, \dim(\Delta_J)=n-q} l^*(\Delta_J).$$

Our assumption that $\{\Delta_i, 0 \le i \le n\}$ is essential implies that $\dim(\Delta_J) \ge |J|$ when $|J| \le n$. Combining this with $|J| = p$, $\dim(\Delta_J) = n - q$ from (2.4), we see that $n - q \ge p$ when $p \le n$ in the last summation of (2.4). Thus

$$(2.5) \qquad E_1^{p,q} = 0 \text{ when } p + q > n, \text{ except for } E_1^{n+1,0} = S_\rho.$$

When $n = 4$, (2.3) and (2.5) give the picture shown in Figure 1 of all possible nonzero differentials in the spectral sequence.

Now consider $E_r^{n+1,0}$. By (2.2), $E_1^{n,0} \to E_1^{n+1,0} \to E_1^{n+2,0}$ is

$$\bigoplus_k S_{\rho - \alpha_k} \longrightarrow S_\rho \longrightarrow 0.$$

Since this comes from the Koszul complex of the f_i, the image of the first map is I_ρ. It follows that

$$E_2^{n+1,0} = (S/I)_\rho.$$

All differentials starting from $E_2^{n+1,0}$ obviously vanish, and the only differentials which can map to this position are those on the "diagonal" $p+q = n$ as in Figure 1. Furthermore, on the diagonal, the only nonzero differentials $d_r^{n-q,q}$ are $d_{q+1}^{n-q,q}$: $E_{q+1}^{n-q,q} \to E_{q+1}^{n+1,0}$. Since the spectral sequence converges to zero, it follows easily that

$$\dim((S/I)_\rho) \le \sum_{q=1}^{n} \dim(E_1^{n-q,q}).$$

By (2.3) and (2.4), we get the upper bound of part (1) of the theorem. As for the lower bound, note that the differentials

$$d_{n+1}^{0,n} : E_{n+1}^{0,n} \to E_{q+1}^{n+1,0} \quad \text{and} \quad d_n^{1,n-1} : E_n^{1,n-1} \to E_{q+1}^{n+1,0}$$

must be injective since the spectral sequence converges to 0 and nothing can map to these positions. (In Figure 1, these correspond to the differentials d_5 and d_4.) Using (2.3) and (2.4) again, we get the lower bound of part (1).

Turning to part (2) of the theorem, suppose that every differential d_r mapping to $E_r^{p,q}$ is zero for $p+q = n$ and $q > 0$. Since the spectral sequence converges to zero, this implies that

$$\dim((S/I)_\rho) = \sum_{q=1}^{n} \dim(E_1^{n-q,q}).$$

From this, we easily get the desired formula for the codimension. Hence it suffices to prove that these differentials vanish when condition (a), (b) or (c) is satisfied by any J with $1 \le |J| \le n - 2$. A differential mapping to the diagonal $p + q = n$ originates from the "sub-diagonal" $p + q = n - 1$. Thus we need to prove that all differentials

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

vanish when $p + q = n - 1$ and $q - r + 1 > 0$ (i.e., $r \le q$).

To analyze this, first note that for arbitrary (p, q), Corollary 1.3 implies that

(2.6) $$E_1^{p,q} = \bigoplus_{|J|=p, \dim(\Delta_J)=n-q} H_J,$$

where

(2.7) $$H_J = H^q(X, \mathcal{O}_X(\alpha_J - \beta_0)) = \bigoplus_{m \in M \cap \text{int}(\Delta_J)} \mathbb{C} \cdot \chi^m.$$

In particular, when $p + q = n - 1$, the description of $E_1^{p,q}$ becomes

$$E_1^{p,q} = \bigoplus_{\dim(\Delta_J)=|J|+1=n-q} H_J.$$

The subsets $J \subset \{0, \dots, n\}$ in this direct sum satisfy

(2.8) $|J| = p = n - q - 1 \leq n - 2$ and $\dim(\Delta_J) = n - q = |J| + 1.$

Every J in the discussion which follows will satisfy (2.8).

Now consider $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ for $p + q = n - 1$ and $q \geq 1$. By (2.7) and (2.8), we see that $H_J = 0$ when J satisfies conditions (a) or (b) of part (2) of the theorem. Now suppose J satisfies condition (c). Since d_1 comes from the Koszul complex, (2.6) for $(p + 1, q)$ shows that $d_1^{p,q}$ restricted to H_J is a map

$$d_1^{p,q} : H_J \longrightarrow \bigoplus_{i \notin J, \dim(\Delta_{J \cup \{i\}}) = n - q} H_{J \cup \{i\}}.$$

By condition (c), we know that $\dim(\Delta_{J \cup \{i\}}) > |J| + 1$ for $i \notin J$. But $|J| = n - q - 1$ by (2.8), so that $\dim(\Delta_{J \cup \{i\}}) > n - q$. Comparing this to the above description of $d_1^{p,q}$ shows that $d_1^{p,q} = 0$ when $p + q = n - 1$ and $q \geq 1$, as claimed.

Next consider $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$, where $p + q = n - 1$ and $q \geq 2$. To understand this map, we will recall its definition which follows from the Snake Lemma. Let $\mathcal{U}$ be a Leray covering of X, so that the cohomology groups $H^q(X, \mathcal{F}^p)$ can be identified with the Čech cohomology of X with respect to this covering. As usual, we call δ the maps between cochain groups. Also let

$$Z^q(\mathcal{U}, \mathcal{F}^p) \subset C^q(\mathcal{U}, \mathcal{F}^p)$$

denote the corresponding Čech group and its subgroup of cocycles. Then we compute d_2 using the following diagram:

$$
\begin{array}{ccccc}
& & C^{q-1}(\mathcal{U}, \mathcal{F}^{p+1}) & \xrightarrow{d_1} & C^{q-1}(\mathcal{U}, \mathcal{F}^{p+2}) \\
& & \delta \downarrow & & \\
& Z^q(\mathcal{U}, \mathcal{F}^p) & \xrightarrow{d_1} & Z^q(\mathcal{U}, \mathcal{F}^{p+1}) & \\
& \downarrow & & \downarrow & \\
0 \to \ \ker(d_1) & \to & H^q(X, \mathcal{F}^p) & \xrightarrow{d_1} & H^q(X, \mathcal{F}^{p+1}) \\
\downarrow & & \downarrow & & \downarrow \\
E_2^{p,q} & & 0 & & 0 \\
\downarrow & & & & \\
0 & & & &
\end{array}
$$

In this diagram, $s \in E_2^{p,q}$ lifts to an element of $E_1^{p,q} = H^q(X, \mathcal{F}^p)$ represented by $s' \in Z^q(\mathcal{U}, \mathcal{F}^p)$. Then

(2.9) $d_1(s') \in Z^q(\mathcal{U}, \mathcal{F}^{p+1})$

represents $0 \in H^q(X, \mathcal{F}^{p+1})$. Thus $d_1(s')$ lifts to $Z^{q-1}(\mathcal{U}, \mathcal{F}^{p+1})$, i.e., $d_1(s') = \delta(s'')$ for some $s'' \in Z^{q-1}(\mathcal{U}, \mathcal{F}^{p+1})$. It is then easy to see that

(2.10) $d_1(s'') \in Z^{q-1}(\mathcal{U}, \mathcal{F}^{p+2}),$

so that $d_1(s'')$ represents an element of $E_1^{p+2,q-1} = H^{q-1}(X, \mathcal{F}^{p+2})$. This is killed by d_1 and represents $d_2^{p,q}(s) \in E_2^{p+2,q-1}$. It follows that d_2 is constructed by applying d_1 twice to suitable liftings.

Since $d_1^{p,q}$ vanishes, $E_2^{p,q}$ is a quotient of $E_1^{p,q}$. So we may assume that $s \in E_2^{p,q}$ comes from H_J for some J satisfying (2.8). As before, this vanishes if J satisfies conditions (a) or (b) of part (2). Now assume J satisfies condition (c). Then as above s gives $s' \in Z^q(\mathcal{U}, \mathcal{F}^p)$ where we only use the summand of $\mathcal{F}^p$ corresponding to J. Since d_1 comes from the Koszul complex, we see from (2.9) that $d_1(s')$ only

involves summands of $\mathcal{F}^{p+1}$ corresponding to $J \cup \{i\}$ for $i \notin J$. Then the same is true for s'', so that in (2.10), $d_1(s'')$ only involves summands of $\mathcal{F}^{p+1}$ corresponding to $J \cup \{i,j\}$ for $i,j \notin J$ and $i \neq j$.

This shows that $d_2^{p,q}(s) \in E_2^{p+2,q-1}$ is represented by an element of $E_1^{p+2,q-1}$ involving only summands $H_{J \cup \{i,j\}}$ for $i,j \notin J$ and $i \neq j$. Yet to actually appear in $E_1^{p+2,q-1}$, the summand must satisfy $n - (q-1) = \dim(\Delta_{J \cup \{i,j\}})$ by (2.6). This is impossible since condition (c) implies that

$$\dim(\Delta_{J \cup \{i,j\}}) > |J| + 2 = (n - q - 1) + 2 = n - (q - 1).$$

It follows that $d_2^{p,q} = 0$ when $p + q = n - 1$ and $q \geq 2$.

The argument for general r is similar. Here, $p + q = n - 1$, $q \geq r$, and $d_1^{p,q}, \ldots, d_{r-1}^{p,q}$ vanish. Then $E_r^{p,q}$ is a quotient of $E_1^{p,q}$, so we may assume that $s \in E_r^{p,q}$ comes from H_J for J as in (2.8). Since d_1 comes from the Koszul complex and d_r is obtained by applying d_1 r times to suitable liftings, we see that $d_r^{p,q}(s) \in E_r^{p+r,q-r+1}$ is represented by an element of $E_1^{p+r,q-r+1}$ involving only summands $H_{J \cup I}$ for $|I| = r$ and $I \cap J = \emptyset$. In order to appear in $E_1^{p+r,q-r+1}$, the summand must satisfy $n - (q - r + 1) = \dim(\Delta_{J \cup I})$ by (2.6). This is impossible since condition (c) implies that

$$\dim(\Delta_{J \cup I}) > |J| + |I| = (n - q - 1) + r = n - (q - r + 1).$$

It follows that $d_r^{p,q} = 0$ when $p + q = n - 1$ and $q \geq r$. This completes the proof of the theorem. $\qquad\square$

Theorem 2.2 has the following corollaries.

Corollary 2.3. *Let $\Delta_0, \ldots, \Delta_n$ be an essential family of lattice polytopes in $\mathbb{R}^n$. Then the Minkowski sum $\Delta_0 + \cdots + \Delta_n$ has at least one interior lattice point.*

Proof. If X_Δ is the toric variety determined by $\Delta = \Delta_0 + \cdots + \Delta_n$, then $\dim(S_\rho)$ is the number of interior lattice points of Δ. The lower bound given by Theorem 2.2 implies that $\dim(S_\rho) \geq \dim((S/I)_\rho) \geq 1$, and the corollary follows. $\qquad\square$

Remark 2.4. Corollary 2.3 may fail if $\Delta_0, \ldots, \Delta_n$ are not essential. For example, suppose that $\Delta_1 + \cdots + \Delta_n$ has dimension $n-1$ and Δ_0 is an interval of length one relative to an integral linear functional constant on the affine hyperplane containing $\Delta_1 + \cdots + \Delta_n$. It is easy to see that $\Delta_0 + \cdots + \Delta_n$ has no interior lattice points. We are grateful to Günter Ziegler for this observation.

Corollary 2.5. *Suppose that X is a complete toric variety of dimension n. Let $\alpha_i \in \mathrm{Pic}(X)$, $0 \leq i \leq n$, be globally generated and assume that $f_i \in S_{\alpha_i}$, $0 \leq i \leq n$, do not vanish simultaneously on X. Then:*

(1) *If the polytopes Δ_i all have dimension n, then*

$$\dim((S/I)_\rho) = 1.$$

(2) *If $n = 2$ and $\{\Delta_0, \Delta_1, \Delta_2\}$ is essential, then*

$$\dim((S/I)_\rho) = 1 + \sum_{\dim(\Delta_i)=1} l^*(\Delta_i).$$

Proof. This follows immediately from part (2) of Theorem 2.2. $\qquad\square$

Remark 2.6. In part (1) of Corollary 2.5, the hypothesis that α_i is globally generated and Δ_i has dimension n is equivalent to assuming that α_i is the class of a big and nef divisor on X.

When $n = 3$, the conditions of part (2) of Theorem 2.2 are equivalent to the assumption that if $\dim(\Delta_i) = 2$, then either Δ_i has no interior lattice points or $\dim(\Delta_i + \Delta_j) = 3$ for all $j \neq i$. It follows that Theorem 2.2 computes the codimension in the critical degree for many but not all cases of essential supports when X has dimension 3.

There is one case where further general results are possible.

Theorem 2.7. *Suppose that X is a complete toric variety of dimension $n \geq 3$. Let $\alpha_i \in \operatorname{Pic}(X)$, $0 \leq i \leq n$, be globally generated and assume that the corresponding family of polytopes $\{\Delta_i, 0 \leq i \leq n\}$ is essential and satisfies*

$$(2.11) \qquad \dim(\Delta_i) \in \{1, n-1, n\}, \quad i = 0, \ldots, n.$$

Let $f_i \in S_{\alpha_i}$, $0 \leq i \leq n$, and assume the following two conditions:

1. *The f_i do not vanish simultaneously on X.*
2. *For every $J \subset \{0, \ldots, n\}$ with $\dim(\Delta_J) = |J| = n-1$ and $\dim(\Delta_i) = n-1$ for at least one $i \in J$, the equations on X_{Δ_J} given by $f_j = 0$ for $j \in J$ have only finitely many solutions, all of which lie in the torus of X_{Δ_J}.*

Then the codimension of I_ρ in S_ρ is given by

$$(2.12) \qquad \dim((S/I)_\rho) = 1 + \sum_{\dim(\Delta_J) = |J| < n} \left(\sum_{\mathcal{J} \subset J} (-1)^{|J| - |\mathcal{J}|} l^*_{|J|} \left(\sum_{\ell \in \mathcal{J}} P_\ell \right) \right),$$

where, for a lattice polytope Δ and integer $k \geq 0$,

$$l^*_k(\Delta) = \begin{cases} 0, & \dim(\Delta) \neq k, \\ l^*(\Delta), & \dim(\Delta) = k. \end{cases}$$

The condition (2.11) is still rather restrictive, yet examples with $n = 4$ show that the formula (2.12) can fail when we omit (2.11). Furthermore, when we do assume (2.11), condition (2) on the f_i is probably unnecessary, yet we cannot figure out how to prove the theorem without using this hypothesis. For these reasons, we omit the proof of Theorem 2.7.

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