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On the Laplace Transforms of Retarded, Lorentz-Invariant Functions

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Let $\phi(t)$ ($t \in \mathbb{R}^n$) be a retarded, Lorentz-invariant function which satisfies, in addition, condition (c). We call "R" the family of such functions. Let $f(z)$ be the Laplace transform of $\phi(t) \in \mathbb{R}$. We prove (Theorem 1) that $f(z)$ can be expressed as a K-transform (formula (I, 2; 1)). We apply this formula to evaluate several Laplace transforms. We show that it affords simple proofs of important known results. Formula (I, 2; 1) is an effective complement to L. Schwartz' method of evaluating Fourier transforms via Laplace transforms ("Théorie des distributions," p. 264, Hermann, Paris, 1966). We think this is the most useful application of our formula.

I. THE BASIC FORMULA

Let $\phi(t)$ ($t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$) be a radial integrable function: $\phi(t) = F(r^2)$, where $r^2 = \sum_{i=1}^n t_i^2$ and $F(\lambda)$ is a function of the scalar variable λ.

Let $F[\phi]$ designate the Fourier transform of ϕ:

$$F[\phi] = \int_{\mathbb{R}^n} e^{-i\langle t,x\rangle}\phi(t)\, dt.$$

A classical Bochner formula [1, p. 187] expresses $F\{\phi\}$ by means of a Hankel transform:

$$F\{\phi\} = \frac{(2\pi)^{n/2}}{\{x_0^2 + x_1^2 + \cdots + x_{n-1}^2\}^{(n-2)/4}} \times \int_0^\infty F(\lambda)\, \lambda^{n/2} J_{(n-2)/2}\{\lambda(x_0^2 + \cdots + x_{n-1}^2)^{1/2}\}\, d\lambda.$$

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Here $J_\nu(z)$ is the well-known Bessel function [2, Vol. II, p. 246].

In this article we prove (Theorem 1) an analog of Bochner's formula for Laplace transforms of the form (I, 1; 1), where ϕ is a function of the Lorentz distance, whose support is contained in the closure of the domain $t_0 > 0$, $t_0^2 - t_1^2 - \cdots - t_{n-1}^2 > 0$. This is our main result (formula (I, 2; 1)), from which we derive important known results (with simpler proofs) and also new ones.

When referring in the sequel to formula (I, 2; 1) we shall call it the "basic formula."

I.1. We begin with some definitions.

Let $t = (t_0, t_1, ..., t_{n-1})$ be a point of $\mathbb{R}^n$. We shall write $t_0^2 - t_1^2 - \cdots - t_{n-1}^2 = u$. By Γ_+ we designate the interior of the forward cone: $\Gamma_+ = \{t \in R^n / t_0 > 0,\ u > 0\}$; and by $\bar{\Gamma}_+$ we designate its closure. Similarly, Γ_- designates the domain $\Gamma_- = \{t \in R^n / t_0 < 0,\ u > 0\}$, and $\bar{\Gamma}_-$ designates its closure. We put $z = (z_0, z_1, ..., z_{n-1}) \in C^n$, where $z_\nu = x_\nu + iy_\nu$, $\nu = 0, 1, 2, ..., n-1$; $\langle t, z\rangle = t_0 z_0 + t_1 z_1 + \cdots + t_{n-1} z_{n-1}$; and $dt = dt_0\, dt_1 \cdots dt_{n-1}$. The tube T_- is defined by $T_- = \{z \in C^n / y \in V_-\}$, where $V_- = \{y \in R^n / y_0 < 0,\ y_0^2 - y_1^2 - \cdots - y_{n-1}^2 > 0\}$.

Similarly we put

$$T_+ = \{z \in C^n / y \in V_+\},$$

where

$$V_+ = \{y \in \mathbb{R}^n / y_0 > 0, y_0^2 - y_1^2 - \cdots - y_{n-1}^2 > 0\}.$$

The Laplace transform of $\phi(t)$ is

$$f(z) = L\{\phi\} = \int_{\mathbb{R}^n} e^{-i\langle t,z\rangle} \phi(t)\, dt. \tag{I, 1; 1}$$

Let $F(\lambda)$ be a function of the scalar variable λ, and let $\phi(t)$ be a function endowed with the following properties:

(a) $\phi(t) = F(u)$,

(b) $\operatorname{supp} \phi(t) \subset \bar{\Gamma}_+$,

(c) $e^{\langle t,y\rangle}\phi(t) \in L_1$ if $y \in V_-$.

We call R the family of functions $\phi(t)$ which satisfies conditions (a), (b), and (c). Similarly we call A the family of functions which satisfy conditions

(a′) $\phi(t) = F(u)$,

(b′) $\operatorname{supp} \phi(t) \in \bar{\Gamma}_-$,

(c′) $e^{\langle t,y\rangle}\phi(t) \in L_1$ if $y \in V_+$.

I.2. Now we can state our main theorem.

THEOREM 1. *Hypothesis*:

(α) $\phi(t) \in R$,

(β) $z \in T_-$.

Thesis:

$$f(z) = L\{\phi\} = \frac{(2\pi)^{(n-2)/2}}{\{z_1^{\,2} + z_2^{\,2} + \cdots + z_{n-1}^2 - z_0^{\,2}\}^{(n-2)/4}}$$
$$\times \int_0^\infty F(\lambda)\, \lambda^{(n-2)/4} K_{(n-2)/2}\{(\lambda(z_1^{\,2} + z_2^{\,2} + \cdots + z_{n-1}^2 - z_0^{\,2}))^{1/2}\}\, d\lambda. \tag{I, 2; 1}$$

Here $K_\nu(z)$ designates the modified Bessel function of the third kind [2, *Vol.* II, *p.* 427].

Proof of Theorem 1. We make the change of variables

$$\begin{aligned} u_0 &= t_0^{\,2} - t_1^{\,2} - \cdots - t_{n-1}^2, \\ u_1 &= t_1, \\ &\vdots \\ u_{n-1} &= t_{n-1}. \end{aligned} \tag{I, 2; 2}$$

The corresponding Jacobian is $2t_0 > 0$. With this change of variable $f(z)$ becomes

$$f(z) = \int_0^\infty F(u_0)\, du_0 \int_{\mathbb{R}^{n-1}}$$
$$\times \frac{\exp\{-i(u_1 z_1 + \cdots + u_{n-1} z_{n-1})\} \exp\{-iz_0(u_0 + u_1^{\,2} + \cdots + u_{n-1}^2)^{1/2}\}}{2(u_0 + u_1^{\,2} + u_2^{\,2} + \cdots + u_{n-1}^2)^{1/2}}\, du. \tag{I, 2; 3}$$

Let us write

$$\frac{\exp\{-iz_0(u_0 + u_1^{\,2} + u_2^{\,2} + \cdots + u_{n-1}^2)^{1/2}\}}{2(u_0 + u_1^{\,2} + u_2^{\,2} + \cdots + u_{n-1}^2)^{1/2}} = g_{u_0, z_0}(u_1, u_2, \ldots, u_{n-1}). \tag{I, 2; 4}$$

With this notation the interior integral of the right-hand member of (I, 2; 1) becomes

$$I_{z_0, u_0}(z_1, z_2, \ldots, z_{n-1})$$
$$= \int_{R^{n-1}} e^{-i(u_1 z_1 + \cdots + u_{n-1} z_{n-1})} g_{u_0, z_0}(u_1, \ldots, u_{n-1})\, du_1 \ldots du_{n-1}. \tag{I, 2; 5}$$

I.3. We now get an explicit formula for the right-hand member of (I, 2; 5). This is a convergent Laplace transform which converges when $y = 0$:

$$I_{z_0,u_0}(x_1 + i0, \ldots, x_{n-1} + i0)$$

$$= \int_{R^{n-1}} e^{-i(u_1x_1+\cdots+u_{n-1}x_{n-1})} g_{u_0,z_0}(u_1, \ldots, u_{n-1})\, du_1 \ldots du_{n-1}$$

$$= F[g_{u_0,z_0}(u_1, \ldots, u_{n-1})]. \qquad \text{(I, 3; 1)}$$

From (I, 2; 4) we conclude that the right-hand member of (I, 3; 1) is the Fourier transform of a radial integrable function (we have, by hypothesis, $\operatorname{Im} z_0 = y_0 < 0$, $u_0 > 0$). Consequently, it can be expressed by means of Bochner's formula. We obtain

$$F[g] = (2\pi)^{(n-1)/2} \frac{1}{(x_1^{\,2} + \cdots + x_{n-1}^2)^{(n-3)/4}}$$

$$\times \int_0^\infty \frac{e^{-iz_0(u_0+\lambda^2)1/2}}{2(u_0 + \lambda^2)^{1/2}} \lambda^{(n-1)/2} J_{(n-3)/2}[\lambda(x_1^{\,2} + \cdots + x_{n-1}^2)^{1/2}]\, d\lambda. \qquad \text{(I, 3; 2)}$$

The right-hand member can be evaluated by a known formula [2, Vol. II, p. 31, formula 22].

One obtains

$$F[g] = (2\pi)^{(n-2)/2} u_0^{(n-2)/4} \frac{1}{(x_1^{\,2} + \cdots + x_{n-1}^2 - z_0^{\,2})^{(n-2)/4}}$$

$$\times K_{(n-2)/2}[(u_0)^{1/2}(x_1^{\,2} + \cdots + x_{n-1}^2 - z_0^{\,2})^{1/2}]. \qquad \text{(I, 3; 3)}$$

If we put $x_\nu + iy_\nu = z_\nu$ instead of x_ν, $\nu = 1, 2, \ldots, n-1$, in the left-hand member of this formula, we recuperate our Laplace transform $L[g]$. We also obtain a regular function of variables z_ν if we proceed analogously with the right-hand member.

This is a consequence of the fact that the expression $z_1^{\,2} + \cdots + z_{n-1}^2 - z_0^{\,2}$ never vanishes [4, p. 38]. Therefore, the following formula is valid:

$$I_{z_0,u_0}(z_1, \ldots, z_{n-1})$$

$$= \frac{(2\pi)^{(n-2)/2} u_0^{(n-2)/4}}{(z_1^{\,2} + \cdots + z_{n-1}^2 - z_0^{\,2})^{(n-2)/4}} K_{(n-2)/2}[u_0^{1/2}(z_1^{\,2} + \cdots + z_{n-1}^2 - z_0^{\,2})^{1/2}].$$

$$\text{(I, 3; 4)}$$

From (I, 2; 3) and (I, 3; 4) we conclude that

$$f(z) = \frac{(2\pi)^{(n-2)/2}}{(z_1{}^2 + \cdots + z_{n-1}^2 - z_0{}^2)^{(n-2)/4}}$$

$$\times \int_0^\infty F(u_0)\, u_0^{(n-2)/4} K_{(n-2)/2}[u_0^{1/2}(z_1{}^2 + \cdots + z_{n-1}^2 - z_0{}^2)^{1/2}]\, du_0 ,$$

which is identical to (I, 2; 1). This finishes the proof of Theorem 1.

Note. We remark, for later purposes, that the following asymptotic formula is valid for $t \to \infty$ (cf. [5, p. 202, formula (1)]):

$$K_{(n-2)/2}(\rho t^{1/2}) \sim \left(\frac{\pi}{2}\right)^{1/2} \frac{e^{-\rho t^{1/2}}}{(\rho t^{1/2})^{1/2}} ; \qquad \text{(I, 3; 5)}$$

here we have written $\rho^2 = z_1{}^2 + \cdots + z_{n-1}^2 - z_0{}^2$.

This formula is a consequence of the relation $|\arg \rho| < \pi/2$.

I.4. The two following representation formulas are easy consequences of the basic formula.

THEOREM 2. *Hypothesis: the same as that of Theorem* 1.

Thesis:

(a) *If* $n = 2m + 2$, $m = 0, 1, \ldots,$

$$f(z) = L[\phi] = (-1)^m\, 2^{2m} \pi^m \frac{d^m}{ds^m} \int_0^\infty F(\lambda)\, K_0[(\lambda s)^{1/2}]\, d\lambda. \qquad \text{(I, 4; 1)}$$

(b) *If* $n = 2m + 1$, $m = 1, 2, \ldots,$

$$f(z) = L\{\phi\} = (-1)^m\, 2^{2m-1} \pi^m \frac{d^m}{ds^m} \int_0^\infty F(\lambda)\, \lambda^{-1/2} e^{-(\lambda s)^{1/2}}\, d\lambda. \qquad \text{(I, 4; 2)}$$

Proof of Theorem 2. We shall need the formula

$$\frac{K_p(us^{1/2})}{s^{p/2}} = (-1)^m \left(\frac{2}{u}\right)^m \frac{d^m}{ds^m} \left\{\frac{K_{p-m}(us^{1/2})}{s^{(p-m)/2}}\right\}, \qquad \text{(I, 4; 3)}$$

which follows, by iteration, from the well-known relation

$$\frac{K_p(z)}{z^p} = -\frac{1}{z}\frac{d}{dz}\frac{K_{p-1}(z)}{z^{p-1}} .$$

The basic formula can be written as

$$f(z) = 2(2\pi)^{(n-2)/2} \int_0^\infty F(u^2)\, u^{n/2} \frac{K_{(n-2)/2}(us^{1/2})}{s^{(n-2)/4}}\, du. \tag{I, 4; 4}$$

From (I, 4; 3) and (I, 4; 4) we get

$$f(z) = (-1)^m\, 2^{(n-2)/2+m+1} \pi^{(n-2)/2} \frac{d^m}{ds^m} \int_0^\infty F(u^2)\, u^{(n/2)-m} \frac{K_{(n-2)/2-m}(us^{1/2})}{s^{(n-2)/4-m/2}}\, du. \tag{I, 4; 5}$$

If we put $n = 2m + 2$ here, we get (I, 4; 1).

If in the same formula we put $n = 2m + 1$ and use the known relation

$$K_{1/2}(z) = (\pi/2)^{1/2}\, z^{-1/2} e^{-z},$$

we obtain (I, 4; 2). This proves the theorem.

Note 1. Formula (I, 4; 2) seems to be equivalent to a very interesting result due to Leray, [4, p. 41, formulas (19, 11)], which he proves by a completely different method.

Note 2. Formulas (I, 4; 1) and (I, 4; 2) are analogs, for Laplace transforms, of two Bochner formulas [1, p. 187, formulas (17) and (18)], valid for Fourier transforms of radial functions.

II. Applications of the Basic Formula

II.1. We begin by considering the following functions of the family R introduced by Riesz [6, p. 17; cf. also 7, p. 89; 8, p. 179; and 9, p. 72]:

$$\begin{aligned} W(t, \alpha, m, n) &= \frac{(m^{-2}u)^{(\alpha-n)/4}}{\pi^{(n-2)/2} 2^{(\alpha+n-2)/2} \Gamma(\alpha/2)} J_{(\alpha-n)/2}\{(m^2u)^{1/2}\} && \text{if } t \in \Gamma_+, \\ &= 0 && \text{if } t \notin \Gamma_+. \end{aligned} \tag{II, 1; 1}$$

Here α is a complex parameter, m a real nonnegative number and n the dimension of the space. $W(t, \alpha, m, n)$, which is an ordinary function if $\operatorname{Re}\alpha$ is $\geqslant n$, is an entire distributional function of α.

From the basic formula one obtains immediately, for $\operatorname{Re}\alpha \geqslant n$

$$L[W] = \frac{2(2\pi)^{(n-2)/2}\, m^{(n-\alpha)/2}}{\rho^{(n-1)/2} \pi^{(n-2)/2} 2^{(\alpha+n-2)/2} \Gamma(\alpha/2)} \times \int_0^\infty \lambda^{(\alpha-1)/2} (\lambda\rho)^{1/2} J_{(\alpha-n)/2}(m\lambda)\, K_{(n-2)/2}(\lambda\rho)\, d\lambda, \tag{II, 1; 2}$$

where we have put $\rho^2 = z_1{}^2 + \cdots + z_{n-1}^2 - z_0{}^2$.

This integral can be explicitly evaluated by means of a known formula [2, Vol. II, p. 137, formula (16)], thus obtaining

$$L[W(t, \alpha, m, n)] = (\rho^2 + m^2)^{-(\alpha/2)}. \tag{II, 1; 3}$$

This formula is valid for

$$\operatorname{Re} \alpha > 2n - 4 \tag{II, 1; 4}$$

and

$$\operatorname{Re} \rho > 0. \tag{II, 1; 5}$$

This last condition effectively holds as a consequence of our assumption $z \in T_-$ (in this connection see [4, p. 38]).

II.2. Formula (II, 1; 3) has useful applications. It follows from (II, 1; 5) that $\rho^2 + m^2$ never vanishes. Consequently, the function which appears at the right-hand member of (II, 1; 3) is an entire function of α. We conclude, by appealing to the principle of analytical in continuation, that (II, 1; 3) is valid for every α. If we put $\alpha = -2k$, $k = 0, 1, 2,\ldots$ in (II, 1; 3) we get

$$L\{t, \alpha = -2k, m, n\} = \{\rho^2 + m^2\}^k. \tag{II, 2; 1}$$

From this formula, in conjunction with the identity

$$L\{(\square + m^2)^k \delta\} = (\rho^2 + m^2)^k$$

we conclude, appealing to the uniqueness theorem for Laplace transforms, that

$$W(t, \alpha = -2k, m, n) = \{\square + m^2\}^k \delta. \tag{II, 2; 2}$$

It is well known that the convolution $W_\alpha * W_\beta$ exists for every couple α, β of complex numbers [8, p. 177]. From this one concludes, taking into account (II, 1; 3) and the uniqueness theorem for Laplace transforms, that

$$W(t, \alpha, m, n) * W(t, \beta, m, n) = W(t, \alpha + \beta, m, n). \tag{II, 2; 3}$$

This formula appears in [7, p. 89]. From (II, 2; 3) and (II, 2; 2) we get

$$\{(\square + m^2)^k\} W_\beta = W_{\beta - 2k}. \tag{II, 2; 4}$$

This formula (for $k = 1$) appears in [1, p. 89].

II.3. Formula (II, 1; 1) simplifies when $m = 0$. Indeed, if we replace $J_{(\alpha-n)/2}$ by its Taylor series in the right-hand member of (II, 1; 1) we get immediately

$$\begin{aligned} W(t, \alpha, m = 0, n) = R_\alpha(u) &= \frac{u^{(\alpha-n)/2}}{H_n(\alpha)} \qquad \text{if} \quad t \in \Gamma^+, \\ &= 0 \qquad\qquad\quad \text{if} \quad t \notin \Gamma^+. \end{aligned} \tag{II, 3; 1}$$

Here we have put

$$H_n(\alpha) = \pi^{(n-2)/2} 2^{\alpha-1} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha-n+2}{2}\right). \qquad \text{(II, 3; 2)}$$

The $R_\alpha(u)$ were introduced by Riesz [7, p. 31]. From formulas (II, 1; 3), (II, 2; 2), (II, 2; 3) and (II, 2; 4) we obtain, respectively, putting in them $m = 0$,

$$L[R_\alpha] = \{z_1{}^2 + \cdots + z_{n-1}^2 - z_0{}^2\}^{-(\alpha/2)}, \qquad \text{(II, 3; 3)}$$

$$R_{-2k}(u) = \square^k \delta, \qquad \text{(II, 3; 4)}$$

$$R_\alpha * R_\beta = R_{\alpha+\beta}, \qquad \text{(II, 3; 5)}$$

$$\square^k R_\alpha = R_{\alpha-2k}. \qquad \text{(II, 3; 6)}$$

A formula equivalent to (II, 3; 3) appears in [8, p. 264, formula (VII, 7; 37)]. Formula (II, 3; 4) is due to Schwartz [8, p. 50, formula (II, 3; 32)]. Formulas (II, 3; 5) and (II, 3; 6) are due to Riesz.

II.4. We shall give a last application of the basic formula.

Let m be a nonnegative number and let α be a complex parameter.

We write

$$G(x, m^2, \alpha) = \frac{(x-m^2)_+^{\alpha-1}}{\Gamma(\alpha)} = \frac{(x-m^2)^{\alpha-1}}{\Gamma(\alpha)} \quad \text{if} \quad x - m^2 > 0, \qquad \text{(II, 4; 1)}$$
$$= 0 \quad \text{if} \quad x - m^2 \leqslant 0.$$

G is an ordinary function of α when $\operatorname{Re} \alpha > 0$, and it is well known that the following formula is valid [10, p. 57]:

$$\left\{\frac{(x-m^2)_+^{\alpha-1}}{\Gamma(\alpha)}\right\}_{\alpha=-k} = \delta_{m^2}^{(k)}, \qquad \text{(II, 4; 2)}$$

where $k = 0, 1, 2, \ldots$.

Starting from $G(x, m^2, \alpha)$ we define by composition the n-dimensional function

$$G_R(t, m^2, \alpha, n) = \frac{(u-m^2)_+^{\alpha-1}}{\Gamma(\alpha)} = \frac{(u-m^2)^{\alpha-1}}{\Gamma(\alpha)} \quad \text{if} \quad u - m^2 > 0 \text{ and } t_0 > 0,$$
$$= 0 \quad \text{if } t \text{ belongs to the complementary set.} \qquad \text{(II, 4; 3)}$$

We now evaluate, by means of the basic formula, the Laplace transform of G_R. Assuming that $\operatorname{Re}\alpha \geqslant 1$, remembering that $K_\mu = K_{-\mu}$ and putting again $\rho^2 = z_1^2 + \cdots + z_{n-1}^2 - z_0^2$ we arrive at the formula

$$L[G_R(t, m^2, n, \alpha)] = \frac{2(2\pi)^{(n-2)/2}}{\Gamma(\alpha)\, \rho^{(n-1)/2}}$$

$$\times \int_m^\infty (s^2 - m^2)^{\alpha-1}\, s^{(n-1)/2}(s\rho)^{1/2}\, K_{(2-n)/2}(s\rho)\, ds. \qquad \text{(II, 4; 4)}$$

This integral is known [2, Vol. II, p. 129, formula (13)], and we obtain

$$L[G_R(t, m^2, \alpha, n)] = 2^\alpha(2\pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2} \cdot \rho^{-\alpha+(2-n)/2} K_{\alpha+(n-2)/2}(m\rho). \qquad \text{(II, 4; 5)}$$

This formula, which we have proved on the assumption that $\operatorname{Re}\alpha \geqslant 1$, is valid, by analytical continuation, for every complex α.

If we put $m = 0$ in (II, 4; 5) we obtain

$$L[G_R(t, m = 0, \alpha, n)] = (2\pi)^{(n-2)/2}\, 2^{2\alpha+(n/2)-2} \rho^{-2\alpha+2-n}\Gamma\left(\alpha + \frac{n-2}{2}\right). \qquad \text{(II, 4; 6)}$$

We remark that in deriving (II, 4; 6) we have used the known asymptotic formula (valid for $s \to 0$)

$$K_\nu(s) \sim 2^{\nu-1}\Gamma(\nu)\, s^{-\nu}. \qquad \text{(II, 4; 7)}$$

II.5. We shall register some particular cases of (II, 4; 5). Putting $\alpha = 1$ in it, we get

$$L[G_R(t, m^2, \alpha = 1, n)] = 2(2\pi)^{(n-2)/2}\, m^{n/2}\rho^{-(n/2)} K_{n/2}(m\rho). \qquad \text{(II, 5; 1)}$$

This is the Laplace transform of the characteristic function of the volume bounded by the forward sheet of the hyperboloid $u = m^2$. Putting $m = 0$ in (II, 5; 1) we get (appealing again to the asymptotic formula (II, 4; 7))

$$L[G_R(t, m = 0, \alpha = 1, n)] = 2^{n/2}(2\pi)^{(n-2)/2}\, \rho^{-n}\Gamma(1/2). \qquad \text{(II, 5; 2)}$$

The Laplace transform (II, 5; 2) is the so-called "Bochner kernel for the forward cone" which appears in the theory of several complex variables [11, p. 299, formula (139); 12, p. 168].

Another particular case of (II, 4; 5) is obtained by putting $\alpha = -k$, $k = 0, 1,\ldots$ in it.

We obtain

$$\begin{aligned} &L[G_R(t, m^2, \alpha = -k, n)] \\ &\quad = L[\delta_R^{(k)}(u = m^2)] \\ &\quad = 2^{-k}(2\pi)^{(n-2)/2}\, m^{-k+(n-2)/2}\rho^{k+(2-n)/2} K_{-k+(n-2)/2}(m\rho). \end{aligned} \qquad \text{(II, 5; 3)}$$

If we put $\alpha = -k$ in (II, 4; 6) we get

$$L[\delta_R^{(k)}(u)] = (2\pi)^{(n-2)/2}\, 2^{-2k+n/2-2}\Gamma(-k+(n-2)/2)\, \rho^{-n+2k+2}. \qquad \text{(II, 5; 4)}$$

We observe that, in constrast with formula (II, 5; 3), which is valid for every $k = 0, 1, 2,\ldots$, formula (II, 5; 4) requires, for its validity, that $-k + (n-2)/2 \neq 0, -1, -2, \ldots$.

The particular cases, corresponding to $k = 0$, of formulas (II, 5; 3) and (II, 5; 4) read as follows:

$$L[\delta_R(u - m^2)] = (2\pi)^{(n-2)/2}\, m^{(n-2)/2}\rho^{(2-n)/2}K_{(n-2)/2}(m\rho), \qquad \text{(II, 5; 5)}$$

$$L[\delta_R(u)] = 2^{n/2-2}(2\pi)^{(n-2)/2}\,\Gamma\left(\frac{n-2}{2}\right)\rho^{2-n}. \qquad \text{(II, 5; 6)}$$

A formula equivalent to (II, 5; 6) appears in [11, p. 299].

II.6. We now evaluate by a different method the Laplace transform of $\delta_R^{(k)}(u - m^2)$.

The idea consists in expressing the right-hand member of the basic formula as a scalar product. This may be done when the support of $F(\lambda)$ does not contain the origin. In the case of the function $G_R(t, m^2, \alpha, n)$ we get

$$L[G_R(t, m^2, \alpha, n)]$$

$$= \frac{(2\pi)^{(n-2)/2}}{(z_1{}^2 + \cdots + z_{n-1}^2 - z_0{}^2)^{(n-2)/4}}$$

$$\times \left\langle \frac{(\lambda - m^2)_+^{\alpha-1}}{\Gamma(\alpha)},\, \lambda^{(n-2)/4}K_{(n-2)/2}(\lambda^{1/2}(z_1{}^2 + \cdots + z_{n-1}^2 - z_0{}^2)^{1/2})\right\rangle.$$

$$\text{(II, 6; 1)}$$

The function $\phi(\lambda) = \lambda^{(n-2)/4}K_{(n-2)/2}[\lambda^{1/2}(z_1{}^2 + \cdots + z_{n-1}^2 - z_0{}^2)^{1/2}]$ is indeed a test function in the interval $[m^2, \infty)$ when $m^2 > 0$ (cf. the asymptotic formula (I, 3; 5)).

If we put in $\alpha = -k$, $k = 0, 1,\ldots$ (II, 6; 1), we get

$$L[\delta_R^{(k)}(u - m^2)]$$

$$= \frac{(2\pi)^{(n-2)/2}}{(z_1{}^2 + \cdots + z_{n-1}^2 - z_0{}^2)^{(n-2)/4}}$$

$$\times \langle \delta^{(k)}(\lambda - m^2),\, \lambda^{(n-2)/2}K_{(n-2)/2}[\lambda^{1/2}(z_1{}^2 + \cdots + z_{n-1}^2 - z_0{}^2)^{1/2}]\rangle$$

or, equivalently,

$$L[\delta_R^{(k)}(u - m^2)]$$

$$= \frac{(2\pi)^{(n-2)/2}}{(z_1{}^2 + \cdots + z_{n-1}^2 - z_0{}^2)^{(n-2)/4}}$$

$$\times (-1)^k D_{\lambda=m^2}^{(k)}[\lambda^{(n-2)/2} K_{(n-2)/2}[\lambda^{1/2}(z_1{}^2 + \cdots + z_{n-1}^2 - z_0{}^2)^{1/2}]]. \tag{II, 6; 2}$$

For this formula to hold for every k it is essential that $m^2 > 0$; but if $k = 0$ it also holds in the case $m = 0$. The formula at which one arrives is, naturally, identical to (II, 5; 6).

We remark that the formulas we have obtained for the Laplace transforms of functions of the family R are also valid for functions of the class A (this applies in particular to the basic formula); the only difference is that, for functions of the class A, the formulas are valid on the assumption $\operatorname{Im} z_0 > 0$.

II.7. We shall refer briefly to a last application of the basic formula, namely, the evaluation of Fourier transforms as limits of Laplace transforms. Schwartz [8, p. 264] has evaluated the Fourier transform of the functions $R_\alpha(x, n)$ of Riesz, by evaluating their Laplace transforms (first step), and then passing to the limit (in S') for $y \to 0$, where $y \in V_-$ (second step). The method was later employed by Lavoine [13] and Vladimirov [11, p. 299–302]. It works generally for any $\phi(t) \in R$ which is, besides, a continuous function of slow growth. It is clear that the basic formula greatly facilitates the use of Schwartz' method, since it disposes of its first step.

One of us (S.E.T.) has evaluated by Schwartz' method the Fourier transforms of all the functions (or distributions) whose Laplace transforms appear in the preceding pages. The results will appear in a separate article.

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