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HOCHSCHILD COHOMOLOGY OF FROBENIUS ALGEBRAS

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Abstract. Let k be a field, A a finite-dimensional Frobenius k-algebra and $\rho\colon A \to A$, the Nakayama automorphism of A with respect to a Frobenius homomorphism $\varphi\colon A \to k$. Assume that ρ has finite order m and that k has a primitive m-th root of unity w. Consider the decomposition $A = A_0 \oplus \cdots \oplus A_{m-1}$ of A, obtained by defining $A_i = \{a \in A : \rho(a) = w^i a\}$, and the decomposition $\mathsf{HH}^*(A) = \bigoplus_{i=0}^{m-1} \mathsf{HH}^*_i(A)$ of the Hochschild cohomology of A, obtained from the decomposition of A. In this paper we prove that $\mathsf{HH}^*(A) = \mathsf{HH}^*_0(A)$ and that if the decomposition of A is strongly $\mathbb{Z}/m\mathbb{Z}$-graded, then $\mathbb{Z}/m\mathbb{Z}$ acts on $\mathsf{HH}^*(A_0)$ and $\mathsf{HH}^*(A) = \mathsf{HH}^*_0(A) = \mathsf{HH}^*(A_0)^{\mathbb{Z}/m\mathbb{Z}}$.

1. Introduction

Let k be a field, A a finite-dimensional k-algebra and $DA = \mathsf{Hom}_k(A, k)$ be endowed with the usual A-bimodule structure. Recall that A is said to be a Frobenius algebra if there exists a linear form $\varphi\colon A \to k$, such that the map $A \to DA$, defined by $x \mapsto x\varphi$ is a left A-module isomorphism. This linear form $\varphi\colon A \to k$ is called a Frobenius homomorphism. It is well known that this is equivalent to saying that the map $x \mapsto \varphi x$, from A to DA, is an isomorphism of right A-modules. From this it follows easily that there exists an automorphism ρ of A, called the Nakayama automorphism of A with respect to φ, such that $x\varphi = \varphi\rho(x)$, for all $x \in A$. It is easy to check that a linear form $\widetilde{\varphi}\colon A \to k$ is another Frobenius homomorphism if and only if there exists $x \in A$ invertible, such that $\widetilde{\varphi} = x\varphi$. It is also easy to check that the Nakayama automorphism of A with respect to $\widetilde{\varphi}$ is the map given by $a \mapsto \rho(x)^{-1}\rho(a)\rho(x)$.

Let A be a Frobenius k-algebra, $\varphi\colon A \to k$ a Frobenius homomorphism and $\rho\colon A \to A$ the Nakayama automorphism of A with respect to φ.

Definition 1.1. We say that ρ has order $m \in \mathbb{N}$ and we write $\mathsf{ord}_\rho = m$ if $\rho^m = id_A$ and $\rho^r \neq id_A$, for all $r < m$.

Assume that ρ has finite order and that k has a primitive ord_ρ-th root of unity w. Since the polynomial $X^{\mathsf{ord}_\rho} - 1$ has distinct roots w^i $(0 \le i < \mathsf{ord}_\rho)$, the algebra A becomes a $\frac{\mathbb{Z}}{\mathsf{ord}_\rho \mathbb{Z}}$-graded algebra

$$A = A_0 \oplus \cdots \oplus A_{\mathsf{ord}_\rho - 1}, \quad \text{where } A_i = \{a \in A : \rho(a) = w^i a\}.$$

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Let $(\mathsf{Hom}_k(A^{\otimes *}, A), b^*)$ be the cochain Hochschild complex of A with coefficients in A. For each $0 \le i < \mathsf{ord}_\rho$, we let $(\mathsf{Hom}_k(A^{\otimes *}, A)_i, b^*)$ denote the subcomplex of $(\mathsf{Hom}_k(A^{\otimes *}, A), b^*)$, defined by

$$\mathsf{Hom}_k(A^{\otimes n}, A)_i = \bigoplus_{\widetilde{B}_{i,n}} \mathsf{Hom}_k(A_{u_1} \otimes \cdots \otimes A_{u_n}, A_v),$$

where $\widetilde{B}_{i,n} = \{(u_1, \dots, u_n, v) \text{ such that } v - u_1 - \cdots - u_n \equiv i \pmod{\mathsf{ord}_\rho}\}$. The cochain Hochschild complex $(\mathsf{Hom}_k(A^{\otimes^*}, A), b^*)$ decomposes as the direct sum

$$(\mathsf{Hom}_k(A^{\otimes^*}, A), b^*) = \bigoplus_{i=0}^{\mathsf{ord}_\rho - 1} (\mathsf{Hom}_k(A^{\otimes^*}, A)_i, b^*).$$

Thus, the Hochschild cohomology $\mathsf{HH}^n(A)$, of A with coefficients in A, decomposes as the direct sum

$$\mathsf{HH}^n(A) = \bigoplus_{i=0}^{\mathsf{ord}_\rho - 1} \mathsf{HH}^n_i(A),$$

where $\mathsf{HH}^n_i(A) = H^n(\mathsf{Hom}_k(A^{\otimes^*}, A)_i, b^*)$.

The aim of this paper is to prove the following results:

Theorem 1.2. *Let A be a Frobenius k-algebra, $\varphi\colon A \to k$ a Frobenius homomorphism and $\rho\colon A \to A$ the Nakayama automorphism of A with respect to φ. If ρ has finite order and k has a primitive ord_ρ-th root of unity w, then*

$$\mathsf{HH}^n(A) = \mathsf{HH}^n_0(A), \quad \textit{for all } n \ge 0.$$

Recall that $A = A_0 \oplus \cdots \oplus A_{\mathsf{ord}_\rho - 1}$ is said to be strongly $\mathbb{Z}/\mathsf{ord}_\rho\mathbb{Z}$-graded if $A_i A_j = A_{i+j}$, for all $i, j \in \{0, \dots, \mathsf{ord}_\rho - 1\}$, where $i + j$ denotes the sum of i and j in $\mathbb{Z}/\mathsf{ord}_\rho\mathbb{Z}$.

Theorem 1.3. *Let A be a Frobenius k-algebra, $\varphi\colon A \to k$ a Frobenius homomorphism and $\rho\colon A \to A$ the Nakayama automorphism of A with respect to φ. If ρ has finite order, k has a primitive ord_ρ-th root of unity w and $A = A_0 \oplus \cdots \oplus A_{\mathsf{ord}_\rho - 1}$ is strongly $\mathbb{Z}/\mathsf{ord}_\rho\mathbb{Z}$-graded, then*

$$\mathsf{HH}^n(A) = \mathsf{HH}^n(A_0)^{\mathbb{Z}/\mathsf{ord}_\rho\mathbb{Z}}, \quad \textit{for all } n \ge 0.$$

Corollary 1.4. *Assume that the hypotheses of Theorem 1.3 are verified. If the Hochschild cohomology $\mathsf{HH}^2(A_0) = 0$, then A is rigid.*

Remark 1.5. As is well known, every finite-dimensional Hopf algebra H is Frobenius, and being a Frobenius homomorphism, any right integral $\varphi \in H^* \setminus \{0\}$. Moreover, by Proposition 3.6 of [S], the composition inverse of the Nakayama map ρ with respect to φ is given by

$$\rho^{-1}(h) = \alpha(h_{(1)})\overline{S}^2(h_{(2)}),$$

where $\alpha \in H^*$ is the modular element of H^* and $\overline{S}$ is the composition inverse of S (note that the automorphism of Nakayama considered in [S] is the composition inverse of the one considered by us). Using this formula and that $\alpha \circ S^2 = \alpha$, it is easy to check that $\rho(h) = \alpha(S(h_{(1)}))S^2(h_{(2)})$ and, more generally, that

$$\rho^l(h) = \alpha^{*l}(S(h_{(1)}))S^{2l}(h_{(2)}),$$

where α^{*l} denotes the l-fold convolution product of α. Since α has finite order with respect to the convolution product and, by the Radford formula for S^4 (see Theorem 3.8 of [S]), the antipode S has finite order with respect to the composition, we have that ρ has finite order. So, the above theorems apply to finite-dimensional Hopf algebras.

We think that the decomposition of H associated with ρ can be useful for studying the structure of finite-dimensional Hopf algebras. In this paper we exploit it in a cohomological level. Recently another decomposition of H has been considered, similar to this one, but distinct, namely, the one associated to S^2 (see [R-S]).

Example 1.6. Let k a field and N a natural number. Assume that k has a primitive N-th root of unity w. Let H be the Taft algebra of order N. That is, H is the algebra generated over k by two elements g and x subject to the relations $g^N = 1$, $x^N = 0$ and $xg = wgx$. The Taft algebra H is a Hopf algebra with comultiplication Δ, counity ϵ and antipode S given by

$$\begin{aligned} \Delta(g) &= g \otimes g, & \Delta(x) &= 1 \otimes x + x \otimes g, \\ \epsilon(g) &= 1, & \epsilon(x) &= 0, \\ S(g) &= g^{-1}, & S(x) &= -xg^{-1}. \end{aligned}$$

Using that $t = \sum_{j=0}^{N-1} w^j g^j x^{N-1}$ is a right integral of H, it is easy to see that the modular element $\alpha \in H^*$ verifies $\alpha(g) = w^{-1}$ and $\alpha(x) = 0$. By the remark above, the Nakayama map $\rho\colon H \to H$ is given by $\rho(g) = wg$ and $\rho(x) = w^{-1}x$. Hence, $H = H_0 \oplus \cdots \oplus H_{N-1}$, where

$$\begin{aligned} H_i &= \{a \in H : \rho(a) = w^{-i}a\} \\ &= \langle x^i, x^{i+1}g, \ldots, x^{N-1}g^{N-i-1}, g^{N-i}, xg^{N-i+1}, \ldots, x^{i-1}g^{N-1} \rangle. \end{aligned}$$

Let $C_N = \{1, t, \ldots, t^{N-1}\}$ be the cyclic group of order N. It is easy to see that C_N acts on H_0 via $t \cdot x^i g^i = w^i x^i g^i$ and that H is isomorphic to the skew product of $H_0 \# C_N$. By Theorem 1.3,

$$\mathsf{HH}^n(H) = \mathsf{HH}^n(H_0)^{C_N} \qquad \text{for all } n \geq 0,$$

where the action of C_N on $\mathsf{HH}^n(H_0)$ is induced by the one of C_N on $\mathsf{Hom}_k(H_0^{\otimes n}, H_0)$, given by

$$t \cdot \varphi(x^{i_1} g^{i_1} \otimes \cdots \otimes x^{i_n} g^{i_n}) = g^{N-1} \varphi(t \cdot x^{i_1} g^{i_1} \otimes \cdots \otimes t \cdot x^{i_q} g^{i_q}) g.$$

2. Proof of Theorems 1.2 and 1.3

Let k be a field, A a k-algebra and V a k-module. To begin, we fix some notation:

(1) As in the introduction, we let DA denote $\mathsf{Hom}_k(A, k)$, endowed with the usual A-bimodule structure.
(2) We let $V^{\otimes n}$ denote the n-fold tensor product $V \otimes \cdots \otimes V$.

(3) Given $x \in A \cup DA$, we write

$$\pi_A(x) = \begin{cases} x & \text{if } x \in A, \\ 0 & \text{if } x \in DA, \end{cases} \qquad \text{and} \qquad \pi_{DA}(x) = \begin{cases} x & \text{if } x \in DA, \\ 0 & \text{if } x \in A. \end{cases}$$

(4) For $n \geq 1$, we let $B^n \subseteq (A \oplus DA)^{\otimes n}$ denote the vector subspace spanned by n-tensors $x_1 \otimes \cdots \otimes x_n$ such that exactly 1 of the x_i's belongs to DA, while the other x_i's belong to A.

(5) Given $i < j$ and $x_i, x_{i+1}, \dots, x_j \in A \cup DA$, we write $\mathbf{x}_{i,j} = x_i \otimes \cdots \otimes x_j$.

(6) For each map $f\colon X \to Y$ and each element $x \in X$, we let $\langle f, x\rangle$ denote the evaluation of f in x.

2.1. **The complex $X^{*,*}(A)$.** For each k-algebra A, we consider the double complex

$$X^{*,*}(A) := \quad \begin{array}{ccc} \vdots & & \vdots \\ \uparrow b^{0,4} & & \uparrow b^{1,4} \\ \mathsf{Hom}_k(A^{\otimes 3}, A) & \xrightarrow{\delta^{1,3}} & \mathsf{Hom}_k(B^4, DA) \\ \uparrow b^{0,3} & & \uparrow b^{1,3} \\ \mathsf{Hom}_k(A^{\otimes 2}, A) & \xrightarrow{\delta^{1,2}} & \mathsf{Hom}_k(B^3, DA) \\ \uparrow b^{0,2} & & \uparrow b^{1,2} \\ \mathsf{Hom}_k(A, A) & \xrightarrow{\delta^{1,1}} & \mathsf{Hom}_k(B^2, DA) \\ \uparrow b^{0,1} & & \uparrow b^{1,1} \\ \mathsf{Hom}_k(k, A) & \xrightarrow{\delta^{1,0}} & \mathsf{Hom}_k(B^1, DA), \end{array}$$

where

$$\begin{aligned} \langle\langle b^{0,n+1}, f\rangle, \mathbf{x}_{1,n+1}\rangle &= x_1\langle f, \mathbf{x}_{2,n+1}\rangle + \sum_{i=1}^{n}(-1)^i\langle f, \mathbf{x}_{1,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+1,n+1}\rangle \\ &\quad + (-1)^{n+1}\langle f, \mathbf{x}_{1,n}\rangle x_{n+1}, \\ \langle\langle b^{1,n}, g\rangle, \mathbf{y}_{1,n+1}\rangle &= \langle \pi_A, y_1\rangle\langle g, \mathbf{y}_{2,n+1}\rangle + \sum_{i=1}^{n}(-1)^i\langle g, \mathbf{y}_{1,i-1} \otimes y_i y_{i+1} \otimes \mathbf{y}_{i+1,n+1}\rangle \\ &\quad + (-1)^{n+1}\langle g, \mathbf{y}_{1,n}\rangle\langle \pi_A, y_{n+1}\rangle, \\ \langle\langle \delta^{1,n}, f\rangle, \mathbf{y}_{1,n+1}\rangle &= \langle \pi_{DA}, y_1\rangle\langle f, \mathbf{y}_{2,n+1}\rangle + (-1)^{n+1}\langle f, \mathbf{y}_{1,n}\rangle\langle \pi_{DA}, y_{n+1}\rangle, \end{aligned}$$

for $f \in \mathsf{Hom}_k(A^{\otimes n}, A)$, $g \in \mathsf{Hom}_k(B^n, DA)$, $\mathbf{x}_{1,n+1} = x_1 \otimes \cdots \otimes x_{n+1} \in A^{\otimes n+1}$ and $\mathbf{y}_{1,n+1} = y_1 \otimes \cdots \otimes y_{n+1} \in B^n$.

Proposition 2.1. *Let $X^*(A)$ be the total complex of $X^{*,*}(A)$. It is true that*

$$H^n(X^*(A)) = \begin{cases} H^0(X^{0,*}(A)) & \text{if } n = 0, \\ H^n(X^{0,*}(A)) \oplus H^{n-1}(X^{0,*}(A)) & \text{if } n \geq 1. \end{cases}$$

Proof. Let

$$\delta^{1,*}\colon (\mathsf{Hom}_k(A^{\otimes *}, A), -b^{0,*+1}) \to (\mathsf{Hom}_k(B^{*+1}, DA), b^{1,*+1})$$

be the map defined by

$$\langle\langle\delta^{1,n}, f\rangle, \mathbf{x}_{1,n+1}\rangle = \langle\pi_{DA}, x_1\rangle\langle f, \mathbf{x}_{2,n+1}\rangle + (-1)^{n+1}\langle f, \mathbf{x}_{1,n}\rangle\langle\pi_{DA}, x_{n+1}\rangle.$$

Since $X^*(A)$ is the mapping cone of $\delta^{1,*}$, in order to obtain the result it suffices to check that $\delta^{1,*}$ is null homotopic. Let $\sigma_*\colon \mathsf{Hom}_k(A^{\otimes *}, A) \to \mathsf{Hom}_k(B^*, DA)$ be the family of maps defined by

$$\langle\langle\langle\sigma_n, f\rangle, \mathbf{x}_{1,n}\rangle, a\rangle = (-1)^{jn+1}\langle x_j, \langle f, \mathbf{x}_{j+1,n} \otimes a \otimes \mathbf{x}_{1,j-1}\rangle\rangle \quad \text{if } x_j \in DA.$$

We assert that σ_* is a homotopy from $\delta^{1,*}$ to 0. By definition,

$$\begin{aligned}\langle\langle b^{1,n}, \langle\sigma_n, f\rangle\rangle, \mathbf{x}_{1,n+1}\rangle &= \langle\pi_A, x_1\rangle\langle\langle\sigma_n, f\rangle, \mathbf{x}_{2,n+1}\rangle \\ &\quad + \sum_{i=1}^{n}(-1)^i\langle\langle\sigma_n, f\rangle, \mathbf{x}_{1,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+1}\rangle \\ &\quad + (-1)^{n+1}\langle\langle\sigma_n, f\rangle, \mathbf{x}_{1,n}\rangle\langle\pi_A, x_{n+1}\rangle.\end{aligned}$$

Hence, if $x_1 \in DA$, then

$$\begin{aligned}\langle\langle\langle b^{1,n}, \langle\sigma_n, f\rangle\rangle, \mathbf{x}_{1,n+1}\rangle, x_{n+2}\rangle &= (-1)^{n+2}\langle x_1, x_2\langle f, \mathbf{x}_{3,n+2}\rangle\rangle \\ &\quad - \sum_{i=2}^{n+1}(-1)^{n+i}\langle x_1, \langle f, \mathbf{x}_{2,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+2}\rangle\rangle;\end{aligned}$$

if $x_j \in DA$ for $1 < j \leq n$, then

$$\begin{aligned}\langle\langle\langle b^{1,n}, \langle\sigma_n, f\rangle\rangle, \mathbf{x}_{1,n+1}\rangle, x_0\rangle &= (-1)^{(j-1)n+j}\langle x_j, \langle f, \mathbf{x}_{j+1,n+1} \otimes \mathbf{x}_{0,j-2}\rangle x_{j-1}\rangle \\ &\quad - \sum_{i=0}^{j-2}(-1)^{(j-1)n+i}\langle x_j, \langle f, \mathbf{x}_{j+1,n+1} \otimes \mathbf{x}_{0,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,j-1}\rangle\rangle \\ &\quad - (-1)^{jn+j}\langle x_j, x_{j+1}\langle f, \mathbf{x}_{j+2,n+1} \otimes \mathbf{x}_{0,j-1}\rangle\rangle \\ &\quad - \sum_{i=j+1}^{n}(-1)^{jn+i}\langle x_j, \langle f, \mathbf{x}_{j+1,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+1} \otimes \mathbf{x}_{0,j-1}\rangle\rangle \\ &\quad + (-1)^{jn+n}\langle x_j, \langle f, \mathbf{x}_{j+1,n} \otimes x_{n+1}x_0 \otimes \mathbf{x}_{1,j-1}\rangle\rangle;\end{aligned}$$

and if $x_{n+1} \in DA$, then

$$\begin{aligned}\langle\langle\langle b^{1,n}, \langle\sigma_n, f\rangle\rangle, \mathbf{x}_{1,n+1}\rangle, x_0\rangle &= \sum_{i=0}^{n-1}(-1)^{n+i+1}\langle x_{n+1}, \langle f, \mathbf{x}_{0,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n}\rangle\rangle \\ &\quad - \langle x_{n+1}, \langle f, \mathbf{x}_{0,n-1}\rangle x_n\rangle.\end{aligned}$$

On the other hand, if $x_1 \in DA$, then

$$\begin{aligned}\langle\langle\langle\sigma_{n+1}, -\langle b^{0,n+1}, f\rangle\rangle, \mathbf{x}_{1,n+1}\rangle, x_{n+2}\rangle &= (-1)^{n+1}\mathbf{x}_1\langle\langle b^{0,n+1}, f\rangle, \mathbf{x}_{2,n+2}\rangle \\ &= (-1)^{n+1}\langle x_1, x_2\langle f, \mathbf{x}_{3,n+2}\rangle\rangle + \langle x_1, \langle f, \mathbf{x}_{2,n+1}\rangle x_{n+2}\rangle \\ &\quad + \sum_{i=2}^{n+1}(-1)^{n+i}\langle x_1, \langle f, \mathbf{x}_{2,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+2}\rangle\rangle;\end{aligned}$$

if $x_j \in DA$ for $1 < j \le n$, then

$$\begin{aligned}
&\langle\langle\langle \sigma_{n+1}, -\langle b^{0,n+1}, f\rangle\rangle, \mathbf{x}_{1,n+1}\rangle, x_0\rangle \\
&\qquad = (-1)^{j(n+1)}\langle x_j, \langle\langle b^{0,n+1}, f\rangle, \mathbf{x}_{j+1,n+1} \otimes \mathbf{x}_{0,j-1}\rangle\rangle \\
&\qquad = (-1)^{j(n+1)}\langle x_j, x_{j+1}\langle f, \mathbf{x}_{j+2,n+1} \otimes \mathbf{x}_{0,j-1}\rangle\rangle \\
&\qquad + \sum_{i=j+1}^{n} (-1)^{j(n+1)+i-j}\langle x_j, \langle f, \mathbf{x}_{j+1,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+1} \otimes \mathbf{x}_{0,j-1}\rangle\rangle \\
&\qquad + (-1)^{j(n+1)+n-j+1}\langle x_j, \langle f, \mathbf{x}_{j+1,n} \otimes x_{n+1}x_0 \otimes \mathbf{x}_{1,j-1}\rangle\rangle \\
&\qquad + \sum_{i=0}^{j-2}(-1)^{j(n+1)+i+n-j}\langle x_j, \langle f, \mathbf{x}_{j+1,n+1} \otimes \mathbf{x}_{0,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,j-1}\rangle\rangle \\
&\qquad + (-1)^{j(n+1)+n+1}\langle x_j, \langle f, \mathbf{x}_{j+1,n+1} \otimes \mathbf{x}_{0,j-2}\rangle x_{j-1}\rangle;
\end{aligned}$$

and if $x_{n+1} \in DA$, then

$$\begin{aligned}
\langle\langle\langle \sigma_{n+1}, -\langle b^{0,n+1}, f\rangle\rangle, \mathbf{x}_{1,n+1}\rangle, x_0\rangle &= (-1)^{n+1}\langle x_{n+1}, \langle\langle b^{0,n+1}, f\rangle, \mathbf{x}_{0,n}\rangle\rangle \\
&= (-1)^{n+1}\langle x_{n+1}, x_0\langle f, \mathbf{x}_{1,n}\rangle\rangle + \langle x_{n+1}, \langle f, \mathbf{x}_{0,n-1}\rangle x_n\rangle \\
&+ \sum_{i=0}^{n-1}(-1)^{n+i}\langle x_{n+1}, \langle f, \mathbf{x}_{0,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n}\rangle\rangle.
\end{aligned}$$

The assertion follows immediately from these equalities. □

2.2. **The complex $Y^{*,*}(A)$.** From now on we fix a Frobenius algebra A, a Frobenius homomorphism $\varphi\colon A \to k$ of A and we let ρ denote the Nakayama automorphism of A with respect to φ. Let A_ρ be A, endowed with the A-bimodule structure given by $a \cdot x \cdot b := \rho(a)xb$. Let $\Theta\colon DA \to A_\rho$ be the A-bimodule isomorphism given by $\Theta(\varphi x) = x$, and let

$$A_\rho \xleftarrow{\mu} A \otimes A_\rho \xleftarrow{b'_1} A^{\otimes 2} \otimes A_\rho \xleftarrow{b'_2} A^{\otimes 3} \otimes A_\rho \xleftarrow{b'_3} A^{\otimes 4} \otimes A_\rho \xleftarrow{b'_4} \cdots$$

be the bar resolution of A_ρ.

Proposition 2.2. *The following assertions hold:*

(1) *The complex*

$$DA \xleftarrow{\mu'} A \otimes B^1 \otimes A \xleftarrow{b''_1} A \otimes B^2 \otimes A \xleftarrow{b''_2} A \otimes B^3 \otimes A \xleftarrow{b''_3} \cdots,$$

where $\langle \mu', x_0 \otimes x_1 \otimes x_2\rangle = x_0x_1x_2$ *and*

$$\begin{aligned}
\langle b''_n, \mathbf{x}_{0,n+2}\rangle = x_0\langle \pi_A, x_1\rangle \otimes \mathbf{x}_{2,n+2} &+ \sum_{i=1}^{n}(-1)^i \mathbf{x}_{0,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+2} \\
&+ (-1)^{n+1}\mathbf{x}_{0,n} \otimes \langle \pi_A, x_{n+1}\rangle x_{n+2},
\end{aligned}$$

is a projective resolution of DA.

(2) *There is a chain map* $\Psi'_*\colon (A^{\otimes *+1} \otimes A_\rho, b'_*) \to (A \otimes B^{*+1} \otimes A, b''_*)$, *given by*

$$\langle \Psi'_n, \mathbf{x}_{0,n+1}\rangle = \sum_{i=0}^{n}(-1)^{i+n}\mathbf{x}_{0,i} \otimes \varphi \otimes \langle \rho, x_{i+1}\rangle \otimes \cdots \otimes \langle \rho, x_n\rangle \otimes x_{n+1}.$$

(3) $\Theta \circ \mu' \circ \Psi'_0 = \mu$.

Proof. Items (2) and (3) follow by a direct computation and item (1) is well known. For instance, the family of maps

$$\sigma_0\colon DA \to A\otimes B^1\otimes A \quad \text{and} \quad \sigma_n\colon A\otimes B^n\otimes A \to A\otimes B^{n+1}\otimes A \ \ (n\geq 1),$$

given by

$$\langle\sigma_0, x\rangle = 1\otimes x\otimes 1,$$
$$\langle\sigma_{n+1}, \mathbf{x}_{0,n+1}\rangle = \begin{cases} 1\otimes \mathbf{x}_{0,n+1} + (-1)^{n+1}\otimes x_0x_1\otimes \mathbf{x}_{2,n+1}\otimes 1 & \text{if } x_1\in DA,\\ 1\otimes \mathbf{x}_{0,n+1} & \text{if } x_1\notin DA,\end{cases}$$

where $\mathbf{x}_{0,n+1} = x_0\otimes\cdots\otimes x_{n+1}\in A\otimes B^n\otimes A$, is a contracting homotopy of the complex of item (1) as a k-module complex. □

Let $Y^{*,*}(A)$ be the double complex

$$Y^{*,*}(A) := \begin{array}{ccc} \vdots & & \vdots \\ \uparrow{\scriptstyle \widetilde{b}^{0,4}} & & \uparrow{\scriptstyle \widetilde{b}^{1,4}} \\ \mathsf{Hom}_k(A^{\otimes 3},A) & \xrightarrow{\widetilde{\delta}^{1,3}} & \mathsf{Hom}_k(A^{\otimes 3},A) \\ \uparrow{\scriptstyle \widetilde{b}^{0,3}} & & \uparrow{\scriptstyle \widetilde{b}^{1,3}} \\ \mathsf{Hom}_k(A^{\otimes 2},A) & \xrightarrow{\widetilde{\delta}^{1,2}} & \mathsf{Hom}_k(A^{\otimes 2},A) \\ \uparrow{\scriptstyle \widetilde{b}^{0,2}} & & \uparrow{\scriptstyle \widetilde{b}^{1,2}} \\ \mathsf{Hom}_k(A,A) & \xrightarrow{\widetilde{\delta}^{1,1}} & \mathsf{Hom}_k(A,A) \\ \uparrow{\scriptstyle \widetilde{b}^{0,1}} & & \uparrow{\scriptstyle \widetilde{b}^{1,1}} \\ \mathsf{Hom}_k(k,A) & \xrightarrow{\widetilde{\delta}^{1,0}} & \mathsf{Hom}_k(k,A), \end{array}$$

with boundary maps

$$\langle\langle \widetilde{b}^{u,n}, f\rangle, \mathbf{x}_{1,n}\rangle = x_1 f(\mathbf{x}_{2,n}) + \sum_{i=1}^{n-1}(-1)^i\langle f, \mathbf{x}_{1,i-1}\otimes x_ix_{i+1}\otimes \mathbf{x}_{i+2,n}\rangle$$
$$+ (-1)^n\langle f, \mathbf{x}_{1,n-1}\rangle x_n,$$
$$\langle\langle \widetilde{\delta}^{1,n-1}, f\rangle, \mathbf{x}_{1,n-1}\rangle = (-1)^n\langle f, \mathbf{x}_{1,n-1}\rangle$$
$$+ (-1)^{n-1}\langle \rho^{-1}, \langle f, \langle\rho, x_1\rangle\otimes\cdots\otimes\langle\rho, x_{n-1}\rangle\rangle\rangle,$$

where $u = 0,1$, $f\in \mathsf{Hom}_k(A^{\otimes n-1}, A)$ and $\mathbf{x}_{1,n} = x_1\otimes\cdots\otimes x_n\in A^{\otimes n}$.

Proposition 2.3. *The double complexes $X^{*,*}(A)$ and $Y^{*,*}(A)$ are quasi-isomorphic.*

Proof. It is immediate that $X^{1,*}(A)\simeq \mathsf{Hom}_{A^e}((A\otimes B^{*+1}\otimes A, b''_*), DA)$. Moreover, by Proposition 2.2, the map $\Psi^* := \mathsf{Hom}_{A^e}(\Psi'_*, DA)$ is a quasi-isomorphism from $\mathsf{Hom}_{A^e}((A\otimes B^{*+1}\otimes A, b''_*), DA)$ to $\mathsf{Hom}_{A^e}\big((A^{\otimes *+1}\otimes A_\rho, b'_*), DA\big)$. On the other hand, the family of bijective maps

$$\Upsilon^n : Y^{1,n}(A)\to \mathsf{Hom}_{A^e}(A^{\otimes n+1}\otimes A_\rho, DA) \qquad (n\geq 0),$$

defined by $\langle\langle \Upsilon^n, f\rangle, \mathbf{x}_{0,n+1}\rangle = x_0\langle f, \mathbf{x}_{1,n}\rangle\varphi x_{n+1}$, is an isomorphism of complexes from $Y^{1,*}(A)$ to $\mathsf{Hom}_{A^e}\big((A^{\otimes *+1}\otimes A_\rho, b'_*), DA\big)$. In fact, we have

$$\begin{aligned}
\langle\langle \Upsilon^{n+1}, \langle \widetilde{b}^{1,n+1}, f\rangle\rangle, \mathbf{x}_{0,n+2}\rangle &= x_0\langle\langle \widetilde{b}^{1,n+1}, f\rangle, \mathbf{x}_{1,n+1}\rangle\varphi x_{n+2}\\
&= x_0x_1\langle f, \mathbf{x}_{2,n+1}\rangle\varphi x_{n+2}\\
&+ \sum_{i=1}^{n}(-1)^i x_0\langle f, \mathbf{x}_{1,i-1}\otimes x_ix_{i+1}\otimes \mathbf{x}_{i+2,n+1}\rangle\varphi x_{n+2}\\
&+ (-1)^{n+1}x_0\langle f, \mathbf{x}_{1,n}\rangle x_{n+1}\varphi x_{n+2}\\
&= x_0x_1\langle f, \mathbf{x}_{2,n+1}\rangle\varphi x_{n+2}\\
&+ \sum_{i=1}^{n}(-1)^i x_0\langle f, \mathbf{x}_{1,i-1}\otimes x_ix_{i+1}\otimes \mathbf{x}_{i+2,n+1}\rangle\varphi x_{n+2}\\
&+ (-1)^{n+1}x_0\langle f, \mathbf{x}_{1,n}\rangle\varphi\langle \rho, x_{n+1}\rangle x_{n+2}\\
&= \sum_{i=0}^{n}(-1)^i\langle\langle \Upsilon^n, f\rangle, \mathbf{x}_{0,i-1}\otimes x_ix_{i+1}\otimes \mathbf{x}_{i+2,n+2}\rangle\\
&+ (-1)^{n+1}\langle\langle \Upsilon^n, f\rangle, \mathbf{x}_{0,n}\otimes \langle \rho, x_{n+1}\rangle x_{n+2}\rangle\\
&= \langle\langle \Upsilon^n, f\rangle, \langle b'_{n+1}, \mathbf{x}_{0,n+2}\rangle\rangle.
\end{aligned}$$

Hence, to finish the proof it suffices to check that $\Upsilon^*\circ\widetilde{\delta}^{1,*} = \Psi^*\circ\delta^{1,*}$. But,

$$\begin{aligned}
&\langle\langle \Psi^n, \langle \delta^{1,n}, f\rangle\rangle, \mathbf{x}_{0,n+1}\rangle\\
&= \sum_{i=0}^{n}(-1)^{i+n}x_0\langle\langle \delta^{1,n}, f\rangle, \mathbf{x}_{1,i}\otimes\varphi\otimes\langle \rho, x_{i+1}\rangle\otimes\cdots\otimes\langle \rho, x_n\rangle\rangle x_{n+1}\\
&= (-1)^n x_0\varphi\langle f, \langle \rho, x_1\rangle\otimes\cdots\otimes\langle \rho, x_n\rangle\rangle x_{n+1} + (-1)^{n+1}x_0\langle f, \mathbf{x}_{1,n}\rangle\varphi x_{n+1}\\
&= (-1)^n x_0\langle \rho^{-1}, \langle f, \langle \rho, x_1\rangle\otimes\cdots\otimes\langle \rho, x_n\rangle\rangle\rangle\varphi x_{n+1} + (-1)^{n+1}x_0\langle f, \mathbf{x}_{1,n}\rangle\varphi x_{n+1}\\
&= x_0\langle\langle \widetilde{\delta}^{1,n}, f\rangle, \mathbf{x}_{1,n}\rangle\varphi x_{n+1}\\
&= \varphi\langle\langle \Upsilon^n, \langle \widetilde{\delta}^{1,n}, f\rangle\rangle, \mathbf{x}_{0,n+1}\rangle,
\end{aligned}$$

as desired. □

Proposition 2.4. *Let $Y^*(A)$ denote the total complex of $Y^{*,*}(A)$. If the Nakayama automorphism ρ has finite order and k has a primitive ord_ρ-th root of unity w, then*

$$H^n(Y^*(A)) = \begin{cases}\mathsf{HH}_0^0(A) & \text{if } n=0,\\ \mathsf{HH}_0^n(A)\oplus\mathsf{HH}_0^{n-1}(A) & \text{if } n\ge 1.\end{cases}$$

Proof. For each $0\le i<\mathsf{ord}_\rho$, let $Y_i^{*,*}(A)$ be the subcomplex of $Y^{*,*}(A)$ defined by

$$Y_i^{u,n} = \bigoplus_{B_{i,n}}\mathsf{Hom}(A_{u_1}\otimes\cdots\otimes A_{u_n}, A_v),$$

where $B_{i,n} = \{(u_1,\dots,u_n,v)$ such that $v-u_1-\cdots-u_n\equiv i \pmod{\mathsf{ord}_\rho}\}$. It is clear that $Y^{*,*}(A) = \bigoplus_{i=0}^{\mathsf{ord}_\rho} Y_i^{*,*}(A)$. Let $f\in Y_i^{0,n}(A)$. A direct computation shows that

$$\langle\langle \widetilde{\delta}_{1,n}, f\rangle, \mathbf{x}_{1,n}\rangle = (-1)^{n+1}(1-w^{-i})\langle f, \mathbf{x}_{1,n}\rangle.$$

Hence, the horizontal boundary maps of $Y_i^{*,*}(A)$ are isomorphisms if $i \neq 0$, and they are zero maps if $i = 0$. So,

$$H^n(Y_i^*(A)) = \begin{cases} 0 & \text{if } i \neq 0, \\ H^0(Y_i^{0,*}(A)) & \text{if } i = 0 \text{ and } n = 0, \\ H^n(Y_i^{0,*}(A)) \oplus H^{n-1}(Y_i^{1,*}(A)) & \text{if } i = 0 \text{ and } n > 0, \end{cases}$$

where $Y_i^*(A)$ is the total complex of $Y_i^{*,*}(A)$. The result follows easily from this fact, since $Y_0^{0,*}(A) = Y_0^{1,*}(A) \simeq (\mathsf{Hom}_k(A^{\otimes *}, A)_0, b^*)$. □

Proof of Theorem 1.2. By Proposition 2.3,

$$H^n(Y^*(A)) = H^n(X^*(A)) \quad \text{and} \quad H^n(Y^{u,*}(A)) = H^n(X^{u,*}(A)), \text{ for } u = 0, 1.$$

Hence, by Propositions 2.1 and 2.4,

$$\begin{aligned} \mathsf{HH}_0^0(A) &= H^0(Y^*(A)) = H^0(X^*(A)) \\ &= H^0(X^{0,*}(A)) = H^0(Y^{0,*}(A)) = \mathsf{HH}^0(A) \end{aligned}$$

and

$$\begin{aligned} \mathsf{HH}_0^n(A) \oplus \mathsf{HH}_0^{n-1}(A) &= H^n(Y^*(A)) = H^n(X^*(A)) \\ &= H^n(X^{0,*}(A)) \oplus H^{n-1}(X^{1,*}(A)) \\ &= H^n(Y^{0,*}(A)) \oplus H^{n-1}(Y^{1,*}(A)) \\ &= \mathsf{HH}^n(A) \oplus \mathsf{HH}^{n-1}(A), \end{aligned}$$

for all $n \geq 1$. From this it follows easily that $\mathsf{HH}^n(A) = \mathsf{HH}_0^n(A)$, for all $n \geq 0$, as desired. □

Proof of Theorem 1.3. By [St] or the cohomological version of [L], $\mathbb{Z}/\mathsf{ord}_\rho\mathbb{Z}$ acts on $\mathsf{H}^*(A_0, A)$ and there is a converging spectral sequence

$$E_2^{pq} = \mathsf{H}^p(\mathbb{Z}/\mathsf{ord}_\rho\mathbb{Z}, \mathsf{H}^q(A_0, A)) \Rightarrow \mathsf{HH}^{p+q}(A).$$

Since k has a primitive ord_ρ-th root of unity, ord_ρ is invertible in k. Hence, the above spectral sequence gives the isomorphisms

$$\mathsf{HH}^n(A) = \mathsf{H}^n(A_0, A)^{\mathbb{Z}/\mathsf{ord}_\rho\mathbb{Z}} \qquad (n \geq 0).$$

These maps are induced by the canonical inclusion of A_0 in A, and the action of $i \in \mathbb{Z}/\mathsf{ord}_\rho\mathbb{Z}$ on $\mathsf{H}^n(A_0, A)$ is induced by the map of complexes

$$\theta_i^* \, (\mathsf{Hom}_k(A_0^{\otimes *}, A), b^*) \to (\mathsf{Hom}_k(A_0^{\otimes *}, A), b^*),$$

defined by

$$\begin{aligned} &\langle \langle \theta_i^n, \varphi\rangle, a_1 \otimes \cdots \otimes a_n\rangle \\ &= \sum_{j_1,\dots,j_{n+1} \in J_i} s'_{i,j_1} \langle \varphi, s_{i,j_1} a_1 s'_{i,j_2} \otimes s_{i,j_2} a_2 s'_{i,j_3} \otimes \cdots \otimes s_{i,j_n} a_n s'_{i,j_{n+1}}\rangle s_{i,j_{n+1}}, \end{aligned}$$

where $(s_{i,j})_{j\in J_i}$ and $(s'_{i,j})_{j \in J_i}$ are families of elements of A_i and A_{n-i}, respectively, that satisfy $\sum_{j\in J_i} s'_{i,j} s_{i,j} = 1$. From this it follows easily that we have the isomorphisms

$$\mathsf{HH}_i^n(A) = \mathsf{H}^n(A_0, A_i)^{\mathbb{Z}/\mathsf{ord}_\rho\mathbb{Z}} \qquad (n \geq 0,\ 0 \leq i < \mathsf{ord}_\rho).$$

By combining this result with Theorem 1.2, we obtain the desired result. □

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