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# Singular Integral Operators with Non-necessarily Bounded Kernels on Spaces of Homogeneous Type*

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## INTRODUCTION

The purpose of this paper is twofold. First, we intend to clarify the relevance of conditions of the type considered in [A, DJS, MT] on the measure of coronas in the study of singular integral operators. The main result in this direction is given in Theorem (1.19), where we show that for a space of homogeneous type satisfying condition $(H_\alpha)$, see (1.5), a normalization can be given to satisfy condition $(L_\alpha)$, see (1.3). This result allows us to interpret $(H_\alpha)$ as a quantitative property ensuring that the order of the normalized space is at least equal to $\alpha$. Examples show that, in general, $\alpha$ cannot be improved. An approximation of the identity of R. Coifman's type is obtained for normalized spaces of order $\alpha$ without restrictions on the measure of the whole space $X$ or the existence of atoms for the measure. This allows us to get rid of the condition $(H_\alpha)$ in the results of Chapter II.

Second, in Chapter II we study singular integral operators with conditions on the associated kernel which generalize those of [A, DJS, MT], allowing the kernel to be unbounded, see [KW].

The conditions we assume on the kernel are stated in (2.3), (2.4), (2.5),

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and (2.6). They are inspired in the $L^r$-Dini condition of [KW]. The main result of the paper is to show that $T$ is weakly bounded if and only if $T\psi$ is a function given by an explicit formula involving the kernel associated to $T$ and $T1 = g$, see Theorem (2.27). By a systematic use of this formula we obtain the following results:

If $T$ is a weakly bounded singular integral operator and $T1$ belongs to B.M.O., then

(a) The kernel associated to $T$ is equal to zero if and only if there exist $h(x) \in L^\infty$ and $Tf(x) = h(x)f(x)$ (see (2.31).

(b) $T$ maps Lipschitz functions into bounded Lipschitz functions if and only if $T1 = 0$ (see (2.32)). For related results see [L].

(c) If $T^*1$ also belongs to B.M.O., then $T$ satisfies estimates of the type given in Lemma 2.3 of [DJS], which allow the $L^2$ theory to develop (see (2.34)).

Finally, we give an application to operators defined by principal value integrals, see (2.37), obtaining a priori Lipschitz estimates for some parabolic partial differential equations.

## I. Geometry of Spaces of Homogeneous Type

We say that a real valued function $d(x, y)$ defined on $X \times X$ is a quasi-distance on $X$ if

$$\begin{aligned} &\text{(i)} \quad d(x, y) \geqslant 0 \text{ and } d(x, y) = 0 \text{ if and only if } x = y, \\ &\text{(ii)} \quad d(x, y) = d(y, x), \text{ and} \\ &\text{(iii)} \quad d(x, y) \leqslant K[d(x, z) + d(z, y)], \end{aligned} \tag{1.1}$$

hold for every $x$, $y$, and $z$ in $X$ and $K$ a finite constant. The set $\{y : d(x, y) \leqslant r\}$ is denoted by $B_d(x, r)$. This quasi-distance defined a uniform structure on $X$, the family $\{(x, y) : d(x, y) < \varepsilon\}$ being a basis of the uniformity. Let $\mu$ be a positive measure on a $\sigma$-algebra of subsets of $X$ which contains the open sets and the balls $B_d(x, r)$. We say that $(X, d, \mu)$ is a space of homogeneous type if there exists a finite constant $A$ such that

$$\mu(B_d(x, 2Kr)) \leqslant A\mu(B_d(x, r)) \tag{1.2}$$

holds for every $x \in X$ and $r > 0$. It is known [MS1] that it is always possible to find a quasi-distance $d'(x, y)$ equivalent to $d(x, y)$ and $0 < \beta \leqslant 1$, such that

$$(L_\beta) \mid d'(x, z) - d'(y, z)| \leqslant Cr^{1-\beta} d(x, y)^\beta \tag{1.3}$$

holds for whenever $d'(x, z)$ and $d'(y, z)$ are smaller than or equal to $r$, with $C$ a finite constant. Thus we can assume that $d(x, y)$ satisfies condition $(L_\beta)$ for some $0<\beta\leqslant 1$.

We say that a triple $(X, d, \mu)$ is a normalized space if there exist constants $K_1$, $K_2$, $A_1$, and $A_2$ such that

$$\begin{aligned}&\text{(i)}\quad \text{if } K_1\mu(\{x\})\leqslant r\leqslant K_2\mu(X), \text{ then } A_1 r\leqslant \mu(B_d(x, r))\leqslant A_2 r,\\ &\text{(ii)}\quad \text{if } r<K_1\mu(\{x\}), \text{ then } B_d(x, r)=\{x\}, \text{ and}\\ &\text{(iii)}\quad \text{if } r>K_2\mu(X), \text{ then } B_d(x, r)=X.\end{aligned} \tag{1.4}$$

These there conditions imply that $(X, d, \mu)$ is a space of homogeneous type.

Let $(X, d, \mu)$ be a space of homogeneous type, with its quasi-distance satisfying condition $(L_\beta)$. Then we shall say that this space satisfies the condition $(H_\alpha)$, $0<\alpha\leqslant 1$, if

$$\begin{aligned}\mu(B_d(x, r+r^{1-\beta}s^\beta))-\mu(B_d(x, r-r^{1-\beta}s^\beta))\\ \leqslant C\mu(B_d(x, r))^{1-\alpha}\,\mu(B_d(x, s))^\alpha\end{aligned} \tag{1.5}$$

holds for $0\leqslant s\leqslant r$ and $x\in X$, with $C$ a finite constant.

The main purpose of this chapter is to prove that in a space of homogeneous type satisfying condition $(H_\alpha)$, (1.5), a normalization can be found such that its quasi-distance satisfies condition $(L_\alpha)$, (1.4). Also, an approximation of the identity, made of Lipschitz functions of order $\alpha$, of the type introduced by R. Coifman is given.

(1.6) LEMMA. *Let $(X, d, \mu)$ satisfy condition $(H_\alpha)$. Then either $\mu(\{x\})=0$ for every $x\in X$ or $\mu(\{x\})>0$ for every $x\in X$.*

This result is proved in [MT]. We give a proof here for the sake of completeness.

*Proof.* Let us assume that there is a point $x\in X$ such that $\mu(\{x\})=0$. Let $y\in X$, $y\neq x$. Then $y$ belongs to $B_d(x, d(x, y)+d(x, y)^{1-\beta}s^\beta)\sim B_d(x, d(x, y)-d(x, y)^{1-\beta}s^\beta)$, for every $s\leqslant d(x, y)$. By condition $(H_\alpha)$, we have

$$\mu((\{y\}))\leqslant C\mu(B_d(x, d(x, y)))^{1-\alpha}\,\mu(B_d(x, s))^\alpha.$$

Since $\lim_{s\to 0}\mu(B_d(x, s))=\mu(\{x\})=0$, we get $\mu(\{y\})=0$.

Let $(X, d, \mu)$ be a space of homogeneous type and define

$$\delta(x, x)=0 \qquad \text{and} \qquad \text{if } x\neq y,\ \delta(x, y)=\mu(B_d(x, d(x, y))). \tag{1.7}$$

(1.8) PROPOSITION. *The function $\delta(x, y)$ satisfies*

$$\begin{aligned} &\text{(i)} \quad \delta(x, y) \geqslant 0 \text{ and } \delta(x, y) = 0 \quad \text{if and only if } x = y, \\ &\text{(ii)}' \quad \delta(x, y) \leqslant A\delta(y, x), \quad \text{and} \\ &\text{(iii)} \quad \delta(x, y) \leqslant A^2 \, |\delta(x, z) + \delta(y, z)|, \end{aligned}$$

*for every $x$, $y$, and $z$ in $X$.*

*Proof.* Part (i) is obvious. Let us consider (ii)'. If $v \in B_d(x, d(x, y))$, we have $d(v, y) \leqslant K\,|d(v, x) + d(x, y)| \leqslant 2Kd(x, y)$; then $\delta(x, y) = \mu(B_d(x, d(x, y))) \leqslant A(B_d(y, d(x, y))) = A\delta(y, x)$. Let us consider (iii). If $d(x, z) \leqslant d(z, y)$, we have that $u \in B_d(x, d(x, y))$ implies $d(u, y) \leqslant K\,|d(u, x) + d(x, y)| \leqslant 2Kd(x, y)$ and since $d(x, y) \leqslant K\,|d(x, z) + d(z, y)| \leqslant 2Kd(z, y)$, it follows that $d(u, y) \leqslant (2K)^2\, d(z, y)$. Thus,

$$\delta(x, y) \leqslant \mu(B_d(x, d(x, y)) \leqslant A^2 \mu(B_d(y, d(y, z))) = A^2 \delta(y, z).$$

Analogously, if $d(z, y) \leqslant d(x, z)$ it turns out that $\delta(x, y) \leqslant A^2\delta(x, z)$. This proves part (iii).

We observe that $\delta(x, y)$ does not necessarily satisfy condition (ii) of (1.1), but it does satisfy (ii)' of (1.8). We shall call this $\delta(x, y)$ the non-necessarily symmetric quasi-distance associated to $(X, d, \mu)$. We denote by $B_\delta(x, r)$ the set $\{y : \delta(x, y) \leqslant r\}$.

(1.9) PROPOSITION. *Let $(X, d, \mu)$ be a space of homogeneous type and $\delta(x, y)$ the non-necessarily symmetric quasi-distance associated to $(X, d, \mu)$. Then the following properties hold:*

$$\begin{aligned} &\text{(i)} \quad \text{if } 0 < r < \mu(\{x\}), \text{ then } B_\delta(x, r) = \{x\}, \\ &\text{(ii)} \quad \text{if } \mu(\{x\}) \leqslant r, \text{ then } \mu(B_\delta(x, r)) \leqslant r, \\ &\text{(iii)} \quad \text{if } \mu(X) \leqslant r, \text{ then } B_\delta(x, r) = X, \quad \text{and} \\ &\text{(iv)} \quad \text{if } r < \mu(X), \text{ then } A^{-2} r \leqslant \mu(B_\delta(x, r)). \end{aligned}$$

*Proof.* Part (i): if $y \in B_\delta(x, r)$ and $y \neq x$, then $r < \mu(\{x\}) \leqslant \mu(B(x, d(x, y))) = \delta(x, y) \leqslant r$, which is a contradiction. Then $B_\delta(x, r) = (\{x\})$. Part (ii): if $\mu(\{x\}) \leqslant r$, since

$$B_\delta(x, r) = \bigcup \{B_d(x, d(x, y)) : y \in B_\delta(x, r)\},$$

it turns out that $\mu(B_\delta(x, r))) \leqslant r$. Part (iii): let $y \in X$; since $\mu(B_d(x, d(x, y)) \leqslant \mu(X) \leqslant r$, it follows that $y \in B_\delta(x, r)$. Part (iv): assume that $B_\delta(x, r) = \{x\}$. This implies that for every $y \neq x$, $\mu(B_d(x, d(x, y))) > r$. Let $\{y_n\}$ be a sequence of points of $X$ such that

$$m = \lim d(x, y_n) = \inf\{d(x, y) : y \in X, y \neq x\}.$$

If this limit $m$ is equal to zero, we have $\mu(\{x\}) = \lim \mu(B(x, d(x, y_n))) \geqslant r$ and therefore $\mu(B_\delta(x, r)) = \mu(\{x\}) \geqslant r > A^{-2}r$. If $m$ is positive, then $B_\delta(x, 3m/4) = \{x\}$ and $\mu(B_d(x, 2k3m/4)) > r$. Thus,

$$r < A\mu(B_d(x, 3m/4)) = A\mu(\{x\}) = A\mu(B_\delta(x, r)),$$

verifying (iv). Let us assume now that $B_\delta(x, r) \neq \{x\}$. Let $s = \sup\{d(x, y) : x \neq y, y \in B_\delta(x, r)\}$. Then $s > 0$, and moreover $s$ is finite, since otherwise $B_\delta(x, r) = X$ and then $r < \mu(X) = \mu(B_\delta(x, r)) \leqslant r$, which is a contradiction. Let $t < s < 2t$. If $A^{-2}r > \mu(B_\delta(x, r))$, we shall show that for every positive integer $n$, $B_d(x, (2K)^n t) = B_d(x, s)$ holds. For $n = 1$, we have

$$\mu(B_d(x, 2Kt)) \leqslant A\mu(B_d(x, t)) \leqslant A\mu(B_\delta(x, r)) \leqslant A^{-1}r < r.$$

If there were $y \in B_d(x, 2Kt) \sim B_d(x, s)$, there would exist $y \in B_\delta(x, r)$ and $d(x, y) > s$, contradicting the definition of $s$. For $n + 1$, we have

$$\begin{aligned}\mu(B_d(x, (2K)^{n+1} t)) &\leqslant A\mu(B_d(X, (2K)^n t)) = A\mu(B_d(x, s)) \\ &\leqslant A\mu(B_d(x, 2Kt)) \leqslant A^2\mu(B_d(x, t)) \\ &\leqslant A^2\mu(B_\delta(x, r)) < r.\end{aligned}$$

Again, since $(2K)^{n+1}t > s$, it follows that $B_d(x, (2K)^{n+1} t) = B_d(x, s)$. Therefore, we have $B_d(x, s) = X$. From

$$\begin{aligned}r < \mu(X) = \mu(B_d(x, s)) = \mu(B_d(x, 2Kt)) \leqslant A\mu(B_d(x, t)) \\ \leqslant A\mu(B_\delta(x, r)),\end{aligned}$$

it follows that

$$A^{-2}r < A^{-1}r \leqslant \mu(B_\delta(x, r)),$$

which is a contradiction and (iv) is proved.

(1.10) LEMMA. *Let $K' = (C + K)^{2/\beta}$, where $C$ is the constant in condition $(L_\beta)$ of* (1.3). *Then, if $(X, d, \mu)$ saatisfies conditions $(L_\beta)$ and $(H_\alpha)$ of* (1.3) *and* (1.5), *respectively, we have*

$$\begin{aligned}&|\mu(B_d(x, d(x', y))) - \mu(B_d(x', d(x', y)))| \\ &\quad\leqslant C''\mu(B_d(x, d(x', y)))^{1-\alpha}\, \mu(B_d(x, d(x, x')))^\alpha,\end{aligned}$$

*provided that $K'd(x, x') \leqslant d(x', y)$.*

*Proof.* Let us assume first that $\mu(B_d(x, d(x', y)))$ is larger than $\mu(B_d(x', d(x', y)))$. If $z \in B(x, d(x', y))$, we have

$$d(z, x') \leqslant K\,|d(z, x) + d(x, x')| \leqslant 2Kd(x', y).$$

Then, by condition $(L_\beta)$ of (1.3),

$$d(z, x') \leqslant d(z, x) + C(2K)^{1-\beta}\, d(x', y)^{1-\beta}\, d(x, x')^\beta,$$

or

$$d(z, x') \leqslant d(x', y) + d(x', y)^{1-\beta}\,(C^{1/\beta}(2K)^{(1-\beta)/\beta}\, d(x, x'))^\beta.$$

Since $C^{1/\beta}(2K)^{(1-\beta)/\beta}\, d(x, x') \leqslant K'd(x, x') \leqslant d(x', y)$, condition $(H_\alpha)$ implies

$$\begin{aligned}&\mu(B_d(x, d(x', y))) - \mu(B_d(x', d(x', y)))\\ &\quad\leqslant C''\mu(B_d(x', d(x', y)))^{1-\alpha}\, \mu(B_d(x', d(x, x')))^\alpha\\ &\quad\leqslant C''\mu(B_d(x, d(x', y)))^{1-\beta}\, \mu(B_d(x, d(x, x')))^\alpha.\end{aligned}$$

The case $\mu(B_d(x, d(x', y))) \leqslant \mu(B_d(x', d(x', y)))$ is similar and even simpler.

(1.11) PROPOSITION. *Let $(X, d, \mu)$ be a space of homogeneous type satisfying conditions $(L_\beta)$ and $(H_\alpha)$. Then, the non-necessarily symmetric quasi-distance $\delta(x, y)$ associated to the space satisfies*

(i) $|\delta(x, y) - \delta(x', y)| \leqslant Cr^{1-\alpha}\delta(x, x')^\alpha$, *whenever $\delta(x, y)$ and $\delta(x', y)$ are less than or equal to $r$, and*

(ii) *for every $x \in X$, $\delta(x, y)$ is a continuous function of $y$.*

*Proof.* We can assume that $d(x, y) \geqslant d(x', y)$. Let $r = [d(x, y) + d(x', y)]/2$ and $s = [d(x, y) - d(x', y)]^{1/\beta}$. $[d(x, y) + d(x, y)]^{1-1/\beta}/2$. It is easy to see that

$$(s/r)^\beta = [d(x, y) - d(x', y)]/[d(x, y) - d(x', y)] \leqslant 1,$$

that is to say, $s \leqslant r$. Moreover,

$$r + r^{1-\beta}s^\beta = d(x, y) \qquad \text{and} \qquad r - r^{1-\beta}s^\beta = d(x', y).$$

By condition $(L_\beta)$, we have

$$d(x, y) - d(x', y) \leqslant Cd(x, y)^{1-\beta}\, d(x, x')^\beta;$$

therefore, $s \leqslant Cd(x, x')$. It is also evident that $r \leqslant d(x, y)$. Applying condition $(H_\alpha)$ with the given $r$,

$$\mu(B_d(x, d(x, y))) - \mu(B_d(x, d(x', y)) \leqslant C\mu(B_d(x, d(x, y)))^{1-\alpha} \mu(B_d(x, d(x, x')))^\alpha.$$

On the other hand, by Lemma (1.10), it follows that if $K'd(x, x')^\beta < d(x', y)^\beta$,

$$|\mu(B_d(x, d(x', y))) - \mu(B_d(x', d(x', y)))| \leqslant C\mu(B_d(x, d(x', y)))^{1-\alpha} \mu(B_d(x, d(x, x')))^\alpha.$$

If we assume that $K'd(x, x')^\beta \geqslant d(x', y)^\beta$, we have

$$\begin{aligned}\mu(B_d(x, d(x', y))) &= \mu(B_d(x, d(x', y)))^{1-\alpha} \mu(B_d(x, d(x', y)))^\alpha \\ &\leqslant C\mu(B_d(x, d(x, y)))^{1-\alpha} \mu(B_d(x, d(x, x')))^\alpha.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mu(B_d(x', d(x', y))) &= \mu(B_d(x', d(x', y)))^{1-\alpha} \mu(B_d(x', d(x', y)))^\alpha \\ &\leqslant C\mu(B_d(x', d(x', y)))^{1-\alpha} \mu(B_d(x', d(x, x')))^\alpha.\end{aligned}$$

Let $u \in B_d(x', d(x', y))$; we have

$$d(u, y) \leqslant K[d(u, x') + d(x', y)] \leqslant 2Kd(x', y) \leqslant 2Kd(x, y),$$

showing that $B_d(x', d(x', y) \subset B_d(y, 2Kd(x, y))$. Therefore,

$$\mu(B_d(x', d(x', y))) \leqslant A\mu(B_d(y, d(x, y))) \leqslant C'\mu(B_d(x, d(x, y)))^\alpha.$$

Thus, we have

$$\mu(B_d(x', d(x', y)) \leqslant C''\mu(B_d(x, d(x, y)))^{1-\alpha} \mu(B_d(x, d(x, x')))^\alpha.$$

Collecting results, it follows that

$$\begin{aligned}|\delta(x, y) - \delta(x', y)| &\leqslant |\mu(B_d(x, d(x, y))) - \mu(B_d(x, d(x', y)))| \\ &\quad + |\mu(B_d(x, d(x', y))) - \mu(B_d(x', d(x', y)))| \\ &\leqslant C\mu(B_d(x, d(x', y)))^{1-\alpha} \mu(B_d(x, d(x, x')))^\alpha \\ &= C\delta(x, y)^{1-\alpha} \delta(x, x')^\alpha,\end{aligned}$$

which implies (i).

As for part (ii), by virtue of Lemma (1.6) we have two possible cases. First, for every $x \in X$, $\mu(\{x\}) > 0$. In this case $X$ is a discrete space for both $d$ and $\delta$ and therefore, every function on $X$ is continuous. The second case is when $\mu(\{x\}) = 0$. Then, if $d(x, y) > d(x, y')$, choosing $r$ and $s$ as

$$r + r^{1-\beta} s^{\beta} = d(x, y), \qquad \text{and} \qquad r - r^{1-\beta} s^{\beta} = d(x, y'),$$

we get

$$\begin{aligned} r &= [d(x, y) + d(x, y')]/2 \\ s &= \{([d(x, y) - d(x, y')]/2([d(x, y) + d(x, y')]/2)^{1-\beta}\}^{1/\beta}, \end{aligned}$$

$s \leqslant r$, and $r \leqslant d(x, y)$. Thus, by condition $(H_\alpha)$, it follows that

$$|\delta(x, y) - \delta(x, y')| \leqslant C\mu(B_d(x, d(x, y)))^{1-\alpha}\, \mu(B_d(x, s))^{\alpha}.$$

Since $y'$ tending to $y$ implies that $s$ tends to zero and $\lim \mu(B_d(x, s)) = \mu(\{x\}) = 0$, the continuity of $\delta(x, y)$ is proved.

In the rest of this chapter, $(X, \delta, \mu)$ will be a triple satisfying the following conditions:

$$\begin{array}{rll} \text{(i)} & 0 \leqslant \delta(x, y) < \infty \text{ and } \delta(x, y) = 0 & \text{if and only if } x = y \\ \text{(ii)} & \delta(x, y) \leqslant K\delta(y, x), & \\ \text{(iii)} & \delta(x, y) \leqslant K[\delta(x, z) + \delta(z, y)], & \\ \text{(iv)} & \text{if } K_1\mu(\{x\}) \leqslant r \leqslant K_2\mu(X), \quad \text{then} & \\ & rA_1 \leqslant \mu(B_\delta(x, r)) \leqslant rA_2, & \\ \text{(v)} & \text{if } r < K_1\mu(\{x\}), \quad \text{then } B_\delta(x, r) = \{x\} \text{ and} & \\ \text{(vi)} & \text{if } r > K_2\mu(X), \quad \text{then } B_\delta(x, r) = X, & \end{array} \tag{1.12}$$

where $K$, $K_1$, $K_2$, $A_1$, and $A_2$ are constants. These conditions imply the existence of a constant satisfying (1.2), i.e., $\mu(B_\delta(x, 2Kr)) \leqslant A\mu(B_\delta(x, r))$. We shall call a triple $(X, \delta, \mu)$ satisfying conditions (1.12) a non-necessarily symmetric normalized space. The only difference between this and a normalized space is that instead of assuming $\delta$ to be symmetric, we assume that (ii) of (1.12) holds with $K$ non-necessarily equal to one.

(1.13) THEOREM (Approximation of the Identity). *Let $(X, \delta, \mu)$ be a non-necessarily symmetric normalized space of order $\alpha$, that is to say*

$$|\delta(x, y) - \delta(x', y)| \leqslant Cr^{1-\alpha}\delta(x, x')^{\alpha} \tag{1.14}$$

*holds for an $\alpha$, $0 < \alpha \leqslant 1$, whenever $\delta(x, y) < r$ and $\delta(x', y) < r$. If $\delta(x, y)$ is* *non-symmetric, we assume that* $\delta(x, y)$ *is a continuous function of* $y$. *Then, for every* $t$, $0 < t < C\mu(X)$, *there is a function* $s_t(x, y)$ *satisfying*

(i) $0 \leqslant s_t(x, y) \leqslant C[\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(y, t))^{-1}]$,

(ii) *if* $\delta(x, y) < C^{-1}t$,
*then* $s_t(x, y) \geqslant C^{-1}[\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(y, t))^{-1}]$,

(iii) $s_t(x, y) = s_t(y, x)$

(iv) $\operatorname{supp} s_t \subset \{(x, y) : \delta(x, y) < Ct\}$

(v) $|s_t(x, y) - s_t(x', y)|$
$\leqslant C\delta(x, x')^\alpha\, [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^{1+\alpha}$

(vi) $\int s_t(x, y)\, d\mu(y) = 1$,

*where* $C$ *is a finite constant. If necessary,* $C$ *can be chosen as large as desired.*

In order to prove this theorem, we shall need some lemmas.

Let $h(t)$ be a $C^\infty$ function defined on $|0, \infty)$ that satisfies $h(t) = 1$ if $0 \leqslant t \leqslant 1$, $h(t) = 0$ if $t \geqslant A$, and $0 \leqslant h(t) \leqslant 1$ for every $t \geqslant 0$.

(1.15) LEMMA. *If* $u_t(x, y) = h(\delta(x, y)/t)$, *then*

$$|u_t(x, y) - u_t(x', y)| \leqslant C\delta(x, x')^\alpha\, [\mu(B_\delta(x, t))^{-1} + (B_\delta(x', t))^{-1}\}^\alpha.$$

*Proof.* Let $\delta(x, y) \leqslant 2KAt$ and $\delta(x', y) \leqslant 2KAt$. Then, by (1.14), we have

$$|u_t(x, y) - u_t(x', y)| \leqslant \|h'\|_\infty\, |\delta(x, y) - \delta(x', y)|/t \leqslant C(\delta(x, x')/t)^\alpha.$$

If $\delta(x, y) > 2KAt$ and $\delta(x', y) \leqslant At$, then

$$2KAt < \delta(x, y) \leqslant K(\delta(x, x') + \delta(x', y)) \leqslant K\delta(x, x') + KAt;$$

thus, $t \leqslant At \leqslant \delta(x, x')$. Therefore

$$|u_t(x, y) - u_t(x', y)| = 1 \leqslant (\delta(x, x')/t)^\alpha.$$

The other possible cases are trivial. Now, if $K_2\mu(X) \geqslant t \geqslant \min(K_1 A^{-1}\mu(\{x\}), K_1 A^{-1}\mu(\{x'\}))$ then

$$|u_t(x, y) - u_t(x', y)| \leqslant C'\delta(x, x')^\alpha\, [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^\alpha.$$

If $t < \min(K_1 A^{-1}\mu(\{x\}), K_1 A^{-1}\mu(\{x'\}))$, then $B_\delta(x, t) = \{x\}$, $B_\delta(x', t) = \{x'\}$, and

$$u_t(x, y) = 1 \text{ if } x = y \quad \text{and} \quad u_t(x, y) = 0 \text{ if } x \neq y,$$
$$u_t(x', y) = 1 \text{ if } x' = y \quad \text{and} \quad u_t(x', y) = 0 \text{ if } x \neq y.$$

Assume $x \neq x'$. Then $K_1 \mu(\{x\}) \leqslant \delta(x, x')$ and $K_1 \mu(\{x\}) \leqslant \delta(x', x) < K\delta(x, x')$, yielding

$$|u_t(x, y) - u_t(x', y)| \leqslant 1 \leqslant C\delta(x, x')^\alpha \, [\mu(\{x\})^{-1} + \mu(\{x'\})^{-1}]^\alpha$$
$$\leqslant C'\delta(x, x')^\alpha \, [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^\alpha.$$

(1.16) LEMMA. *Let*

$$m_t(x) = \int u_t(x, y) \, d\mu(y).$$

*Then $m_t(x)$ is well defined and*

(i) $|m_t(x) - m_t(x')| \leqslant C\delta(x, x')^\alpha \, [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^\alpha \cdot [\mu(B_\delta(x, t)) + \mu(B_\delta(x', t)];$

*moreover,*

(ii) $\mu(B_\delta(x, t)) \leqslant m_t(x) \leqslant \mu(B_\delta(x, At)).$

*Proof.* The function $m_t(x)$ is well defined since we assume that $\delta(x, y)$ is a continuous function of $y$. On the other hand, by Lemma (1.15), we have

$$|m_t(x) - m_t(x')| \leqslant \int |u_t(x, y) - u_t(x', y)| \, d\mu(y)$$
$$\leqslant C'\delta(x, x')^\alpha \, [\mu(B_\delta(x, t))^{-1} + (\mu(B_\delta(x', t))^{-1}]^\alpha$$
$$\times [\mu(B_\delta(x, t)) + \mu(B_\delta(x', t))].$$

As for (ii), since $u_t(x, y) = 1$ if $y \in B_\delta(x, t)$ and $u_t(x, y) = 0$ if $y \notin B(x, t)$, (ii) follows.

(1.17) LEMMA. *Let*

$$v_t(x, y) = m_t(x)^{-1} \, u_t(x, y).$$

*Then,*

(i) $|v_t(x, y) - v_t(x', y)|$
$\leqslant C\delta(x, x')^\alpha \, [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^{1+\alpha}$,

(ii) $\int v_t(x, y) \, d\mu(y) = 1$, *and*

(iii) $C^{-1} \leqslant \int v_t(x, y) \, d\mu(x) \leqslant C$,

*where $C$ is a finite constant.*

*Proof.* We can assume that $m_t(x') \leqslant m_t(x)$. Then

$$v_t(x, y) - v_t(x', y) = m_t(x)^{-1}\,[u_t(x, y) - u_t(x', y)]$$
$$+ u_t(x', y)[m_t(x') - m_t(x)]\, m_t(x)^{-1}\, m_t(x')^{-1}.$$

By Lemmas (1.15) and (1.16), it follows that

$$|v_t(x, y) - v_t(x', y)| \leqslant C'\delta(x, x')^\alpha\, [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^{1+\alpha}.$$

As for (ii), it is apparent from the definition of $v_t(x, y)$. In order to prove (iii), we observe that

$$C^{-1} m_t(y) \leqslant m_t(x) \leqslant C m_t(y),$$

for $x \in B_\delta(y, At)$. This implies (iii).

*Proof of Theorem* (1.13). Let

$$w(z) = \left(\int v_k(x, z)\, d\mu(x)\right)^{-1}.$$

We define

$$s_t(x, y) = \int v_t(x, z)\, w(z)\, v_t(y, z)\, d\mu(z).$$

Part (i) By definition of $v_t$ and from part (iii) of Lemma (1.17), we get

$$0 \leqslant s_t(x, y) \leqslant (m_t(x)^{-1}\, m_t(y)^{-1}\, [\mu(B_\delta(x, t)) + \mu(B_\delta(y, t))]$$
$$\leqslant C[\mu(B(x, t))^{-1} + \mu(B_\delta(y, t))^{-1}].$$

Part (ii). If $\delta(x, z) < C^{-1}t$ and $\delta(x, y) < C^{-1}t$, then $\delta(y, z) \leqslant K(\delta(y, x) + \delta(x, z)) \leqslant 2KAC^{-1}t < t$, if $C$ is chosen to be $2KA < C$. Then

$$s_t(x, y) \geqslant C' m_t(x)^{-1}\, m_t((y)[\mu(B_\delta(x, t)) + \mu(B_\delta(y, t))]$$
$$\geqslant C^{-1}[\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(y, t))^{-1}].$$

Part (iii). follows from the definition of $s_t(x, y)$.

Part (iv). If $s_t(x, y) > 0$, there exists $z$ such that $\delta(x, z) < At$ and $\delta(y, z) < At$, therefore $\delta(x, y) \leqslant Ct$.

Part (v). By Lemma (1.17) we have

$$|s_t(x, y) - s_t(x', y)| \leqslant \int |v_t(x, z) - v_t(x', z)|\, w(z)\, v_t(y, z)\, d\mu(z)$$
$$\leqslant C\delta(x, x')^\alpha\, [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^{1+\alpha}.$$

Part (vi). By Lemma (1.7) we have

$$\int s_t(x, y)\, d\mu(y) = \int v_t(x, z)\, w(z) \left( \int v_t(y, z)\, d\mu(y) \right) d\mu(z)$$

$$= \int v_t(x, z)\, d\mu(z) = 1.$$

(1.18) THEOREM. *If* $(X, \delta, \mu)$ *is a non-necessarily symmetric normalized space of order* $\alpha$, *then there exists* $\delta'$, *symmetric and equivalent to* $\delta$, *such that* $(X, \delta', \mu)$ *is a normalized space of order* $\alpha$, *that is to say, it satisfies conditions* (1.4) *and* $(L_\alpha)$.

*Proof.* Let $C$ be the constant of Theorem (1.13). If $x \neq y$, let $i$ be the integer such that $cA^{-i-1} < \delta(x, y) \leqslant CA^{-i}$. Let $p$ be the integer satisfying

$$C^{-1}A^{-p-2} < K_2 \mu(X) \leqslant C^{-1}A^{-p-1},$$

and let $n$ be the positive integer satisfying

$$C^2 A^{-n} < 1 \leqslant C^2 A^{-n+1}.$$

Then, if $k \leqslant i$, we have

$$CA^{-k} \geqslant CA^{-i} \geqslant \delta(x, y) \geqslant K_1 \mu(\{x\});$$

thus, $\mu(B_\delta(x, A^{-i})) \approx \mu(B_\delta(x, CA^{-i})) \approx A^{-i}$. On the other hand, we have

$$CA^{-i-1} < \delta(x, y) \leqslant K_2 \mu(X) \leqslant C^{-1}A^{-p-1},$$

therefore,

$$1 < C^2 A^{-n+1} < A^{i-p-n},$$

thus, $i \geqslant p + n$.

Moreover,

$$\delta(x, y) \leqslant CA^{-i} = C^2 A^{-n} C^{-1} A^{-i-n} < C^{-1} A^{-(i-n)}$$

and if $k \geqslant i + 1$, then

$$\delta(x, y) > CA^{-i-1} \geqslant CA^{-k}.$$

We have that

$$s(x, y) = \sum_{k=p}^{s} s_{A^{-k}}(x, y)$$

satisfies

$$s(x, y) = \sum_{k=p}^{i} s_{A^{-k}}(x, y) \leqslant C' \sum_{k=p}^{i} A^k \leqslant C'' A^i \leqslant C''' \delta(x, y)^{-1}$$

and

$$s(x, y) \geqslant s_{A^{-(i-n)}}(x, y) \geqslant C A^i \geqslant C \delta(x, y)^{-1}.$$

Next, we estimate $|s(x, y) - s(x', y)|$. We can assume that $0 < \delta(x, y) \leqslant \delta(x', y)$. Let $m$ be an integer satisfying $A^m \geqslant 2K$. Then, if $A^m \delta(x, y) \leqslant \delta(x', y)$, we have

$$A^m \delta(x, y) \leqslant \delta(x', y) \leqslant K\delta(x', x) + K\delta(x, y),$$

which implies $\delta(x', y)/2 \leqslant K\delta(x, x')$. Then

$$|s(x, y) - s(x', y)| \leqslant C' \delta(x, y)^{-1} \leqslant C'' \frac{\delta(x', x)}{\delta(x, y)\, \delta(x', y)^\alpha}.$$

If $\delta(x, y) \leqslant \delta(x', y) \leqslant A^m \delta(x, y) \leqslant C A^{m-i+1}$, and since for $k > i$, $CA^{-k} \leqslant CA^{-i-1} < \delta(x, y) \leqslant \delta(x', y)$, we have $s_{A^{-k}}(x, y) = s_{A^{-k}}(x', y) = 0$; thus

$$|s(x, y) - s(x', y)| \leqslant \sum_{k=p}^{i} |s_{A^{-k}}(x, y) - s_{A^{-k}}(x', y)|,$$

and by Theorem (1.13), we get that

$$\begin{aligned} |s(x, y) - s(x', y)| &\leqslant C' \delta(x, x')^\alpha \sum_{k=p}^{i} A^{k(1+\alpha)} \\ &\leqslant C'' A^{i(1+\alpha)} \delta(x, y)^\alpha \leqslant C''' \delta(x', y)^{-(1+\alpha)}\, \delta(x, x')^\alpha. \end{aligned}$$

Now, let us define

$$\delta'(x, x) = 0 \qquad \text{and}$$
$$\delta(x, y) = s(x, y)^{-1} \qquad \text{for} \quad x \neq y.$$

We have already shown that there exists a constant $C > 0$ such that

$$C^{-1} \delta(x, y) < \delta'(x, y) \leqslant C \delta(x, y).$$

Let us estimate $|\delta'(x, y) - \delta'(x', y)|$. If $x = y$, then

$$|\delta'(x, x) - \delta'(x', x)| \leqslant C r^{1-\alpha} \delta(x, x')^\alpha$$

if $\delta(x, x') < r$. Analogously for $x' = y$. Thus, we can assume that $x \neq x'$, $y \neq x$, and $y \neq x'$. Then

$$|\delta'(x, y) - \delta'(x', y)| \leqslant C' \, |s(x, y) - s(x', y)| \, \delta'(x, y) \, \delta'(x', y),$$

which, by previous estimates on $s(x, y)$, is smaller than or equal to

$$C'' \frac{\delta(x, x')}{\delta(x, y) \, \delta(x', y)^{\alpha}} \, \delta'(x, y) \, \delta(x', y) \leqslant C''' r^{1-\alpha} \delta(x, x')^{\alpha},$$

if $\delta(x, y) \leqslant \delta(x', y) \leqslant r$. This ends the proof of the theorem.

(1.19) THEOREM. *Let $(X, d, \mu)$ be a space of homogeneous type satisfying conditions $(L_{\beta})$ and $(H_{\alpha})$, Then a normalization of order $\alpha$ can be found for this space.*

*Proof.* The normalization is given by the quasi-distance $\delta'(x, y)$ of Theorem (1.18), where $\delta(x, y)$ is the non-necessarily symmetric quasi-distance associated to $(X, d, \mu)$ in (1.7). Propositions (1.9) and (1.11) and Theorem (1.18) show that $(X, \delta', \mu)$ is a normalized space of order $\alpha$.

(1.20) PROPOSITION. *Let $f$ be a Lipschitz function of order $\eta \leqslant \alpha$, with respect to the quasi-distance $\delta$, supported in $B_{\delta}(x_0, r)$, and $(X, \delta, \mu)$ a normalized space of order $\alpha$. Then if $0 < \eta' < \eta$, we have that the functions*

$$f_t(x) = \int S_t(x, y) \, f/y) \, d\mu(y),$$

*for $t < K_2 \mu(X)$, satisfy*

(i) $\operatorname{supp} f_t \subset B(x_0, r + C'' r^{1-\alpha}, t^{\alpha})$, *if $t < r$,*

(ii) $|f_t(x) - f_t(x')| \leqslant C'' t^{-(1+\alpha)} \mu(B(x_0, r))^{1+\eta} \, \delta(x, x')$.

(iii) $|(f_t(x) - f(x)) - (f_t(x') - f(x'))| \leqslant C(t) \, \delta(x, x')^{\eta}$, *where* $\lim_{t \to 0} C(t) = 0$.

*Proof.* The support of $f_t(x)$ is contained in the set of point $x$ such that there exists $y$ satisfying $\delta(x, y) < Ct$ and $\delta(x_0, y) < r$. Then $|\delta(x_0, x) - \delta(x_0, y)| \leqslant C'(t + r)^{1-\alpha} \, \delta(x, y)^{\alpha} \leqslant C'(t + r)^{1-\alpha} \, t^{\alpha} \leqslant C'' r^{1-\alpha} t^{\alpha}$.

Let us consider part (ii). We have

$$|f_t(x) - f_t(x')| \leqslant \int |s_t(x, y) - s_t(x', y)| \, |f(y)| \, d\mu(y).$$

By Theorem (1.13), this is smaller than or equal to

$$C\delta(x, x')^\alpha [\mu(B_\delta(x, t))^{-1} + \mu(B_\delta(x', t))^{-1}]^{1+\alpha} \int |f(y)| \, d\mu(y)$$
$$\leqslant C'\delta(x, x')^\alpha t^{-(1+\alpha)} C\mu(B(x_0, r))^{\eta+1}.$$

As for part (iii), given $\varepsilon > 0$, assume that $t < \varepsilon$; then

$$|f_t(x) - f(x)| \leqslant \int s_t(x, y) f(y) - f(x)| \, d\mu(y)$$
$$\leqslant C \int s_t(x, y) \, \delta(x, y)^\eta \, d\mu(y) \leqslant Ct^\eta.$$

If $\delta(x, x') \geqslant t$, we get

$$|f_t(x) - f(x)| \leqslant C\varepsilon^{\eta - \eta'} \delta(x, x')^{\eta'}.$$

Analogously for $f_t(x') - f(x')$. If $\delta(x, x') < t$, we have

$$|(f_t(x) - f(x)) - (f_t(x') - f(x'))|$$
$$\leqslant |f_t(x) - f_t(x')| + |f(x) - f(x')| = I_1 + I_2.$$

For $I_1$, we have

$$|f_t(x) - f_t(x')| = \left| \int |s_t(x, y) - s_t(x', y)| \, f(y) \, d\mu(y) \right|$$
$$\leqslant \int |s_t(x, y) - s_t(x', y)| \, |f(y) - f(x)| \, d\mu(y)$$
$$\leqslant C\delta(x, x')^\alpha t^{-1-\alpha} \int_{B_\delta(x, At) \cup B_\delta(x', At)} \delta(x, y)^\eta \cdot d\mu(y)$$
$$\leqslant C'\delta(x, x')^\alpha t^{-1-\alpha} t^{\eta+1} \leqslant C''\delta(x, x')^{\eta'} \varepsilon^{\eta-\eta'}.$$

The same estimate holds for $|f(x) - f(x')| = I_2$. This ends the proof of the proposition.

## II. Singular Integral Operators

In this chapter $(X, \delta, \mu)$ will be a triple satisfying the following conditions:

(i) $0 \leqslant \delta(x, y) < \infty$ and $\delta(x, y) = 0$ if and only if $x = y$,

(ii) $\delta(x, y) = \delta(y, x)$,

$$
\begin{aligned}
&\text{(iii)}\quad \delta(x, y) \leqslant K(\delta(x, z) + \delta(z, y)),\\
&\text{(iv)}\quad \text{if } k_1\mu(\{x\}) \leqslant r \leqslant k_2\mu(X) \text{ then } rA_1 \leqslant \mu(B_\delta(x, r)) \leqslant rA_2,\\
&\text{(v)}\quad \text{if } r < k_1\mu(\{x\}) \text{ then } B_\delta(x, r) = \{x\},\\
&\text{(vi)}\quad \text{if } r > k_2\mu(X) \text{ then } B_\delta(x, r) = X, \text{ and}\\
&\text{(vii)}\quad \text{there exists } \alpha,\ 0 < \alpha \leqslant 1, \text{ such that}\\
&\qquad |\delta(x, y) - \delta(x', y)| \leqslant Cr^{1-\alpha}\delta(x, x')^\alpha\\
&\qquad \text{holds, whenever } \delta(x, y) < r \text{ and } \delta(x', y) < r,
\end{aligned}
\tag{2.1}
$$

where $K$, $k_1$, $k_2$, $A_1$, $A_2$, and $C$ are constants. These conditions imply the existence of a constant $A$ satisfying (1.2). For the sake of simplicity we shall assume that $A = K$.

Given a ball $B$ and a number $\gamma$, $0 < \gamma \leqslant \alpha$, we denote by $\Lambda(B)$ the Banach space of complex-valued functions supported on $B$, such that

$$|\psi(x) - \psi(y)| \leqslant C\delta(x, y). \tag{2.2}$$

Given $\psi \in \Lambda(B)$ we shall denote by $|||\psi|||_\gamma$ the infimum of the constants $C$ appearing in (2.2).

We say that $\psi$ belongs to $\Lambda_0^\gamma$ if $\psi \in \Lambda^\gamma(B)$ for some ball $B$. On $\Lambda_0^\gamma$ we define the topology which is the inductive limit of the spaces $\Lambda^\gamma(B)$, see [MS2], and $(\Lambda_0^\gamma)'$ denotes the space of all continuous linear functions on $\Lambda_0^\gamma$. By $\{\Lambda_0^\gamma\}_0$ we denote the subspace of all functions $\psi$ in $\Lambda_0^\gamma$ such that $\int \psi(x)\, d\mu(x) = 0$. $\Lambda_b^\gamma$ stands for the space of bounded functions $\psi$ satisfying (2.2). As usual, B.M.O. is the space of all the locally integrable functions $g$ on $X$ such that

$$\mu(B)^{-1} \int_B |g(x) - m_B g|\, d\mu(x) \leqslant C,$$

where $B$ is any ball and $m_B g = \mu(B)^{-1} \int_B g(x)\, d\mu(x)$.

We consider a continuous linear operator $T$ from $\Lambda_0^\gamma$ into $(\Lambda_0^\gamma)'$ for some $\gamma$, $0 < \gamma \leqslant \alpha$, associated to a kernel $k(x, y)$, that is to say, for any $x$ not in the support of $f$

$$Tf(x) = \int k(x, y) f(y)\, d\mu(y).$$

Let $\tilde{k}(x, y)$ be the function defined by

$$\sup\{\mu(B_\delta(x, \varepsilon))^{-1} \mu(B_\delta(y, \varepsilon))^{-1} \cdot \iint_{\substack{\delta(u, x) < \varepsilon \\ \delta(v, y) < \varepsilon}} |k(u, v)|\, d\mu(u)\, ds(v) : \delta(x, y) > \varepsilon 4A^2\}. \tag{2.3}$$

We say that $k$ satisfies an $L^r$-Dini condition $1 \leqslant r \leqslant \infty$, if the following conditions hold:

$$\left(\int_{R<\delta(x,y)\leqslant AR} (|\tilde{k}(x,y)|^r + |\tilde{k}(y,x)|^r)\, d\mu(y)\right)^{1/r} \leqslant CR^{-1/r'}, \quad \text{for any } R>0, \tag{2.4}$$

there exists $\eta$, $0<\eta\leqslant\alpha$, such that if $A\delta(y,z)\leqslant R$, then

$$\left(\int_{R<\delta(y,x)\leqslant AR} |k(y,x)-k(z,x)|^r\, d\mu(x)\right)^{1/r} \leqslant CR^{-1/r'}\left(\frac{\delta(y,z)}{R}\right)^{\eta}, \tag{2.5}$$

and

$$\left(\int_{R<\delta(y,x)\leqslant AR} |k(x,y)-k(x,z)|^r\, d\mu(x)\right)^{1/r} \leqslant CR^{-1/r'}\left(\frac{\delta(y,z)}{R}\right)^{\eta}. \tag{2.6}$$

(2.7) LEMMA. *Let $k(x,y)$ be a kernel satisfying* (2.4), *and* $\eta$, $0<\eta\leqslant\alpha$, *then*

$$\int_{B_\delta(x,s)} \delta(x,y)^\eta\, \tilde{k}(x,y)\, d\mu(y) \leqslant C\min(s^\eta, \mu(B_\delta(x,s))^\eta)$$

*Proof.* If $s<k_1\mu(\{x\})$, then the integral is equal to zero. It is enough to assume $s<k_2\mu(X)$. Then

$$\begin{aligned}
&\int_{B_\delta(x,s)} \delta(x,y)^\eta\, \tilde{k}(x,y)\, d\mu(y)\\
&\quad\leqslant \sum_{j=0}^{\infty} \int_{A^{-j}s<\delta(x,y)\, A^{-j+1}s} \delta(x,y)^\eta\, \tilde{k}(x,y)\, d\mu(y)\\
&\quad\leqslant \sum_{j=0}^{\infty} \left(\int_{A^{-j}s<\delta(x,y)\leqslant A^{-j+1}} |\tilde{k}(x,y)|^r\, d\mu(y)\right)^{1/r}\\
&\qquad\times\left(\int_{A^{-j}s<\delta(x,y)\leqslant A^{-j+1}s} \delta(x,y)^{\eta r'}\, d\mu(y)\right)^{1/r'}\\
&\quad\leqslant C\sum_{j=0}^{\infty} (A^{-j}s)^{-1/r'}(A^{-j}s)^\eta(A^{-j}s)^{1/r'} \leqslant Cs^\eta.
\end{aligned}$$

(2.8) DEFINITION. We say that $T$ is weakly bounded of order $\gamma$, $0< \leqslant\alpha$, if $T$ is a linear operator from $\Lambda_0^\gamma$ into $(\Lambda_0^\gamma)'$ and

$$|\langle Tf, g\rangle| \leqslant C\mu(B)^{1+2\gamma}\, |||f|||_\gamma\, |||g|||_\gamma \tag{2.9}$$

holds for any ball $B$ and functions $f$ and $g$ with their supports contained in $B$.

(2.10) LEMMA. *Let $T$ be a continuous linear operator from $\Lambda_0^\gamma$ into $(\Lambda_0^\gamma)'$ for some $\gamma$, $0<\gamma<\alpha$, associated to a kernel satisfying* (2.4) *and* (2.5). *Let us assume that $T$ is weakly bounded of order $\eta$, for some $\eta$, $\gamma \leqslant \eta$. Then, for any $f$, $g$, and $h$ in $\Lambda_0^{\gamma'}$, $\gamma' > \gamma$,*

$$\langle Tgh, f\rangle = \langle Th, fg\rangle + \iint f(x)[g(y)-g(x)] \times k(x, y)\, h(y)\, d\mu(x)\, d\mu(y) \tag{2.11}$$

*holds.*

*Proof.* It is clear that (2.11) holds if $T$ is defined by integration against a locally bounded kernel.

In the general case let $T_t$ be defined from $\Lambda_0^{\gamma'}$ into $(\Lambda_0^{\gamma'})'$ by

$$\langle T_t f, g\rangle = \langle Tf_t, g_t\rangle.$$

$f_t$ and $g_t$ are introduced in Proposition (1.20). Let $B = B_\delta(x_0, r)$ be a ball containing the support of $f$; then for $z \in B$, the support of $s_t(\cdot, z)$ is contained in the ball $B_\delta(x_0, C_t r)$. Thus, the application

$$z \to s_t(\cdot, z), \qquad z \in B,$$

is a $\Lambda^{\gamma'}(B_\delta(x_0, C_t r))$-valued Bochner integrable function with respect to the measure $|f(z)|\, d\mu(z)$. Therefore,

$$T_t(z, y) = \langle Ts_t(\cdot, z), s_t(\cdot, y)\rangle$$

is the kernel associated to $T_t$.

Since by Theorem (1.13) $s_t(\cdot, z) \in \Lambda_0^\eta$, then, by (2.9) (weak boundedness) and (2.4), if $t < k_2 \mu(X)$, we get

$$|T_t(z, y)| \leqslant C\, |\mu(B_\delta(z, t)) + \mu(B_\delta(y, t))|^{-1}.$$

Then (2.11) holds for $T_t$. On the other hand, by Proposition (1.20), $f_t$ converges to $f$ in $\Lambda_0^\gamma$ for $f$ in $\Lambda_0^{\gamma'}$ when $t$ goes to zero. Therefore, $\langle T_t f, g\rangle$ converges to $\langle Tf, g\rangle$ for $f$ and $g$ in $\Lambda_0^{\gamma'}$. Moreover, $T_t(x, y)$ converges pointwise to $k(x, y)$. Using again (2.4) and weak boundedness, it follows that for $t$ sufficiently small, $|T_t(x, y)| \leqslant C\tilde{k}(x, y)$. Then, by the Lebesgue dominated convergence theorem, the right hand side of (2.11) is equal to the limit of

$$\iint f(x)\, |g(y) - g(x)|\, T_t(x, y)\, h(y)\, d\mu(x)\, d\mu(y).$$

Given a ball $B=B_\delta(z,s)$ we define

$$h_B(y)=h(\delta(z,y)/4A^2s), \tag{2.12}$$

where $h$ is the function considered in (1.15).

(2.13) LEMMA. *Let $k(x, y)$ be a kernel satisfying* (2.5) *and $B=B_\delta(z,s)$. Then for any $x_1, x_2 \in B$*

$$\left|\int (k(x_1,y)-k(x_2,y))(1-h_B(y))\,d\mu(y)\right| \leqslant C\left(\frac{\delta(x_1,x_2)}{A\mu(B)}\right)^\eta \leqslant C.$$

*Proof.* It is enough to prove the lemma for $k_1\mu(\{z\})\leqslant s\leqslant k_2\mu(X)$. Then

$$\begin{aligned}
&\int_{4A^2s<\delta(z,y)} |k(x_1,y)-k(x_2,y)|\,d\mu(y)\\
&\quad\leqslant \int_{3As<\delta(x_1,y)} |k(x_1,y)-k(x_2,y)|\,d\mu(y)\\
&\quad\leqslant \sum_{j=0}^{\infty}\int_{A^j3As<\delta(x_1,y)\leqslant A^{j+1}3As} |k(x_1,y)-k(x_2,y)|\,d\mu(y).
\end{aligned}$$

Therefore, by (2.5), this is less than

$$\begin{aligned}
&\sum_{j=0}^{\infty} (A^{j+1}3As)^{1/r'}(A^j3As)^{-1/r'}(\delta(x_1,x_2))^\eta (A^j3As)^{-\eta}\\
&\quad\leqslant C\left(\frac{\delta(x_1,x_2)}{As}\right)^\eta \sum_{j=0}^{\infty}\frac{1}{A^{j\eta}} = C\left(\frac{\delta(x_1,x_2)}{As}\right)^\eta\\
&\quad\leqslant C\left(\frac{\delta(x_1,x_2)}{A\mu(B)}\right)^\eta.
\end{aligned}$$

(2.14) LEMMA. *Let $k(x,y)$ be a kernel satisfying* (2.5), *$B=B_\delta(z,s)$, and $\phi\in\Lambda_b^\gamma$, $0<\gamma\leqslant\alpha$. Then*

$$I_B\phi(x)=\int (k(x,y)-k(z,y))\,\phi(y)(1-h_B(y))\,d\mu(y)$$

*is well defined for any $x\in B$. Moreover, $I_B\phi\in\Lambda^\gamma(B)'$ and $I_B$ satisfies* (2.9) *for functions supported on $B$.*

*Proof.* We can assume $s \leqslant k_2 \mu(X)$, since otherwise $I_B \phi = 0$. Let $\psi \in \Lambda^\gamma(B)$. By Lemma (2.13) we get

$$\left| \int I_B \phi(x)\, \psi(x)\, d\mu(x) \right| \leqslant C \, \|\psi\|_\infty \, \|\phi\|_\infty \int_{\delta(x,z) < s} \left( \frac{\delta(x, z)}{s} \right)^\eta d\mu(x)$$

$$\leqslant C \, \|\psi\|_\infty \, \|\phi\|_\infty \, \mu(B) \leqslant C \mu(B)^{1+\gamma} \, \|\phi\|_\infty \, |||\psi|||_\gamma.$$

If $\phi \in \Lambda^\gamma(B)$ then

$$\left| \int I_B \phi(x)\, \psi(x)\, d\mu(x) \right| \leqslant C \mu(B)^{1+2\gamma} \, |||\phi|||_\gamma \, |||\psi|||_\gamma.$$

(2.15) Definition. Let $T$ be a linear operator from $\Lambda_0^\gamma$ into $(\Lambda_0^\gamma)'$. Given $B = B_\delta(z, r)$ we define $T_B$ from $\Lambda_b^\gamma$ into $\Lambda^\gamma(B)'$ as

$$T_B \phi = T(\phi h_B) + I_B \phi.$$

(2.16) Lemma. *Let $T$ be a continuous linear operator from $\Lambda_0^\gamma$ into $(\Lambda_0^\gamma)'$ associated to a kernel satisfying (2.5). Then for any pair of balls $B_1 = B_\delta(z_1, r_1) \subset B_2 = B_\delta(z_2, r_2)$,*

$$\langle T_{B_1} \phi, \psi \rangle = \langle T_{B_2} \phi, \psi \rangle$$

*holds for any $\psi \in \{\Lambda^\gamma(B_1)\}_0$, the set of functions in $\Lambda^\gamma(B_1)$ with integral equal to zero, and $\phi \in \Lambda_b^\gamma$.*

*Proof.* We have

$$\begin{aligned}
\langle T_{B_2} \phi, \psi \rangle &= \langle T(\phi h_{B_2}), \psi \rangle + \langle I_{B_2} \phi, \psi \rangle \\
&= \langle T(\phi h_{B_1}), \psi \rangle + \langle T\phi(h_{B_2} - h_{B_1}), \psi \rangle \\
&\quad + \int I_{B_2} \phi(x)\, \psi(x)\, d\mu(x) \\
&= \langle T(\phi h_{B_1}), \psi \rangle + \int \psi(x) \int k(x, y)[h_{B_2}(y) - h_{B_1}(y)] \\
&\quad \times \phi(y)\, d\mu(y)\, d\mu(x) + \int I_{B_2} \phi(x)\, \psi(x)\, d\mu(x).
\end{aligned}$$

Clearly,

$$T\phi(h_{B_2} - h_{B_1})(z_1) = \int k(z_1, y)\, \phi(y) [h_{B_2} - h_{B_1}(y)]\, dy,$$

and

$$-I_{B_2}\phi(z_1)=\int [k(z_2, y)-k(z_1, y)][1-h_{B_2}(y)]\,\phi(y)\,dy.$$

Then, since $\int \psi=0$, we get

$$\begin{aligned}\langle T_{B_2}\phi, \psi\rangle &= \langle T(\phi h_{B_1}), \psi\rangle \\ &\quad+\int \psi(x)\int [k(x, y)-k(z_1, y)]\,\phi(y)[1-h_{B_1}(y)]\,d\mu(y)\,d\mu(x) \\ &= \langle T(\phi h_{B_1}), \psi\rangle+\langle I_{B_1}\phi, \psi\rangle=\langle T_{B_1}\phi, \psi\rangle.\end{aligned}$$

It is clear that

$$\langle T_B\phi, \psi\rangle=\langle T\phi, \psi\rangle,$$

whenever $\operatorname{supp}(\phi)\subset B_1$. Then Lemma (2.16) allows us to introduce the following extension of $T$.

(2.17) DEFINITION. Let $T$ be a continuous linear operator from $\Lambda_0^\gamma$ into $(\Lambda_0^\gamma)'$ associated to a kernel satisfying (2.5). For any $\phi\in\Lambda_b^\gamma$ and $\psi\in\{\Lambda_0^\gamma\}_0$ with $\operatorname{supp}\psi\subset B$, we define

$$\langle T\phi, \psi\rangle=\langle T_B\phi, \psi\rangle.$$

(2.18) LEMMA. *Let $T$ be a continuous linear operator from $\Lambda_0^\gamma$ into $(\Lambda_0^\gamma)'$ associated to a kernel $k(x, y)$ satisfying* (2.5), *and such that $T$ is weakly bounded of order $\gamma$. Assume that $T1=g$ with $g\in$ B.M.O. Then, given a ball $B=B_\delta(z, r)$, there exists a constant $c_B$ such that for any $\phi\in\Lambda^\gamma(B)$*

$$\langle Th_B, \phi\rangle=\int (g(x)-m_B(g))\,\phi(x)\,d\mu(x)+c_B\int \phi(x)\,d\mu(x)$$
$$-\int I_B1(x)\,\phi(x)\,d\mu(x).$$

*Moreover,* $\sup_B|c_B|\leqslant C$, *where $C$ is an absolute constant depending on the constants appearing in* (2.5), (2.9), *and* $\|g\|_{\mathrm{BMO}}$.

*Proof.* Given the ball $B=B_\delta(z, r)$, consider the function

$$h'_B(y)=h(A^2\delta(z, y)/r),$$

where $h$ is the function considered in (1.15). This function is supported in $B_\delta(z, r/A)$. Therefore the function

$$l_B(y) = \left(\int h'_B(y)\, d\mu(y)\right)^{-1} h'_B(y)$$

is supported in $B_\delta(z, r/A)$ and its integral is equal to one.

Then, given $\phi \in \Lambda^\gamma(B)$, we have

$$\begin{aligned}
&\langle Th_B + I_B 1, \phi \rangle \\
&= \left\langle Th_B + I_B 1, \phi - \left(\int \phi\right) l_B \right\rangle + \left\langle Th_B + I_B 1, \left(\int \phi\right) l_B \right\rangle \\
&= \left\langle g, \phi - \left(\int \phi\right) l_B \right\rangle + \langle Th_B + I_B 1, l_B \rangle \int \phi(x)\, d\mu(x) \\
&= \int (g(x) - m_B g)\, \phi(x)\, d\mu(x) + m_B g \int \phi(x)\, d\mu(x) \\
&\quad - \langle g, l_B \rangle \int \phi(x)\, d\mu(x) + \langle Th_B + I_B 1, l_B \rangle \int \phi(x)\, d\mu(x) \\
&= \int (g(x) - m_B g)\, \phi(x)\, d\mu(x) + c_B \int \phi(x)\, d\mu(x),
\end{aligned}$$

where

$$c_B = \langle Th_B + I_B 1 - (g - m_B(g)), l_B \rangle.$$

It is easy to check that

$$|||h'_B|||_\gamma \leqslant C\mu(B)^{-\gamma} \qquad \text{and} \qquad |||l_B|||_\gamma \leqslant C\mu(B)^{-(1+\gamma)};$$

then, by weak boundedness (2.9),

$$|\langle Th_B, l_B \rangle| \leqslant C\mu(B)^{1+2\gamma}\, |||h_B|||_\gamma\, |||l_B|||_\gamma \leqslant C,$$

and, by Lemma (2.13),

$$|\langle I_B 1, l_B \rangle| \leqslant C\mu(B)^{1+\gamma}\, |||l_B|||_\gamma \leqslant C.$$

Finally, it is clear that

$$|\langle g - m_B g, l_B \rangle| \leqslant C\, \|g\|_{\mathrm{BMO}}.$$

These estimates show that $|c_B|$ is bounded by a constant $C$ not depending on $B$.

(2.19) COROLLARY. *Let $T$ be an operator satisfying all the conditions of Lemma* (2.18). *Then $g \in L^\infty$ if and only if $|\langle Th_B, \phi\rangle| \leqslant C \|\phi\|_1$ for any $\phi \in \Lambda^\gamma(B)$, where $C$ is an absolute constant not depending on $B$.*

(2.20) DEFINITION. Let $T$ be an operator satisfying the conditions of Lemma (18.1). Given $\phi \in \Lambda^\gamma(B)$ and $x \in B$, we define

$$T^B\phi(x) = (g(x) - m_B g)\,\phi(x) + c_B\phi(x) - I_B 1(x)\,\phi(x)$$
$$+ \int [\phi(y) - \phi(x)]\, k(x, y)\, h_B(y)\, d\mu(y).$$

(2.21) LEMMA. *Let $B_1 = B_\delta(z_1, r_1) \subset B_2 = B_\delta(z_2, r_2)$ and $\phi \in \Lambda^\gamma(B_1)$. Then*

$$T^{B_2}\phi(x) = T^{B_1}\phi(x), \qquad \textit{for} \quad x \in B_1,$$

*Proof.* First observe that

$$\begin{aligned} c_{B_2} - c_{B_1} &= \langle Th_{B_2} + I_{B_2}1 - (g - m_2 g), l_{B_2} - l_{B_1}\rangle \\ &\quad + \langle Th_{B_2} + I_{B_2}1 - (g - m_{B_2} g), l_{B_1}\rangle \\ &\quad - \langle Th_{B_1} + I_{B_1}1 - (g - m_{B_1} g), l_{B_1}\rangle \\ &= \langle T(h_{B_2} - h_{B_1}) + I_{B_2}1 - I_{B_1}1, l_{B_1}\rangle + m_{B_2} g - m_{B_1} g. \end{aligned} \tag{2.22}$$

On the other hand

$$\begin{aligned} & I_{B_1}1(x) - I_{B_2}1(x) \\ &\quad = \int k(x, y)(h_{B_2}(y) - h_{B_1}(y))\, d\mu(y) \\ &\qquad + \int (k(z_2, y) - k(z_1, y))(1 - h_{B_2}(y))\, d\mu(y) \\ &\qquad - \int k(z_1, y)(h_{B_2}(y) - h_{B_1}(y))\, d\mu(y) \\ &\quad = T(h_{B_2} - h_{B_1})(x) - I_{B_2}1(z_1) - T(h_{B_2} - h_{B_1})(z_1); \end{aligned} \tag{2.23}$$

consequently,

$$\begin{aligned} \langle T(h_{B_2} - h_{B_1}) - I_{B_2}1 - I_{B_1}1, l_B\rangle &= \langle I_{B_2}1(z_1) + T(h_{B_2} - h_{B_1})(z_1), l_B\rangle \\ &= I_{B_2}1(z_1) + T(h_{B_2} - h_{B_1})(z_1). \end{aligned} \tag{2.24}$$

Moreover,

$$\int |\phi(y)-\phi(x)|\, k(x, y)(h_{B_2}(y)-h_{B_1}(y))\, d\mu(y)$$

$$= -\phi(x) \int k(x, y)(h_{B_2}(y)-h_{B_1}(y))\, d\mu(y)$$

$$= -\phi(x)\, T(h_{B_2}-h_{B_1})(x). \tag{2.25}$$

Then passing up together (2.22), (2.23), (2.24), and (2.25), we obtain the result sought.

Given $\phi \in \Lambda_0^\gamma$, Lemma (2.21) allows us to define $\tilde{T}\phi$ as the function

$$\tilde{T}\phi(x) = T^B\phi(x), \tag{2.26}$$

where $B$ is a ball containing the support of $\phi$ and $x \in B$.

Now we can prove the main result.

(2.27) THEOREM. *Let $T$ be a continuous linear operator from $\Lambda_0^\gamma$ into $(\Lambda_0^\gamma)'$, for every $0<\gamma\leqslant\alpha$, with an associated kernel satisfying* (2.4) *and* (2.5), *and such that $T1=g$, $g\in$ BMO. Then for any $\eta$, $0<\eta\leqslant\alpha$, the following conditions are equivalent*:

$$T \text{ is weakly bounded of order } \eta. \tag{2.28}$$

$$\text{For any } \phi\in\Lambda_0^\eta,\ T\phi=\tilde{T}\phi. \tag{2.29}$$

*Proof.* Let us show that (2.28) implies (2.29). Let $\psi$, $\phi\in\Lambda^\eta(B)$. Then, by Lemma (2.10),

$$\langle T\phi, \psi\rangle = \langle Th_B, \phi\psi\rangle + \iint \psi(x)[\phi(y)-\phi(x)]\, k(x, y)\, h_B(y)\, d\mu(x)\, d\mu(y),$$

and (2.29) follows by applying Lemma (2.18). Let us prove the converse. Given $B=B_\delta(z, s)$, we apply Lemma (2.7), getting

$$\left|\int [\phi(y)-\phi(x)]\, k(x, y)\, h_B(y)\, d\mu(y)\right|$$

$$\leqslant C\, |||\phi|||_\eta \int_{B\delta(z,\, As)} \delta(x, y)^\eta\, \tilde{k}(x, y)\, d\mu(y)$$

$$\leqslant C\, |||\phi|||_\eta \int_{B\delta(x,\, 2A^2 s)} \delta(x, y)^\eta\, \tilde{k}(x, y)\, d\mu(y)$$

$$\leqslant C\, |||\phi|||_\eta\, \mu(B)^\eta;$$

therefore, for $\phi, \psi \in \Lambda^\gamma(B)$,

$$|\langle T\phi, \psi\rangle| \leqslant \left|\int (g(x)-m_B g)\,\phi(x)\,\psi(x)\,d\mu(x)\right| + C\int |\phi(x)\,\psi(x)|\,d\mu(x)$$

$$+\int |I_B 1(x)|\,|\phi(x)\,\psi(x)|\,d\mu(x) + C\,|||\phi|||_\eta\,\mu(B)^\eta \int |\psi(x)|\,d\mu(x)$$

$$\leqslant (\|g\|_{\mathrm{BMO}} + C)\,\|\phi\|_\infty\,\|\psi\|_\infty\,\mu(B) + |||\phi|||_\eta\,\mu(B)^{1+2\eta}\,|||\psi|||_\eta$$

$$\leqslant (\|g\|_{\mathrm{BMO}} + C)\,\mu(B)^{1+2\eta}\,|||\phi|||_\eta\,|||\psi|||_\eta.$$

(2.30) *Remark.* Consider the operator

$$T\phi(x) = g(x)\,\phi(x).$$

If $T$ is weakly bounded of order $\gamma$ then, for every ball $B$,

$$|\langle Th_B, l_B\rangle| \leqslant C\mu(B)^{1+2\gamma}\,|||h_B|||_\gamma\,|||l_B|||_\gamma \leqslant C.$$

This means that for every $B$,

$$\left|\int g(x)\,l_B(x)\,dx\right| \leqslant C,$$

and by differentiation (assuming that it holds) we get $|g(x)| \leqslant C$.

(2.31) COROLLARY. *Let $T$ be an operator satisfying the hypotheses and conclusions of Theorem* (2.27). *Then the kernel $k(x, y)$ is zero if and only if $T\phi(x) = h(x)\,\phi(x)$, with $h \in L^\infty$.*

*Proof.* Assume that the kernel is zero. Then

$$T\phi(x) = (g(x) - m_B g)\,\phi(x) + c_B\phi(x) = (g(x) - m_B g + c_B)\,\phi(x).$$

Therefore, by Remark (2.30), $g(x) - m_B g + c_B$ must be bounded, but since $c_B$ is bounded this tells us that $g$ must be bounded. In other words, $h(x) = g(x) - m_B g + c_B$.

(2.32) THEOREM. *Let $T$ be a continuous linear operator defined from $\Lambda_0^\gamma$ into $(\Lambda_0^\gamma)'$ for every $\gamma$, $0<\gamma\leqslant\alpha$, weakly bounded of order $\eta$ for some $\eta$, $0<\eta\leqslant\alpha$, and with an associated kernel satisfying* (2.4) *and* (2.5) *for $\eta+\varepsilon$ with $\varepsilon>0$. Assume that $T1 = g$ belongs to B.M.O. Then $T$ satisfies*

$$|||T\phi|||_\eta \leqslant C\,|||\phi|||_\eta \qquad \textit{and} \qquad T\phi \textit{ is a bounded function,}$$

*if and only if $T1 = 0$.*

*Proof.* Assume first that $T1=0$. Given $x_1, x_2 \in X$, $\phi \in \Lambda_0^\eta$, and $B_1 = B_\delta(x_1, \delta(x_1, x_2))$, we consider $B = B_\delta(x_1, s)$ such that $x_1$, $x_2 \in B$, $\operatorname{supp} \phi \subset B$, and $A\delta(x_1, x_2) < s$.

We want to show that $T^B\phi$ is a Lipschitz function. Let us estimate the difference

$$\begin{aligned}
&|T^B\phi(x_1) - T^B\phi(x_2)| \\
&\quad \leqslant c_B\, |\phi(x_1) - \phi(x_2)| \\
&\qquad + |I_B 1(x_1)\, \phi(x_1) - I_B 1(x_2)\, \phi(x_2)| \\
&\qquad + \left| \int [\phi(y) - \phi(x_1)]\, k(x_2, y)\, h_B(y)\, d\mu(y) \right. \\
&\qquad \left. - \int [\phi(y) - \phi(x_2)]\, k(x_2, y)\, h_B(y)\, d\mu(y) \right| \\
&\quad = \sigma_1 + \sigma_2 + \sigma_3.
\end{aligned}$$

We have

$$\sigma_1 \leqslant \sup_B |c_B|\ |||\phi|||_\eta\, \delta(x_1, x_2)^\eta.$$

On the other hand, since $I_B 1(x_1) = 0$, by Lemma (2.13) we have

$$\sigma_2 \leqslant C\, \|\phi\|_\infty \left( \frac{\delta(x_1, x_2)}{A\mu(B)} \right)^\eta \leqslant C\, |||\phi|||_\eta\, \delta(x_1, x_2)^\eta.$$

As for $\sigma_3$, we have

$$\begin{aligned}
\sigma_3 &\leqslant \left| \int [\phi(y) - \phi(x_1)]\, k(x_1, y)\, h_B(y)\, h_{B_1}(y)\, d\mu(y) \right| \\
&\quad + \left| \int [\phi(y) - \phi(x_2)]\, k(x_2, y)\, h_B(y)\, h_{B_1}(y)\, d\mu(y) \right| \\
&\quad + \left| \int \{ [\phi(y) - \phi(x_1)]\, k(x_1, y) \right. \\
&\quad \left. - [\phi(y) - \phi(x_2)]\, k(x_2, y) \}\, h_B(y)(1 - h_{B_1}(y))\, d\mu(y) \right| \\
&= \sigma_{31} + \sigma_{31} + \sigma_{33}.
\end{aligned}$$

By Lemma (2.7) we have

$$\sigma_{31} \leqslant C \,|||\phi|||_\eta \int \delta(x_1, y)^\eta \,\tilde{k}(x_1, y)\, h_B(y)\, h_{B_1}(y)\, d\mu(y)$$

$$\leqslant C \,|||\phi|||_\eta \int_{\delta(x_1, y) < A^2\delta(x_1, x_2)} \delta(x_1, y)^\eta \,\tilde{k}(x_1, y)\, d\mu(y)$$

$$\leqslant C \,|||\phi|||_\eta\, \delta(x_1, x_2)^\eta.$$

Analogously,

$$\sigma_{32} \leqslant C \,|||\phi|||_\eta \int_{\delta(x_1, y) < A^2\delta(x_1, x_2)} \delta(x_2, y)^\eta \,\tilde{k}(x_2, y)\, d\mu(y)$$

$$\leqslant C \,|||\phi|||_\eta \int_{\delta(x_2, y) < A^3\delta(x_1, x_2)} \delta(x_2, y)^\eta \,\tilde{k}(x_2, y)\, d\mu(y)$$

$$\leqslant C \,|||\phi|||_\eta\, \delta(x_1, x_2)^\eta.$$

It is clear that

$$\sigma_{33} \leqslant |\phi(x_2) - \phi(x_1)| \left| \int K(x_1, y)\, h_B(y)(1 - h_{B_1}(y))\, d\mu(y) \right|$$

$$+ \int |\phi(y) - \phi(x_2)|$$

$$\times |K(x_1, y) - K(x_2, y)|\, h_B(y)(1 - h_{B_1}(y))\, d\mu(y)$$

$$= \sigma_{331} + \sigma_{332}.$$

By the definition of the associated kernel and Corollary (2.19),

$$\sigma_{331} \leqslant C \,|||\phi|||_\eta\, \delta(x_1, x_2)^\eta \,(|Th_B(x_1)| + |Th_{B_1}(x_1)|)$$
$$\leqslant C \,|||\phi|||_\eta\, \delta(x_1, x_2)^\eta.$$

On the other hand, by (2.5)

$$\sigma_{332} \leqslant |||\phi|||_\eta \int_{A\delta(x_1, x_2) < \delta(x_1, y)} \delta(x_2, y)^\eta\, |k(x_1, y) - k(x_2, y)|\, d\mu(y)$$

$$\leqslant |||\phi|||_\eta \sum_{j=0}^{\infty} \left( \int_{A^jA\delta(x_1, x_2) < \delta(x_1, y) < A^{j+1}A\delta(x_1, x_2)} |k(x_1, y) - k(x_2, y)|^r\, d\mu(y) \right)^{1/r}$$

$$\cdot \left( \int_{A^jA\delta(x_1, x_2) < \delta(x_1, y) < A^{j+1}A\delta(x_1, x_2)} \delta(x_2, y)^{\eta r'}\, d\mu(y) \right)^{1/r'}$$

$$\leqslant C\, |||\phi|||_\eta \sum_{j=0}^{\infty} (A^j\delta(x_1, x_2))^{-1/r'} \left(\frac{\delta(x_1, x_2)}{A^j\delta(x_1, x_2)}\right)^{\eta+\varepsilon}$$
$$\cdot (A^j\delta(x_1, x_2))^\eta \cdot (A^j\delta(x_1, x_2))^{1/r'}$$
$$\leqslant C\, |||\phi|||_\eta\, \delta(x_1, x_2)^\eta \sum_{j=0}^{\infty} A^{-j\varepsilon} \leqslant C\, |||\phi|||_\eta\, \delta(x_1, x_2)^\eta.$$

Finally, we shall prove that if $\operatorname{supp} \phi \subset B_0$,

$$\|T\phi(x)\|_\infty \leqslant C\, |||\phi|||_\eta\, \mu(B_0)^\eta.$$

It is enough to show that

$$\left|\int [\phi(y) - \phi(x)]\, k(x, y)\, h_B(y)\, d\mu(y)\right| \leqslant C\, |||\phi|||_\eta\, (\operatorname{diam}(\operatorname{supp} \phi))^\eta,$$

for any sufficiently large $B$.

Let $B_0 = B_\delta(z, r_0)$, $B_1 = B_\delta(z, A^2 r_0)$, and $B = B_\delta(z, r)$ be such that $\operatorname{supp} \phi \subset B_0$ and $A^3 r_0 < r$.

Assume first that $x \notin B_\delta(z, A^2 r_0)$. Then

$$\left|\int [\phi(y) - \phi(x)]\, k(x, y)\, h_B(y)\, d\mu(y)\right| = \left|\int \phi(y)\, k(x, y)\, h_B(y)\, d\mu(y)\right|$$
$$= \left|\int \phi(y)\, k(x, y)\, d\mu(y)\right|.$$

In this integral the relevant points $y$ satisfy $\delta(z, y) < r_0$, since $y \in \operatorname{supp} \phi$, and $\delta(x, z) > A^2 r_0$.

Then, if $A^j r_0 < \delta(x, z) \leqslant A^{j+1} r_0$, $j \geqslant 2$, we have $A^{j-2}(A-1)\, r_0 < \delta(x, y) \leqslant 2A^{j+2} r_0$.

Therefore, for $x \in B(z, A^{j+1} r_0) \backslash B(z, A^j r_0)$, $j \geqslant 2$, we have

$$\left|\int \phi(y)\, k(x, y)\, d\mu(y)\right|$$
$$= \left|\int_{A^{j-2}(A-1)\, r_0 < \delta(x, y) < 2A^{j+2} r_0} \phi(y)\, k(x, y)\, d\mu(y)\right|$$
$$\leqslant \|\phi\|_\infty \int_{A^{j-2}(A-1)\, r_0 < \delta(x, y) < 2A^{j+2} r_0} \tilde{k}(x, y)\, d\mu(y)$$
$$\leqslant C\, \|\phi\|_\infty \left(\int_{A^{j-2} r_0 < \delta(x, y) < 2A^{j+2} r_0} \tilde{k}(x, y)^r\, d\mu(y)\right)^{1/r} \left(\mu(B_\delta(x, 2A^{j+2} r_0)\right)^{1/r'}$$
$$\leqslant C\, \|\phi\|_\infty \leqslant C\, |||\phi|||_\eta\, \mu(B_0)^\eta.$$

If $x \in B(z, A^2 r_0)$, using (2.4), (2.19), and (2.7), we get

$$\left| \int [\phi(y) - \phi(x)] \, k(x, y) \, h_B(y) \, d\mu(y) \right|$$

$$\leqslant \left| \int [\phi(y) - \phi(x)] \, k(x, y) \, h_B(y) \, h_{B_1}(y) \, d\mu(y) \right|$$

$$+ \left| \int [\phi(y) - \phi(x)] \, k(x, y) \, h_B(y)(1 - h_{B_1}(y)) \, d\mu(y) \right|$$

$$\leqslant \left| C \int_{\delta(x, y) \leqslant 2A^3 r_0} |||\phi|||_\eta \, \delta(x, y)^\eta \, \tilde{k}(x, y) \, d\mu(y) \right|$$

$$+ \left| \phi(x) \int k(x, y)(h_B(y) - h_{B_1}(y)) \, d\mu(y) \right|$$

$$\leqslant C \, |||\phi|||_\eta \, \mu(B_0)^\eta + C \, \|\phi\|_\infty \leqslant C \, |||\phi|||_\eta \, \mu(B_0)^\eta.$$

In order to prove the converse, assume that $T$ is continuous from $\Lambda_0^\eta$ into $\Lambda_b^\eta$. Then, by the computations above, this implies that the function defined for $x \in B$ as

$$(g(x) - m_B \, g)\phi(x)$$

is a Lipschitz function for any $\phi \in \Lambda_0^\eta$; moreover

$$|||(g(\cdot) - m_B \, g) \, \phi(\cdot)|||_\eta \leqslant C \, |||\phi|||_\eta. \tag{2.33}$$

Now take $x_1$, $x_2$, and $B = B_\delta(z, r)$ such that $x_1, x_2 \in B$; then by (2.33),

$$\begin{aligned} |g(x_1) - g(x_2)| &= |(g(x_1) - m_B \, g) - (g(x_2) - m_B \, g)| \\ &= |(g(x_1) - m_B \, g) \, h_B(x_1) - (g(x_2) - m_B \, g) \, h_B(x_2)| \\ &\leqslant C \, |||h_B|||_\eta \leqslant C r^{-\eta}. \end{aligned}$$

Now letting $r \to \infty$ we obtain $g(x_1) = g(x_2)$. In other words, $g(x)$ is constant and $T1 = 0$.

Let us define

$$t_j(x, y) = s_{A^{-j}}(x, y) - s_{A^{-j-1}}(x, y),$$

where $s_t(x, y)$ is the approximation of the identity introduced in Theorem (1.13). We define

$$k_{j_1, j_2}(x, y) = \langle t_{j_1}(x, \cdot), T t_{j_2}(y, \cdot \cdot) \rangle.$$

(2.34) THEOREM. *Let $T$ be a continuous linear operator defined from $\Lambda_0^\gamma$ into $(\Lambda_0^\gamma)'$ for every $\gamma$, $0<\gamma\leqslant\alpha$, weakly bounded of order $\eta$, for some $\eta$, $0<\eta\leqslant\alpha$, and with an associated kernel satisfying* (13.1) *and* (13.2) *with $1/r'+\eta>1$. Assume that $T1=0$. Then the following inequality holds for $j_1\geqslant j_2$:*

$$|k_{j_1,j_2}(x,y)|\leqslant\frac{A^{\eta(j_2-j_1)}A^{j_2}A^{-j_2(1/r'+\eta)}}{\delta(x,y)^{1/r'+\eta}+A^{-j_2(1/r'+\eta)}}.$$

*Proof.* Let $B$ be a ball with radius bigger than $A^{-j_2}$ and such that

$$\{z:\delta(x,z)<CA^{-j_1}\}\cup\{z:\delta(y,z)<CA^{-j_2}\}\subset B.$$

Theorem (2.27) tells us that

$$\begin{aligned}k_{j_1,j_2}(x,y)&=\langle t_{j_1}(x,\cdot),T^B t_{j_2}(y,\cdot)\rangle\\&=c_B\int t_{j_1}(x,z)\,t_{j_2}(y,z)\,d\mu(z)\\&\quad-\int t_{j_1}(x,z)\,I_B1(z)\,t_{j_2}(y,z)\,d\mu(z)\\&\quad+\int t_{j_1}(x,z)\left(\int (t_{j_2}(y,u)-t_{j_2}(y,z))\,k(z,u)\,d\mu(u)\right)d\mu(z).\end{aligned}\tag{2.35}$$

Assume first that $\delta(x,y)\leqslant A(A+1)\,A^{-j_2}$. Then, by Theorem (1.13), we have

$$\begin{aligned}&\left|\int t_{j_1}(x,z)\,t_{j_2}(y,z)\,d\mu(z)\right|\\&\quad=\left|\int t_{j_1}(x,z)(t_{j_2}(y,z)-t_{j_2}(y,x))\,d\mu(z)\right|\\&\quad\leqslant C\int t_{j_1}(x,z)\,A^{j_2(1+\eta)}\delta(x,z)^\eta\,dz\\&\quad<CA^{-j_1\eta}A^{j_2(1+\eta)}\leqslant C\,\frac{A^{-j_1\eta}}{\delta(x,y)^{1+\eta}+A^{-j_2(1+\eta)}}\\&\quad=CA^{\eta(j_2-j_1)}\,\frac{A^{-j_2(1+\eta)}A^{j_2}}{\delta(x,y)^{1+\eta}+A^{-j_2(1+\eta)}}.\end{aligned}$$

Analogously, by Lemma (2.13), we have

$$\begin{aligned}
&\left|\int t_{j_1}(x, z)\, I_B 1(z)\, t_{j_2}(y, z)\, d\mu(z)\right| \\
&\quad = \left|\int t_{j_1}(x, z)[I_B 1(z)\, t_{j_2}(y, z) - I_B 1(x)\, t_{j_2}(y, x)]\, d\mu(z)\right| \\
&\quad \leqslant C \int t_{j_1}(x, z)\, A^{j_2(1+\eta)} \delta(x, z)^\eta\, d\mu(z) \\
&\quad \leqslant C A^{\eta(j_2 - j_1)} \frac{A^{-j_2(1+\eta)}}{\delta(x, y)^{1+\eta} + A^{-j_2(1+\eta)}}.
\end{aligned}$$

Analogously, by Theorem (2.32), we have

$$\begin{aligned}
&\left|\int t_{j_1}(x, z) \left(\int (t_{j_2}(y, u) - t_{j_2}(y, z))\, k(z, u)\, d\mu(u)\right) d\mu(z)\right| \\
&\quad = \left|\int t_{j_1}(x, z) \left(\int (t_{j_2}(y, u) - t_{j_2}(y, z))\, k(z, u)\, d\mu(u)\right.\right. \\
&\qquad \left.\left. - \int (t_{j_2}(y, u) - t_{j_2}(y, x))\, k(x, u)\, d\mu(u)\right) d\mu(z)\right| \\
&\quad \leqslant \int t_{j_1}(x, z)\, A^{j_2(1+\eta)} \delta(x, z)^\eta\, d\mu(z) \\
&\quad \leqslant C A^{\eta(j_2 - j_1)} \frac{A^{-j_2(1+\eta)}}{\delta(x, y)^{1+\eta} + A^{-j_2(1+\eta)}}.
\end{aligned}$$

Let us assume now that $\delta(x, y) > A(A+1)\, A^{-j_2}$. If $t_{j_2}(y, z) \neq 0$, then

$$A(A+1)\, A^{j_2} < \delta(x, y) \leqslant A(\delta(x, z) + \delta(z, y)) \leqslant A(\delta(x, z) + A^{-j_2}).$$

In other words,

$$\delta(x, z) > A A^{-j_2} > A^{-j_2} \geqslant A^{-j_1}.$$

This tells us that $t_{j_1}(x, z) = 0$ and therefore the first two integrals in (2.35) are zero.

We estimate now

$$\int t_{j_1}(x, z) \left(\int (t_{j_2}(y, u) - t_{j_2}(y, z))\, k(z, u)\, d\mu(u)\right) d\mu(z).$$

As we have seen before, if $t_{j_2}(y, z) \neq 0$, then $t_{j_1}(x, z) = 0$. Then it is enough to estimate

$$\int t_{j_1}(x, z) \left( \int t_{j_2}(y, u)\, k(z, u)\, d\mu(u) \right) d\mu(z)$$

$$= \int t_{j_1}(x, z) \left( \int t_{j_2}(y, u)(k(z, u) - k(x, u))\, d\mu(u) \right) d\mu(z).$$

Observe that

$$\delta(x, y) \leqslant A(\delta(x, u) + \delta(u, y)) < A(\delta(x, u) + A^{-j_2})$$

$$\leqslant A\delta(x, u) + \frac{1}{A+1}\, \delta(x, y);$$

then $\delta(x, u)(A+1) \geqslant \delta(x, y)$, and moreover

$$\delta(x, z) < A^{-j_1} \leqslant A^{-j_2} < \frac{1}{A(A+1)}\, \delta(x, y). \tag{2.36}$$

Therefore, if we define

$$E = \{u : \delta(x, y) < (A+1)\, \delta(x, u);\ A(A+1)\, \delta(x, z) < \delta(x, y)\}$$

and

$$E_h = \left\{ u : \frac{A^h}{A+1}\, \delta(x, y) < \delta(x, u) \leqslant \frac{A^{h+1}}{A+1}\, \delta(x, y), \right.$$

$$\left. \delta(x, z) < \frac{1}{A(A+1)}\, \delta(x, y) \right\},$$

we obtain by Hölder's inequality that the last integral is less than or equal to

$$\int t_{j_1}(x, z) \left\{ \left( \int |t_{j_2}(y, u)|^{r'}\, d\mu(\mu) \right)^{1/r'} \right.$$

$$\left. \times \left( \int_E |k(z, u) - k(x, u)|^r\, d\mu(u) \right)^{1/r} \right\} d\mu(z)$$

$$\leqslant C \int t_{j_1}(x, z)\, A^{j_2} A^{-j_2(1/r')}$$

$$\times \left( \sum_h \int_{E_h} |k(z, u) - k(x, u)|^r\, d\mu(u) \right)^{1/r} d\mu(z).$$

By (2.5), this is less than

$$C \int t_{j_1}(x, z)\, A^{j_2} A^{-j_2(1/r')} \left( \sum_h \left( A^h \delta(x, y) \right)^{-r/r'} \left( \frac{\delta(x, z)}{A^h \delta(x, y)} \right)^{\eta r} \right)^{1/r} d\mu(z)$$

$$\leqslant C \int t_{j_1}(x, z)\, A^{j_2} A^{-j_2(1/r')} \delta(x, y)^{-(1/r' + \eta)} A^{-j_1 \eta}$$

$$\times \left( \sum_h A^{-h(r/r' + \eta r)} \right)^{1/r} d\mu(z)$$

$$\leqslant C \frac{A^{j_2} A^{-j_2(1/r')} A^{-j_1 \eta}}{\delta(x, y)^{1/r' + \eta}} \leqslant C \frac{A^{\eta(j_2 - j_1)} A^{-j_2(1/r' + \eta)}}{\delta(x, y)^{1/r' + \eta} + A^{-j_2(1/r' + \eta)}}.$$

(2.37) COROLLARY. *Under the conditions of Theorem* (2.34), *if we define*

$$T_{j_1, j_2} f(x) = \int k_{j_1, j_2}(x, y)\, f(y)\, dy,$$

*then* $T_{j_1, j_2}$ *is a bounded operator from* $L^2(X, d\mu)$ *into* $L^2(X, d\mu)$ *with norm less than or equal to* $A^{\eta(j_2 - j_1)}$.

(2.38) APPLICATION. *Assume that* $k(x, y)$ *is a singular integral kernel* $k(x, y)$ *satisfying* (2.4), (2.5) *for* $\eta + \varepsilon$ *with* $\varepsilon > 0$ *and the following cancellation property*:

$$\text{let } 0 < r < R < \infty, \text{ then}$$
$$\int_{r < \delta(x, y) \leqslant R} k(x, y)\, d\mu(y) = 0, \qquad \text{for every } x \in X. \tag{2.39}$$

*Under these conditions we define for* $\phi \in \Lambda_0^\eta$

$$Tf(x) = \lim_{r \to 0} \int_{r < \delta(x, y)} k(x, y)\, \phi(y)\, dy. \tag{2.40}$$

*Then the operator* $T$ *is well defined and maps* $\Lambda_0^\eta$ *into* $\Lambda_b^\eta$.

In order to prove this result we show that $T$ satisfies the hypotheses of Theorem (2.32) and in addition, $T1 = 0$.

Let $x$ be a fixed point in $X$ and $\phi \in \Lambda_0^\eta$ such that $\operatorname{supp} \phi \subset B(z, s)$, $s \leqslant k_2 \mu(z)$. Then. by (2.39), we have

$$\begin{aligned}
T\phi(x) &= \lim_{r\to 0} \int_{r<\delta(x,y)} k(x,y)\,\phi(y)\,dy \\
&= \lim_{r\to 0} \int_{r<\delta(x,y)\leqslant A(\delta(x,z)+s)} k(x,y)\,\phi(y)\,dy \\
&= \lim_{r\to 0} \int_{r<\delta(x,y)\leqslant A(\delta(x,z)+s)} k(x,y)(\phi(y)-\phi(x))\,dy \\
&= \int_{\delta(x,y)\leqslant A(\delta(x,z)+s)} k(x,y)(\phi(y)-\phi(x))\,dy.
\end{aligned}$$

The last integral converges since, by Lemma (2.7),

$$\begin{aligned}
&\int_{\delta(x,y)\leqslant A(\delta(x,z)+s)} |k(x,y)(\phi(y)-\phi(x))|\,dy \\
&\quad\leqslant |||\phi|||_\eta \int_{\delta(x,y)\leqslant A(\delta(x,z)+s)} \tilde{k}(x,y)\,\delta(x,y)^\eta\,dy \\
&\quad\leqslant C\,|||\phi|||_\eta\,A(\delta(x,z)+s)^\eta.
\end{aligned}$$

Therefore, (2.40) is well defined. Using the same kind of argument, if $(\operatorname{supp}\phi)\cup(\operatorname{supp}\phi)\subset B_\delta(z,s)$, we have

$$\begin{aligned}
|\langle T\phi,\psi\rangle| &= \left|\int\left(\lim_{r\to 0}\int_{r<\delta(x,y)} k(x,y)\,\phi(y)\,dy\right)\psi(x)\,dx\right| \\
&\leqslant C\,|||\phi|||_\eta \int (\delta(x,z)+s)^\eta\,|\psi(x)|\,dx \\
&\leqslant Cs^\eta\,|||\phi|||_\eta \int |\psi(x)|\,dx \\
&\leqslant C\mu(B_\delta(z,s))^{1+2\eta}\,|||\phi|||_\eta\,|||\psi|||_\eta.
\end{aligned}$$

Finally, let us compute $T1$. Assume that $\psi\in\{\Lambda_0^\eta\}_0$ with $\operatorname{supp}\psi\subset B=B_\delta(z,s)$. Then

$$\begin{aligned}
&\langle Th_B,\psi\rangle + \langle I_B 1,\psi\rangle \\
&\quad= \int\left(\lim_{r\to 0}\int_{r<\delta(x,y)} k(x,y)\,h_B(y)\,dy\right)\psi(x)\,dx \\
&\qquad+ \int\left(\int (k(x,y)-k(z,y))\,(1-h_B(y))\,dy\right)\psi(x)\,dx \\
&\quad= \int\left[\lim_{r\to 0}\int_{r<\delta(x,y)} k(x,y)\,h_B(y)\,dy\right. \\
&\qquad\left.+ \int (k(x,y)-k(z,y))(1-h_B(y))\,dy\right]\psi(x)\,dx.
\end{aligned}$$

By (2.39), this integral is equal to

$$\int \left| \lim_{\substack{r \to 0 \\ R \to \infty}} \int_{r < \delta(x,y) \leqslant R} k(z, y)(1 - h_B(y))\, dy \right| \psi(x)\, dx$$

$$= \int \left| \lim_{\substack{r \to 0 \\ R \to \infty}} \int_{r < \delta(x,y) \leqslant R} k(z, y)(h_B(z) - h_B(y))\, dy \right| \psi(x)\, dx$$

$$= \int \left| \int k(z, y)(h_B(z) - h_B(y))\, dy \right| \psi(x)\, dx = 0,$$

since the innermost integral does not depend on $x$ and $\psi \in \{\Lambda_0^\eta\}_0$.

A particular case of this application is the following:

Given a homogeneous polynomial $P(x)$ of even degree $m$, defined on $\mathbb{C}^n$ with negative real part for real $x$, we consider the parabolic differential equation

$$L\,|u| = \frac{\partial}{\partial t} u - (-1)^{m/2} P(D) u = f.$$

In [J] the following expression was considered in order to obtain a priori estimates:

$$D_x^\rho u(x, t) = \lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \int_{\mathbb{R}^n} s(x-y, t-s)\, f(y, s)\, dy\, ds,$$

where $\rho$ is a multi-index, $|\rho| = \rho_1 + \cdots + \rho_n = m$, and $s(x, t)$ is the $\rho$th spatial derivative of a fundamental solution of the homogeneous equation $L(U) = 0$.

It has been observed in [RT] that a priori estimates can be obtained from

$$\lim_{\varepsilon \to 0} \int_{|x-y| + |t-s|^{1/m} > \varepsilon} s(x-y, t-s)\, f(y, s)\, dy\, ds.$$

This limit is viewed as defining a singular integral operator associated to the kernel $k(\bar{x}, \bar{y}) = s(x-y, t-s)$, on the space of homogeneous type $(X, d, \mu)$ given by

$$X = \mathbb{R}^n x \mid 0, \infty),$$

$$d(\bar{x}, \bar{y}) = d((x, t), (y, s)) = |x-y| + |t-s|^{1/m},$$

and $\mu$ the Lebesgue measure on $\mathbb{R}^n x \mid 0, \infty)$.

In [MT] it is proved that the kernel satisfies (2.4), (2.5) for $\gamma=(m+n)^{-1}$, and (2.35); therefore the a priori estimate

$$|||D_x^{p_u}|||_\eta \leqslant C\ |||L(u)|||_\eta$$

holds for any $0<\eta<(m+n)^{-1}$.

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