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Solutions of the divergence operator on John domains ☆

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Abstract

If $\Omega \subset \mathbb{R}^n$ is a bounded domain, the existence of solutions $\mathbf{u} \in W_0^{1,p}(\Omega)$ of $\operatorname{div} \mathbf{u} = f$ for $f \in L^p(\Omega)$ with vanishing mean value and $1 < p < \infty$, is a basic result in the analysis of the Stokes equations. It is known that the result holds when Ω is a Lipschitz domain and that it is not valid for domains with external cusps.

In this paper we prove that the result holds for John domains. Our proof is constructive: the solution $\mathbf{u}$ is given by an explicit integral operator acting on f. To prove that $\mathbf{u} \in W_0^{1,p}(\Omega)$ we make use of the Calderón–Zygmund singular integral operator theory and the Hardy–Littlewood maximal function.

For domains satisfying the separation property introduced in [S. Buckley, P. Koskela, Sobolev–Poincaré implies John, Math. Res. Lett. 2 (5) (1995) 577–593], and $1 < p < n$, we also prove a converse result, thus characterizing in this case the domains for which a continuous right inverse of

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the divergence exists. In particular, our result applies to simply connected planar domains because they satisfy the separation property.

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1. Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^n$, a basic result for the theoretical and numerical analysis of the Stokes equations in Ω is the existence of a solution $\mathbf{u} \in H_0^1(\Omega)^n$ of

$$\operatorname{div} \mathbf{u} = f \tag{1.1}$$

such that

$$\|\mathbf{u}\|_{H^1(\Omega)^n} \leqslant C\|f\|_{L^2(\Omega)} \tag{1.2}$$

for any $f \in L_0^2(\Omega)$, where C is a constant depending only on Ω, and $L_0^2(\Omega)$ denotes the space of functions in $L^2(\Omega)$ with vanishing mean value in Ω. By duality, an equivalent way of stating this result is to say that

$$\|f\|_{L_0^2(\Omega)} \leqslant C\|\nabla f\|_{H^{-1}(\Omega)^n} \tag{1.3}$$

for any $f \in L_0^2(\Omega)$.

This result is of interest also because of its connection with the Korn inequality which is fundamental in the analysis of the elasticity equations. Indeed, the Korn inequality can be deduced from (1.1) and (1.2).

Several arguments have been given to prove the existence of $\mathbf{u} \in H_0^1(\Omega)^n$ satisfying (1.1) and (1.2) (see, for example, [5] and the references therein). In particular, it is known that the result is true for Lipschitz domains.

On the other hand, it is known that the result does not hold if the domain has an external cusp. In fact, this can be deduced from a counterexample given by Friedrichs [6] for a related inequality. Let us recall here this counterexample which seems to be not very well known. Other counterexamples have been given in much more recent papers (see [7] and also [13] where counterexamples for the Korn inequality are given).

Suppose that Ω is a two-dimensional domain and that

$$w(z) = f(x, y) + ig(x, y)$$

is an analytic function of the variable $z = x + iy$ in Ω with f and g real functions and $\int_\Omega f\,dx = 0$. Under suitable assumptions on Ω, Friedrichs proved in [6] that there exists a constant Γ, depending only on Ω, such that

$$\|f\|_{L^2(\Omega)} \leqslant \Gamma\|g\|_{L^2(\Omega)}. \tag{1.4}$$

He also proved that the existence of the constant Γ is equivalent to the existence of a constant $\theta < 1$ such that

$$\left| \int\limits_{\Omega} w^2 \, dx \, dy \right| \leqslant \theta \int\limits_{\Omega} |w|^2 \, dx \, dy \tag{1.5}$$

whenever $\int_\Omega w \, dx \, dy = 0$.

Now, in order to show that the inequality does not hold for a domain with an external cusp, he defined, using polar coordinates (r, ϑ),

$$\Omega = \{(r, \vartheta)\colon \ 0 < r < R, \ \vartheta_1(r) < \vartheta < \vartheta_2(r)\}, \tag{1.6}$$

with

$$\vartheta_1(r) = -kr + O(r^2), \qquad \vartheta_2(r) = kr + O(r^2)$$

where k is a constant. Then, for $\alpha > 0$ he introduced the functions $w_\alpha = (2\alpha)^{1/2} z^{\alpha - 3/2}$ and showed by an elementary computation (see [6, p. 343] for details) that

$$\frac{\left| \int_\Omega w_\alpha^2 \, dx \, dy \right|}{\int_\Omega |w_\alpha|^2 \, dx \, dy} \longrightarrow 1 \tag{1.7}$$

when $\alpha \to 0$. And, since $\int_\Omega w_\alpha \, dx \, dy \to 0$, one can subtract to w_α its average to obtain functions with vanishing mean value and satisfying (1.7). Therefore (1.5) does not hold, and consequently (1.4) does not hold either.

But, on the other hand, observe that (1.4) follows easily from (1.3) together with the fact that f and g satisfy the Cauchy–Riemann equations. Consequently, we conclude that (1.3), and its equivalent forms (1.1) and (1.2), are not valid for the domain defined in (1.6).

An interesting problem is to determine which conditions on the domain Ω are sufficient in order to have the existence of $\mathbf{u}$ satisfying (1.1) and (1.2). In view of the results mentioned above it is clear that we have to consider a class of domains which excludes domains with external cusps. On the other hand, the Lipschitz condition is not necessary. In fact, it is known that if the result holds for two domains then it also holds for the union of them (see, for example, the argument given in [1]), and consequently, domains having internal cusps are allowed although they are not Lipschitz.

Taking into account all the comments made above, it seems that a natural class of domains to be considered for our problem is that of the John domains. For instance, it is known that a two-dimensional domain with a piecewise smooth boundary is a John domain if and only if it does not have external cusps.

These domains where first considered by F. John in his work on elasticity [8] and where named after him by Martio and Sarvas [10]. Further, John domains were used in the study of several problems in Analysis. For example they were used by G. David and S. Semmes [4] in the analysis of quasiminimal surfaces of codimension one and by S. Buckley and P. Koskela [2] for the study of different kind of inequalities. On the other hand, the John domains are closely related with the extension domains of P. Jones [9]. Indeed the (ε, ∞)

domains, also called uniform domains, are John domains (but the converse is not true: a John domain can have an internal cusp while a uniform domain can not).

We will recall in Section 2 the definition of John domains but, roughly speaking, Ω is a John domain with respect to a point $x_0 \in \Omega$ if each point $y \in \Omega$ can be reached by a Lipschitz curve beginning at x_0 and contained in Ω in such a way that, for every point x in the curve, the distance from x to y is proportional to the distance from x to the boundary of Ω (in particular, external cusps are not allowed).

This class contains the Lipschitz domains but it is much larger. In fact, the boundary of a John domain can be very bad: a typical example is the so called snowflake domain which has a fractal boundary.

In this paper we prove the existence of solutions of (1.1) satisfying (1.2) and vanishing at the boundary when Ω is a John domain. More generally, we prove the analogous result in L^p, for $1 < p < \infty$, namely, if $f \in L^p_0(\Omega)$ then our solution of (1.1) satisfies

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)^n} \leqslant C\|f\|_{L^p(\Omega)} \tag{1.8}$$

and $\mathbf{u} \in W^{1,p}_0(\Omega)^n = \overline{C^\infty_0(\Omega)}^n$. Moreover, our proof is constructive: we give an explicit solution of (1.1) defined by an integral operator (actually, a family of solutions because our operator depends on an arbitrary weight function).

For the class of domains satisfying the separation property introduced in [2] we prove a converse result, namely, if for some $1 < p < n$ and any $f \in L^p_0(\Omega)$ there exists a solution $\mathbf{u} \in W^{1,p}_0(\Omega)^n$ of (1.1) satisfying (1.8), then Ω is a John domain. This result applies in particular to simply connected planar domains since, as was proved in [2], these domains satisfy the separation property. To prove this converse result we prove that the existence of solutions $\mathbf{u} \in W^{1,p}_0(\Omega)^n$ of (1.1) satisfying (1.8) for $1 < p < n$ implies the Sobolev–Poincaré inequality for any $1 < p < n$.

Our construction generalizes the one given in [1] (and analyzed also in [5]) for a domain which is star-shaped with respect to a ball. The arguments are rather technical and so, to help the reader, we explain here some of the ideas. Given a function ϕ let us call $\overline{\phi} = \int_\Omega \phi\omega$, where ω is an arbitrary smooth weight such that $\int_\Omega \omega = 1$. Now, a key point in our construction is to recover $\phi - \overline{\phi}$ from its gradient. Suppose that Ω is star-shaped with respect to a ball B centered at x_0 and that $\operatorname{supp}\omega \subset B$. If for any $y \in \Omega$ we call $\gamma(s, y)$ the function defining the segment joining y with x_0, namely, $\gamma(s, y) = y + s(x_0 - y)$, then, for any $z \in B$, the segment joining y with z is parametrized by $\gamma(s, y) + s(z - x_0)$. Therefore, integrating over the segments $[y, z]$, we have

$$\phi(y) - \phi(z) = -\int_0^1 \big(\dot{\gamma}(s, y) + (z - x_0)\big) \cdot \nabla\phi\big(\gamma(s, y) + s(z - x_0)\big)\, ds, \tag{1.9}$$

and so, multiplying by $\omega(z)$ and integrating on z, we obtain

$$(\phi - \overline{\phi})(y) = -\int_\Omega \int_0^1 \big(\dot{\gamma}(s, y) + (z - x_0)\big) \cdot \nabla\phi\big(\gamma(s, y) + s(z - x_0)\big)\omega(z)\, ds\, dz. \tag{1.10}$$

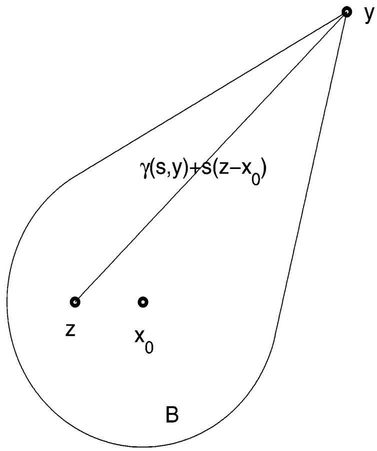

Fig. 1.

Then, we have obtained an expression for $(\phi - \bar{\phi})(y)$ in terms of an integral involving $\nabla\phi$ evaluated at points in the cone formed by all the segments with end points at y and $z \in B$ which is contained in Ω (see Fig. 1).

Suppose now that Ω is not star-shaped but it is a John domain with respect to x_0, with x_0 being as above the center of a ball B which contains the support of ω. We can then generalize formulas (1.9) and (1.10) replacing the segment joining y and x_0 by an appropriate curve given by $\gamma(s, y)$, such that $\gamma(0, y) = y$, $\gamma(1, y) = x_0$ and with the property that the "twisted cone" formed by the curves parametrized by $\gamma(s, y) + s(z - x_0)$ is contained in Ω. In this way we obtain a generalization of (1.10) where now $\nabla\phi$ is evaluated at points in that "twisted cone" (see Fig. 2).

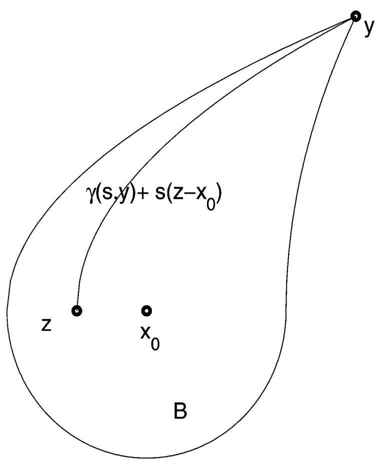

Fig. 2.

The paper is organized as follows: In Section 2 we recall the definition of John domains and prove some of their properties. In particular we construct the curves that will be used to obtain formula (1.10) and, as a byproduct, our solution of (1.1). The arguments of the rest of the paper depend only on the properties of these curves stated and proved in Lemma 2.1 and not on our particular construction. In Section 3 we construct our explicit solution of $\operatorname{div} \mathbf{u} = f$. This solution is given by an integral operator acting on f. In Section 4, we prove that our solution satisfies the estimate (1.8). In order to do that, we first show that the derivatives of $\mathbf{u}$ can be expressed in terms of a singular integral operator acting on f and then we show that this operator can be decomposed in two parts: the first one is a singular integral operator with a kernel that satisfies the conditions of the classic Calderón–Zygmund theory while the second one can be controlled by the Hardy–Littlewood maximal operator. We end Section 4 with an important corollary of our main result: the Korn inequality. Finally, in Section 5 we prove a converse result for the case of planar simply connected domains.

2. Properties of John domains

In this section we recall the definition of John domains and prove some of their properties which will be useful in our construction. We will denote with $d(x)$ the distance of $x \in \Omega$ to the boundary.

Definition 2.1 *(John domains).* Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and $x_0 \in \Omega$. We say that Ω is a John domain with respect to x_0 and with constant L if for any $y \in \Omega$ there exists a Lipschitz mapping $\rho : [0, |y - x_0|] \to \Omega$, with Lipschitz constant L, such that $\rho(0) = y$, $\rho(|y - x_0|) = x_0$ and $d(\rho(t)) \geqslant t/L$ for $t \in [0, |y - x_0|]$.

Clearly, if Ω is a John domain, for each $y \in \Omega$ there are many curves joining y and x_0 satisfying the properties required in Definition 2.1. To construct our solution of the divergence we will choose a family of curves verifying some extra conditions, in particular, we will require that the first part of each curve (i.e., the part closer to y) be a segment, this fact will be important in our analysis. Moreover, we need to have some control of the variability of the curves as functions of y. Indeed, measurability will be enough for our purposes. Also, for convenience we rescale the curves in order to have the parameter in $[0, 1]$.

In the next lemma we state the properties that we will need on the curves and prove the existence of a family of curves satisfying them. We will make use of the Whitney decomposition of an open set which we recall in the next definition (see, for example, [11] for a proof of its existence). In what follows, $d(Q, \partial\Omega)$ denotes the distance of a cube Q to the boundary of Ω and $\operatorname{diam}(Q)$ the diameter of Q.

Definition 2.2. Given an open bounded set $\Omega \subset \mathbb{R}^n$, a *Whitney decomposition* of Ω is a family W of closed dyadic cubes with pairwise disjoint interiors satisfying the following properties:

(1) $\Omega = \bigcup_{Q \in W} Q$;
(2) $\operatorname{diam}(Q) \leqslant d(Q, \partial\Omega) \leqslant 4 \operatorname{diam}(Q)$, $\forall Q \in W$;
(3) $\frac{1}{4} \operatorname{diam}(Q) \leqslant \operatorname{diam}(\tilde{Q}) \leqslant 4 \operatorname{diam}(Q)$, $\forall Q, \tilde{Q} \in W$ such that $Q \cap \tilde{Q} \neq \emptyset$.

Given $Q \in W$, let x_Q be its center and Q^* the cube with the same center but expanded by a factor 9/8, namely,

$$Q^* = \frac{9}{8}(Q - x_Q) + x_Q.$$

We will make use of the following facts which follow easily from the properties given in Definition 2.2,

$$d\big(Q^*, \partial\Omega\big) \sim \operatorname{diam}\big(Q^*\big) \sim d(y) \quad \forall y \in Q^*, \tag{2.1}$$

where $A \sim B$ means that there are constants c and C, which may depend on the dimension n but on nothing else, such that $cA \leqslant B \leqslant CA$.

Lemma 2.1. *Let $\Omega \subset R^n$ be a John domain with respect to x_0 and with constant L. Then, there exists a function $\gamma : [0,1] \times \Omega \to \Omega$ and constants K, δ and C_1 depending only on L,* $\operatorname{diam}(\Omega)$, *$d(x_0)$ and n, such that*:

(1) $\gamma(0, y) = y$, $\gamma(1, y) = x_0$;
(2) $d(\gamma(s, y)) \geqslant \delta s$;
(3) *$\gamma(s, y)$ is Lipschitz in the variable s with constant K*;
(4) *$\gamma(s, y)$ is a segment for* $0 \leqslant s \leqslant C_1 d(y) \leqslant 1$;
(5) *$\gamma(s, y)$ and $\dot{\gamma}(s, y) := \frac{\partial\gamma}{\partial s}(s, y)$ are measurable functions.*

Proof. Let W be a Whitney decomposition of Ω and $Q_0 \in W$ be a cube containing x_0. Given $y \in \Omega$, let $Q \in W$ be such that $y \in Q$. We remark that if y belongs to the boundary of some $Q \in W$ then it belongs to more than one cube. We choose one of them arbitrarily (in any case this is of no importance because the set of those points has measure zero).

Suppose first that $x_0 \in Q^*$. In this case, we can take the curve to be a segment, namely, $\gamma(s, y) = s x_0 + (1 - s) y$. In fact, in view of (2.1), it is easy to see that $\gamma(s, y)$ satisfies (2) and (3) with K and δ proportional to $d(x_0)$. Also (4) is trivially satisfied for any C_1 such that $C_1 d(y) \leqslant 1$, we can take for example $C_1 = 1/\operatorname{diam}(\Omega)$.

Now, if $x_0 \notin Q^*$, let x_Q be the center of Q and take a parametrization $\rho(t)$ of a curve joining x_Q and x_0 satisfying the conditions given in the definition of John domains. First we reparametrize ρ and define

$$\mu(s) = \rho\big(s|x_0 - x_Q|\big).$$

Then, μ is Lipschitz with constant $K = L \operatorname{diam}(\Omega)$ and satisfies $d(\mu(s)) \geqslant \delta s$ with $\delta \sim |x_0 - x_Q|/L$. But, since $x_0 \notin Q^*$, then $Q \neq Q_0$, we obtain from properties (2) and (3) of

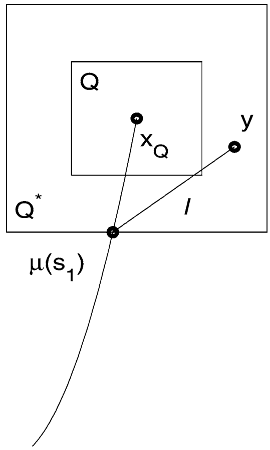

Fig. 3.

Definition 2.2 that $|x_0 - x_Q| \geqslant cd(x_0)$ with c depending only on n. Therefore, (2) holds for μ with $\delta \sim d(x_0)/L$.

To define $\gamma(s, y)$ we modify this curve in the following way. Let s_1 be the first $s \in [0, 1]$ such that $\mu(s) \in \partial Q^*$. Then we define

$$\gamma(s, y) = \begin{cases} \ell(s), & \text{if } s \in [0, s_1], \\ \mu(s), & \text{if } s \in [s_1, 1] \end{cases}$$

where

$$\ell(s) = \frac{s}{s_1}\mu(s_1) + \left(1 - \frac{s}{s_1}\right)y,$$

see Fig. 3.

Now, $|\dot{\ell}(s)| = (\mu(s_1) - y)/s_1$. But, since μ is Lipschitz with constant $L\,\mathrm{diam}(\Omega)$, $\mu(s_1) \in \partial Q^*$ and $\mu(0) = x_Q$, it is easy to check that $s_1 \geqslant c\,\mathrm{diam}(Q^*)/L\,\mathrm{diam}(\Omega)$ with c depending only on n. Therefore, ℓ is Lipschitz with constant $K \sim L\,\mathrm{diam}(\Omega)$.

So, $\gamma(s, y)$ satisfies (2) on the interval $[0, s_1]$ with $K \sim L\,\mathrm{diam}(\Omega)$. On the other hand, for $s \in [0, s_1)$, both $\mu(s)$ and $\gamma(s, y)$ belong to Q^* and so $d(\gamma(s, y)) \sim d(\mu(s))$ which proves that (2) holds on this interval. Since, $\gamma(s, y) = \mu(s)$ on $s \in [s_1, 1]$, (2) and (3) hold on the whole interval $[0, 1]$.

Using again that $s_1 \geqslant c\,\mathrm{diam}(Q^*)/L\,\mathrm{diam}(\Omega)$, (4) follows from (2.1).

Finally, observe that (5) holds because $\gamma(s, y)$ and $\dot{\gamma}(s, y)$ are continuous for y in the interior of each $Q \in W$ and so they are continuous up to a set of measure zero. Therefore, the proof is complete. □

3. Construction of the explicit solutions of the divergence

In this section we construct the explicit solution of the divergence. For any $y \in \Omega$ let $\gamma(s, y)$ be the curve given in Lemma 2.1. We define a new family of curves in the following way.

For $y \in \Omega$ and $z \in B(x_0, \delta)$, where δ is the constant given in Lemma 2.1, define

$$\tilde{\gamma}(s, y, z) := \gamma(s, y) + s(z - x_0), \quad s \in [0, 1]. \tag{3.1}$$

Let us note the following facts, which follow immediately from (1) and (2) of Lemma 2.1,

$$\tilde{\gamma}(0, y, z) = y, \qquad \tilde{\gamma}(1, y, z) = z \quad \text{and} \quad \tilde{\gamma}(s, y, z) \in \Omega \quad \text{for any } s \in [0, 1]. \tag{3.2}$$

In order to simplify the notation, we will assume without loss of generality, that $x_0 = 0$. Let $\omega \in C_0^\infty$ such that $\int_\Omega \omega = 1$ and $\operatorname{supp} \omega \subset B(0, \delta/2)$. Observe that from the proof of Lemma 2.1 it follows that $\delta < d(x_0)$ and so $B(0, \delta/2) \subset \Omega$.

Let us now introduce the function

$$G = (G_1, \ldots, G_n) : \overline{\Omega} \times \Omega \to \mathbb{R}^n$$

which will be the kernel of the right inverse of the divergence. For $x \in \overline{\Omega}$ and $y \in \Omega$ we define

$$G(x, y) := \int_0^1 \left\{ \dot{\gamma}(s, y) + \frac{x - \gamma(s, y)}{s} \right\} \omega\left(\frac{x - \gamma(s, y)}{s} \right) \frac{1}{s^n}\, ds. \tag{3.3}$$

Observe that, from (5) of Lemma 2.1, we know that $G(x, y)$ is a measurable function.

In the rest of the paper it will be important to use that the integral defining $G(x, y)$ can be restricted to $s \geqslant C_2|x - y|$ for some positive constant C_2. Indeed, for $C_2 = 1/(\delta + K)$, we have:

$$\text{if} \quad s < C_2|x - y| \quad \text{then} \quad \omega\left(\frac{x - \gamma(s, y)}{s} \right) = 0. \tag{3.4}$$

In fact, if s is such that $(x - \gamma(s, y))/s \in \operatorname{supp} \omega$, then, $|x - \gamma(s, y)| < \delta s$. Therefore, recalling that $y = \gamma(0, y)$ and that γ is Lipschitz with constant K in the variable s, we have

$$|x - y| \leqslant \left| x - \gamma(s, y) \right| + \left| \gamma(s, y) - \gamma(0, y) \right| \leqslant \delta s + K s$$

and so (3.4) holds.

An important consequence of (3.4) is the bound for $G(x, y)$ given in the following lemma.

Lemma 3.1. *There exists a constant $C = C(n, \delta, K)$ such that*

$$\left|G(x, y)\right| \leqslant C \frac{\|\omega\|_\infty}{|x - y|^{n-1}}. \tag{3.5}$$

Proof. In view of (3.4) we have

$$G(x, y) = \int_{C_2|x-y|}^{1} \left\{\dot{\gamma}(s, y) + \frac{x - \gamma(s, y)}{s}\right\}\omega\left(\frac{x - \gamma(s, y)}{s}\right)\frac{1}{s^n}\, ds.$$

But,

$$\left|\dot{\gamma}(s, y) + \frac{x - \gamma(s, y)}{s}\right| \leqslant \left|\dot{\gamma}(s, y)\right| + \left|\frac{x - y}{s}\right| + \left|\frac{y - \gamma(s, y)}{s}\right|$$

and from property (3) of Lemma 2.1, and recalling that $y = \gamma(0, y)$, we know that the first and last term of the right-hand side are bounded by K, and therefore estimate (3.5) follows easily. □

We will call $\bar{\phi}$ the weighted average of a function ϕ, namely, $\bar{\phi} = \int_\Omega \phi\omega$. The next lemma shows how $\phi - \bar{\phi}$ can be recovered from its gradient by means of the kernel G. As a corollary of this result we obtain our constructive solution of the divergence.

Lemma 3.2. *For $\phi \in C^1(\Omega)$ and for any $y \in \Omega$,*

$$(\phi - \bar{\phi})(y) = -\int_\Omega G(x, y) \cdot \nabla\phi(x)\, dx.$$

Proof. Since $\int_\Omega \omega = 1$, we have, in view of (3.2), that for any $y \in \Omega$,

$$(\phi - \bar{\phi})(y) = \int_\Omega \big(\phi(y) - \phi(z)\big)\omega(z)\, dz = -\int_\Omega \int_0^1 \dot{\tilde{\gamma}}(s, y, z) \cdot \nabla\phi\big(\tilde{\gamma}(s, y, z)\big)\omega(z)\, ds\, dz.$$

But, $\dot{\tilde{\gamma}}(s, y, z) = \dot{\gamma}(s, y) + z$ (recall that we have assumed $x_0 = 0$). Then, making the change of variables $x = \tilde{\gamma}(s, y, z)$, we have $z = (x - \gamma(s, y))/s$ and $dz = dx/s^n$. Hence

$$(\phi - \bar{\phi})(y) = -\int_\Omega \int_0^1 \left\{\dot{\gamma}(s, y) + \frac{x - \gamma(s, y)}{s}\right\}\omega\left(\frac{x - \gamma(s, y)}{s}\right)\frac{1}{s^n} ds \cdot \nabla\phi(x)\, dx$$

which in view of the definition (3.3) concludes the proof. □

Corollary 3.1. *For $f \in L^1(\Omega)$ such that $\int_\Omega f = 0$ define*

$$\mathbf{u}(x) = \int_\Omega G(x, y) f(y)\, dy. \tag{3.6}$$

Then, $\mathbf{u}$ *satisfies*

$$\operatorname{div} \mathbf{u} = f.$$

Proof. For any $\phi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}\int_\Omega f(y)\phi(y)\, dy &= \int_\Omega f(y)(\phi - \bar{\phi})(y)\, dy = -\int_\Omega f(y)\left(\int_\Omega G(x, y) \cdot \nabla\phi(x)\, dx\right) dy \\ &= -\int_\Omega \left(\int_\Omega G(x, y) f(y)\, dy\right) \cdot \nabla\phi(x)\, dx = -\int_\Omega \mathbf{u}(x) \cdot \nabla\phi(x)\, dx\end{aligned}$$

where the change in the order of integration can be easily justified by using the bound (3.5). $\square$

In order to show that the solution defined in (3.6) vanishes on the boundary we will make use of the following lemma.

Lemma 3.3. *If $x \in \partial\Omega$, $G(x, y) = 0$ for all $y \in \Omega$. Moreover, for any $x, y \in \Omega$ and any $0 < \alpha \leqslant 1$, there exists a constant $C = C(n, \delta, K, \omega)$ such that*

$$\left|G(x, y)\right| \leqslant C \frac{d(x)^\alpha}{|x - y|^{n-1+\alpha}}. \tag{3.7}$$

Proof. Observe first that

$$\omega\left(\frac{x - \gamma(s, y)}{s}\right) = 0 \quad \text{for } x \in \partial\Omega,\ y \in \Omega \text{ and } s \in [0, 1]. \tag{3.8}$$

Indeed, in this case we know from property (2) of Lemma 2.1 that

$$\delta s \leqslant d\big(\gamma(s, y)\big) \leqslant \big|\gamma(s, y) - x\big|.$$

Hence,

$$\frac{|\gamma(s, y) - x|}{s} \geqslant \delta$$

and therefore, (3.8) follows immediately since $\operatorname{supp} \omega \subset B(0, \delta/2)$. Then, from the definition of G, it follows that $G(x, y) = 0$ for all $x \in \partial\Omega$ and all $y \in \Omega$.

Now, for $x \in \Omega$, let $\bar{x} \in \partial\Omega$ be such that $d(x) = |x - \bar{x}|$. Since $\omega((\bar{x} - \gamma(s, y))/s) = 0$, we can write

$$G(x, y) = \int_{C_2|x-y|}^{1} \left\{\dot{\gamma}(s, y) + \frac{x - \gamma(s, y)}{s}\right\}\left\{\omega\left(\frac{x - \gamma(s, y)}{s}\right) - \omega\left(\frac{\bar{x} - \gamma(s, y)}{s}\right)\right\}\frac{1}{s^n}\, ds$$

but, since ω and its first derivatives are bounded, ω is a Hölder α function for $0 < \alpha \leqslant 1$. Also, as was shown in the proof of Lemma 3.1, $\dot{\gamma}(s, y) + (x - \gamma(s, y))/s$ is bounded by a constant which depends only on δ and K. Therefore, there exists a constant $C = C(\delta, K, \omega)$ such that

$$\left|G(x, y)\right| \leqslant C \int_{C_2|x-y|}^{1} \left(\frac{|x - \bar{x}|}{s}\right)^{\alpha} \frac{1}{s^n}\, ds,$$

and integrating we conclude the proof of (3.7). □

4. Estimate of the derivatives

The object of this section is to give an estimate of the derivatives of the solution of the divergence defined in (3.6) in terms of the right-hand side. First, we show that the derivatives of $\mathbf{u}$ can be written in terms of a singular integral operator applied to the right-hand side f. With this goal we introduce

$$T_{ik} f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{\partial G_k}{\partial x_i}(x, y) f(y)\, dy \tag{4.1}$$

and

$$T_{ik}^{*} g(y) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{\partial G_k}{\partial x_i}(x, y) g(x)\, dx \tag{4.2}$$

for functions f and g with support in Ω.

In the proof of the next lemma we will use that the operator T_{ik}^{*} is the adjoint of T_{ik}. This is a consequence of the existence in L^p norm of the limit in (4.1) and of the boundedness of T_{ik} in L^p for $1 < p < \infty$. These results will be proved in the last part of the paper. We prefer to present the results in this order for the sake of clarity.

Lemma 4.1. *For* $\mathbf{u} = (u_1, \ldots, u_n)$ *defined as in* (3.6)*, we have*

$$\frac{\partial u_k}{\partial x_i} = T_{ik} f + \omega_{ik} f \quad \text{in } \Omega$$

where

$$\omega_{ik}(y) = \int_{\mathbb{R}^n} \frac{z_k z_i}{|z|^2} \omega\big(-\dot{\gamma}(0, y) + z\big)\, dz,$$

in particular, $\omega_{ik} \in L^\infty(\Omega)$.

Proof. For $\phi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}\int_\Omega \frac{\partial u_k}{\partial x_i}(x)\phi(x)\, dx &= -\int_\Omega u_k(x)\frac{\partial \phi}{\partial x_i}(x)\, dx = -\int_\Omega \bigg(\int_\Omega G_k(x, y) f(y)\, dy\bigg)\frac{\partial \phi}{\partial x_i}(x)\, dx \\ &= -\int_\Omega \bigg(\int_\Omega G_k(x, y)\frac{\partial \phi}{\partial x_i}(x)\, dx\bigg) f(y)\, dy = -\int_\Omega I(y) f(y)\, dy \qquad (4.3)\end{aligned}$$

with

$$I(y) := \int_\Omega G_k(x, y)\frac{\partial \phi}{\partial x_i}(x)\, dx,$$

where again, the change in the order of integration can be done because of (3.5). Also from (3.5) we know that $I(y)$ is finite.

We can write

$$I(y) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} G_k(x, y)\frac{\partial \phi}{\partial x_i}(x)\, dx$$

and integrating by parts we obtain

$$I(y) = \lim_{\varepsilon \to 0} \bigg\{-\int_{|x-y|>\varepsilon} \frac{\partial G_k(x, y)}{\partial x_i}\phi(x)\, dx - \int_{|\zeta-y|=\varepsilon} G_k(\zeta, y)\phi(\zeta)\frac{(\zeta - y)_i}{|\zeta - y|}\, d\zeta\bigg\}. \qquad (4.4)$$

Now, the surface integral can be written as

$$\int_{|\zeta-y|=\varepsilon} G_k(\zeta, y)\phi(\zeta)\frac{(\zeta - y)_i}{|\zeta - y|}\, d\zeta = \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon, \qquad (4.5)$$

where

$$\mathrm{I}_\varepsilon := \phi(y) \int_{|\zeta-y|=\varepsilon} G_k(\zeta, y)\frac{(\zeta - y)_i}{|\zeta - y|}\, d\zeta$$

and

$$\mathrm{II}_\varepsilon := \int\limits_{|\zeta-y|=\varepsilon} G_k(\zeta,y)\big(\phi(\zeta)-\phi(y)\big)\frac{(\zeta-y)_i}{|\zeta-y|}\,d\zeta.$$

But, it is easy to see that

$$\lim_{\varepsilon\to 0}\mathrm{II}_\varepsilon = 0 \tag{4.6}$$

uniformly in y. Indeed, using again the bound (3.5), we have

$$|\mathrm{II}_\varepsilon| \leqslant C \int\limits_{|\zeta-y|=\varepsilon} \|\nabla\phi\|_\infty \frac{1}{|\zeta-y|^{n-2}}\,d\zeta = C\frac{\|\nabla\phi\|_\infty}{\varepsilon^{n-2}} \int\limits_{|\zeta-y|=\varepsilon} d\zeta = O(\varepsilon).$$

Let us now treat I_ε. We have

$$\mathrm{I}_\varepsilon = \phi(y) \int\limits_{|\zeta-y|=\varepsilon} G_k(\zeta,y)\frac{(\zeta-y)_i}{|\zeta-y|}\,d\zeta$$

and so, from the definition of G (see (3.3)), we have

$$\mathrm{I}_\varepsilon = \phi(y)(a_\varepsilon + b_\varepsilon) \tag{4.7}$$

with

$$a_\varepsilon(y) := \int\limits_{|\zeta-y|=\varepsilon} \int\limits_0^1 \left(\frac{\zeta_k-y_k}{s}\right)\omega\left(\frac{\zeta-\gamma(s,y)}{s}\right)\frac{(\zeta-y)_i}{|\zeta-y|}\frac{1}{s^n}\,ds\,d\zeta$$

and

$$b_\varepsilon(y) := \int\limits_{|\zeta-y|=\varepsilon} \int\limits_0^1 \left(\dot\gamma_k(s,y) + \frac{y_k-\gamma_k(s,y)}{s}\right)\omega\left(\frac{\zeta-\gamma(s,y)}{s}\right)\frac{(\zeta-y)_i}{|\zeta-y|}\frac{1}{s^n}\,ds\,d\zeta.$$

We claim that

$$|a_\varepsilon(y)| \leqslant C \quad \text{and} \quad \lim_{\varepsilon\to 0} a_\varepsilon(y) = \int \frac{z_k z_i}{|z|^2}\omega\big(-\dot\gamma(0,y)+z\big)\,dz \tag{4.8}$$

where $C = C(n,K,\delta,\omega)$ and that

$$\big|b_\varepsilon(y)\big| \leqslant C \quad \text{and} \quad \lim_{\varepsilon\to 0} b_\varepsilon(y) = 0 \tag{4.9}$$

where, also here, $C = C(n,K,\delta,\omega)$.

To prove (4.8) we introduce the change of variables $r = \varepsilon/s$ to obtain

$$a_\varepsilon(y) = \int_{|\zeta - y| = \varepsilon} \int_\varepsilon^\infty \left(\frac{\zeta_k - y_k}{\varepsilon}\right) \omega\left(\frac{r}{\varepsilon}\left(\zeta - \gamma\left(\frac{\varepsilon}{r}, y\right)\right)\right) \frac{(\zeta - y)_i}{|\zeta - y|} \frac{r^{n-1}}{\varepsilon^{n-1}}\, dr\, d\zeta.$$

Then, a further change of variables $\sigma = (\zeta - y)/\varepsilon$ yields

$$a_\varepsilon(y) = \int_{|\sigma| = 1} \int_\varepsilon^\infty \sigma_k \sigma_i \omega\left(\frac{r}{\varepsilon}\left(y - \gamma\left(\frac{\varepsilon}{r}, y\right)\right) + r\sigma\right) r^{n-1}\, dr\, d\sigma.$$

Hence, taking $z = r\sigma$, we obtain

$$a_\varepsilon(y) = \int_{|z| > \varepsilon} \frac{z_k z_i}{|z|^2} \omega\left(\frac{|z|}{\varepsilon}\left(y - \gamma\left(\frac{\varepsilon}{|z|}, y\right)\right) + z\right) dz.$$

Now, $(|z|/\varepsilon)(y - \gamma(\varepsilon/|z|, y)) + z \in \operatorname{supp} \omega$ implies

$$\left|\frac{|z|}{\varepsilon}\left(y - \gamma\left(\frac{\varepsilon}{|z|}, y\right)\right) + z\right| \leqslant \delta.$$

But,

$$|z| \leqslant \left|\frac{|z|}{\varepsilon}\left(y - \gamma\left(\frac{\varepsilon}{|z|}, y\right)\right) + z\right| + \left|\frac{|z|}{\varepsilon}\left(y - \gamma\left(\frac{\varepsilon}{|z|}, y\right)\right)\right|$$

and so,

$$|z| \leqslant \delta + \left|\frac{\gamma(0, y) - \gamma(\varepsilon/|z|, y)}{\varepsilon/|z|}\right| \leqslant \delta + K.$$

Therefore, we have shown that

$$a_\varepsilon(y) = \int_{\varepsilon < |z| \leqslant \delta + K} \frac{z_k z_i}{|z|^2} \omega\left(\frac{|z|}{\varepsilon}\left(y - \gamma\left(\frac{\varepsilon}{|z|}, y\right)\right) + z\right) dz,$$

which in particular implies that

$$\left|a_\varepsilon(y)\right| \leqslant C(n, K, \delta) \|\omega\|_\infty.$$

On the other hand, since

$$\lim_{\varepsilon \to 0} \frac{|z|}{\varepsilon}\left(y - \gamma\left(\frac{\varepsilon}{|z|}, y\right)\right) = \lim_{\varepsilon \to 0} \frac{(\gamma(0, y) - \gamma(\varepsilon/|z|, y))}{\varepsilon/z} = -\dot{\gamma}(0, y)$$

(the existence of this limit follows from (4) of Lemma 2.1), the dominated convergence theorem allows us to conclude the proof of (4.8).

To prove (4.9), we make again the change of variables

$$r = \frac{\varepsilon}{s}, \qquad \sigma = \frac{\zeta - y}{\varepsilon}, \qquad z = r\sigma$$

to obtain

$$\begin{aligned} b_\varepsilon(y) = -\int\limits_{|z|\geqslant\varepsilon} &\left(\dot{\gamma}_k\left(\frac{\varepsilon}{|z|}, y\right) + \left(y_k - \gamma_k\left(\frac{\varepsilon}{|z|}, y\right)\right)\frac{|z|}{\varepsilon}\right)\frac{z_i}{|z|} \\ &\times \omega\left(\frac{|z|}{\varepsilon}\left(y - \gamma\left(\frac{\varepsilon}{|z|}, y\right)\right) + z\right) dz. \end{aligned}$$

But now,

$$\left|\dot{\gamma}_k\left(\frac{\varepsilon}{|z|}, y\right) + \left(y_k - \gamma_k\left(\frac{\varepsilon}{|z|}, y\right)\right)\frac{|z|}{\varepsilon}\right| \leqslant 2K$$

and, as in the case of a_ε, taking into account that $\operatorname{supp}\omega \subset B(0,\delta)$ we can restrict the integral defining b_ε to $\varepsilon < |z| \leqslant \delta + K$ and obtain that

$$\left|b_\varepsilon(y)\right| \leqslant C(n, K, \delta)\|\omega\|_\infty.$$

Now, observe that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \dot{\gamma}\left(\frac{\varepsilon}{|z|}, y\right) + \left(y - \gamma\left(\frac{\varepsilon}{|z|}, y\right)\right)\frac{|z|}{\varepsilon} \\ &\quad = \lim_{\varepsilon\to 0} \dot{\gamma}\left(\frac{\varepsilon}{|z|}, y\right) + \lim_{\varepsilon\to 0} \frac{(\gamma(0, y) - \gamma(\varepsilon/|z|, y))}{\varepsilon/|z|} \\ &\quad = \dot{\gamma}(0, y) - \dot{\gamma}(0, y) = 0 \end{aligned}$$

and applying the dominated convergence theorem again we conclude the proof of (4.9).

Now, from (4.7)–(4.9) we conclude that

$$\left|\mathrm{I}_\varepsilon(y)\right| \leqslant \left|\phi(y)\right|\left(\left|a_\varepsilon(y)\right| + \left|b_\varepsilon(y)\right|\right) \leqslant C(n, K, \delta, \omega)\left|\phi(y)\right| \tag{4.10}$$

and

$$\lim_{\varepsilon\to 0} \mathrm{I}_\varepsilon(y) = \phi(y)\omega_{ik}(y). \tag{4.11}$$

Finally, from (4.3)–(4.5), we have

$$\int\limits_\Omega \frac{\partial u_k}{\partial x_i}(x)\phi(x)\,dx = \int\limits_\Omega \lim_{\varepsilon\to 0}\left\{\int\limits_{|\zeta-y|>\varepsilon} \frac{\partial G_k(x,y)}{\partial x_i}\phi(x)\,dx + \mathrm{I}_\varepsilon(y) + \mathrm{II}_\varepsilon(y)\right\} f(y)\,dy$$

but, in view of (4.10) we can apply the dominated convergence theorem to obtain from (4.6) and (4.11) that

$$\int_{\Omega} \frac{\partial u_k}{\partial x_i}(x)\phi(x)\,dx$$
$$= \int_{\Omega} \left\{ \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{\partial G_k(x,y)}{\partial x_i}\phi(x)\,dx \right\} f(y)\,dy + \int_{\Omega} \omega_{ik}(y) f(y)\phi(y)\,dy$$

or, in view of (4.2),

$$\int_{\Omega} \frac{\partial u_k}{\partial x_i}(x)\phi(x)\,dx = \int_{\Omega} T_{ik}^{*}\phi(y) f(y)\,dy + \int_{\Omega} \omega_{ik}(y) f(y)\phi(y)\,dy$$

and, since ϕ is arbitrary, the lemma is proved. $\square$

Our final goal is to prove the estimate

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)} \leqslant C\|f\|_{L^p(\Omega)}$$

for $1 < p < \infty$.

In view of Lemma 4.1, our problem reduces to show that T_{ik} is a bounded operator in L^p for $1 < p < \infty$. To simplify notation we drop the subscripts i, k and introduce the functions

$$\eta = \frac{\partial \omega}{\partial x_i} \quad \text{and} \quad \psi = \frac{\partial (x_k \omega)}{\partial x_i}.$$

Then, we have to prove the continuity of an operator of the form

$$Tf(x) = \lim_{\varepsilon\to 0} T_\varepsilon f(x) \tag{4.12}$$

where, for $\varepsilon > 0$, T_ε is given by

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \left\{ \int_0^1 \left(\dot{\gamma}_k(s,y)\eta\left(\frac{x-\gamma(s,y)}{s}\right) + \psi\left(\frac{x-\gamma(s,y)}{s}\right)\right) \frac{ds}{s^{n+1}} \right\} f(y)\,dy \tag{4.13}$$

with η and ψ bounded and with support contained in that of ω. Moreover, since both are derivatives of functions with compact support, they satisfy

$$\int \eta = 0 \quad \text{and} \quad \int \psi = 0. \tag{4.14}$$

We will use the following

Lemma 4.2. *There exists a constant $C_3 = C_3(K, \delta)$ such that, if $\omega((x - \gamma(s, y))/s) \neq 0$, then*

$$|x - y| \leqslant C_3\, d(x).$$

Proof. Recalling that $\operatorname{supp} \omega \subset B(0, \delta/2)$ and using (2) of Lemma 2.1 we know that

$$\bigl|x - \gamma(s, y)\bigr| \leqslant \frac{\delta s}{2} \leqslant \frac{1}{2} d\bigl(\gamma(s, y)\bigr) \tag{4.15}$$

and so, recalling that $\gamma(0, y) = y$ and that γ is Lipschitz with constant K in the variable s, we obtain

$$|x - y| \leqslant \bigl|x - \gamma(s, y)\bigr| + \bigl|\gamma(s, y) - \gamma(0, y)\bigr| \leqslant \frac{\delta s}{2} + K s.$$

Therefore, using (2) of Lemma 2.1 again, it follows that

$$|x - y| \leqslant \left(\frac{1}{2} + \frac{K}{\delta}\right) d\bigl(\gamma(s, y)\bigr). \tag{4.16}$$

But, the function d is Lipschitz with constant 1 and then, it follows from (4.15) that

$$d\bigl(\gamma(s, y)\bigr) - d(x) \leqslant \bigl|\gamma(s, y) - x\bigr| \leqslant \frac{1}{2} d\bigl(\gamma(s, y)\bigr)$$

and therefore,

$$d\bigl(\gamma(s, y)\bigr) \leqslant 2 d(x)$$

which together with (4.16) concludes the proof. □

In order to prove the continuity of the operator defined in (4.12) and (4.13), in the next lemma we decompose it in two parts. Afterwards, we will show that the first part is a singular integral operator with a kernel satisfying the conditions of the classic theory of Calderón and Zygmund while the second part can be bounded by the Hardy–Littlewood maximal operator. In all our integrals the domain of integration is contained in Ω and so, to simplify notation, we extend the function f by zero outside of Ω.

Lemma 4.3. *The operator T_ε defined in* (4.13) *can be written as*

$$T_\varepsilon = T_{1,\varepsilon} + T_{2,\varepsilon}$$

with

$$T_{1,\varepsilon} f(x) = \int\limits_{\varepsilon < |x-y| \leqslant C_3 d(x)} K_1(x, y) f(y)\, dy$$

where

$$K_1(x, y) = H(y, x - y)$$

and

$$H(y, z) = \int_0^\infty \left\{ \dot{\gamma}_k(0, y)\eta\left(\frac{z}{s} - \dot{\gamma}(0, y)\right) + \psi\left(\frac{z}{s} - \dot{\gamma}(0, y)\right)\right\} \frac{ds}{s^{n+1}}$$

and with

$$T_{2,\varepsilon} f(x) = \int_{\varepsilon < |x-y| \leqslant C_3 d(x)} K_2(x, y) f(y)\, dy$$

where

$$\begin{aligned} K_2(x, y) = -\int_{\max\{C_1 d(y), C_2|x-y|\}}^{\infty} & \left\{ \dot{\gamma}_k(0, y)\eta\left(\frac{x-y}{s} - \dot{\gamma}(0, y)\right)\right. \\ & \left. + \psi\left(\frac{x-y}{s} - \dot{\gamma}(0, y)\right)\right\} \frac{ds}{s^{n+1}} \\ & + \int_{\max\{C_1 d(y), C_2|x-y|\}}^{1} \left\{ \dot{\gamma}_k(s, y)\eta\left(\frac{x-\gamma(s, y)}{s}\right) + \psi\left(\frac{x-\gamma(s, y)}{s}\right)\right\} \frac{ds}{s^{n+1}}. \end{aligned}$$

Proof. From the previous sections (see (4) of Lemma 2.1 and (3.4)) and recalling that the supports of η and ψ are contained in $\operatorname{supp} \omega$, we know that there exist constants $C_1 = C_1(K, \delta)$ and $C_2 = C_2(K, \delta)$ such that

$$\gamma(s, y) = \gamma(0, y) + \dot{\gamma}(0, y)s = y + \dot{\gamma}(0, y)s \quad \text{for } 0 \leqslant s \leqslant C_1 d(y) \tag{4.17}$$

and

$$C_2|x - y| \leqslant s \quad \text{whenever} \quad \eta\left(\frac{x-\gamma(s, y)}{s}\right) \neq 0 \quad \text{or} \quad \psi\left(\frac{x-\gamma(s, y)}{s}\right) \neq 0. \tag{4.18}$$

Therefore, using Lemma 4.2 we can write

$$\begin{aligned} T_\varepsilon f(x) = \int_{\varepsilon < |x-y| \leqslant C_3 d(x)} & \left(\int_{C_2|x-y|}^{1} \left\{ \dot{\gamma}_k(s, y)\eta\left(\frac{x-\gamma(s, y)}{s}\right)\right.\right. \\ & \left.\left. + \psi\left(\frac{x-\gamma(s, y)}{s}\right)\right\} \frac{ds}{s^{n+1}}\right) f(y)\, dy. \end{aligned}$$

Let us call

$$I = \int_{C_2|x-y|}^{1} \left\{ \dot{\gamma}_k(s, y)\eta\left(\frac{x-\gamma(s, y)}{s}\right) + \psi\left(\frac{x-\gamma(s, y)}{s}\right)\right\} \frac{ds}{s^{n+1}}.$$

In view of (4.17) we can decompose this integral as

$$\begin{aligned} I = & \int_{C_2|x-y|}^{\max\{C_1 d(y), C_2|x-y|\}} \left\{ \dot{\gamma}_k(0, y)\eta\left(\frac{x-y}{s} - \dot{\gamma}(0, y)\right) + \psi\left(\frac{x-y}{s} - \dot{\gamma}(0, y)\right)\right\} \frac{ds}{s^{n+1}} \\ & + \int_{\max\{C_1 d(y), C_2|x-y|\}}^{1} \left\{ \dot{\gamma}_k(s, y)\eta\left(\frac{x-\gamma(s, y)}{s}\right) + \psi\left(\frac{x-\gamma(s, y)}{s}\right)\right\} \frac{ds}{s^{n+1}} \end{aligned}$$

and so, using (4.18),

$$\begin{aligned} I = & \int_{0}^{\infty} \left\{ \dot{\gamma}_k(0, y)\eta\left(\frac{x-y}{s} - \dot{\gamma}(0, y)\right) + \psi\left(\frac{x-y}{s} - \dot{\gamma}(0, y)\right)\right\} \frac{ds}{s^{n+1}} \\ & - \int_{\max\{C_1 d(y), C_2|x-y|\}}^{\infty} \left\{ \dot{\gamma}_k(0, y)\eta\left(\frac{x-y}{s} - \dot{\gamma}(0, y)\right) + \psi\left(\frac{x-y}{s} - \dot{\gamma}(0, y)\right)\right\} \frac{ds}{s^{n+1}} \\ & + \int_{\max\{C_1 d(y), C_2|x-y|\}}^{1} \left\{ \dot{\gamma}_k(s, y)\eta\left(\frac{x-\gamma(s, y)}{s}\right) + \psi\left(\frac{x-\gamma(s, y)}{s}\right)\right\} \frac{ds}{s^{n+1}}, \end{aligned}$$

and the lemma is proved. □

Next, we show that the kernel of the operator $T_{1,\varepsilon}$ satisfies the conditions of the classical Calderón–Zygmund theory (see [3]).

Lemma 4.4. *The kernel $H(y, z)$ is homogeneous of degree $-n$ in the variable z, and has vanishing mean value and is uniformly bounded in y on $S = \{|z| = 1\}$.*

Proof. Given $\lambda > 0$, making the change of variable $t = s/\lambda$, we have

$$\begin{aligned} H(y, \lambda z) &= \int_{0}^{\infty} \left\{ \dot{\gamma}_k(0, y)\eta\left(\frac{\lambda z}{s} - \dot{\gamma}(0, y)\right) + \psi\left(\frac{\lambda z}{s} - \dot{\gamma}(0, y)\right)\right\} \frac{ds}{s^{n+1}} \\ &= \lambda^{-n} \int_{0}^{\infty} \left\{ \dot{\gamma}_k(0, y)\eta\left(\frac{z}{t} - \dot{\gamma}(0, y)\right) + \psi\left(\frac{z}{t} - \dot{\gamma}(0, y)\right)\right\} \frac{dt}{t^{n+1}} = \lambda^{-n} H(y, z). \end{aligned}$$

On the other hand, to see that $H(y,z)$ is bounded on $\{|z|=1\}$ uniformly in y, observe that for $|z|=1$, $H(y,z)$ can be written as

$$H(y,z)=\int_{C_4}^{\infty}\left\{\dot{\gamma}_k(0,y)\eta\left(\frac{z}{s}-\dot{\gamma}(0,y)\right)+\psi\left(\frac{z}{s}-\dot{\gamma}(0,y)\right)\right\}\frac{ds}{s^{n+1}}$$

with $C_4=\min\{1/\delta,1/2K\}$. Indeed, since the supports of ψ and η are contained in $B(0,\delta/2)$ it is easy to see that the integrand vanishes for $s<C_4$. Therefore, the boundedness of $H(y,z)$ follows from the fact that $\psi,\eta\in L^\infty$ and also $\dot{\gamma}_k(0,y)$ is a bounded function of y.

Now, making the change of variable $r=1/s$ in the integral defining $H(y,z)$, we obtain

$$H(y,z)=\int_0^{\infty}\left\{\dot{\gamma}(0,y)\eta\big(rz-\dot{\gamma}(0,y)\big)+\psi\big(rz-\dot{\gamma}(0,y)\big)\right\}r^{n-1}\,dr$$

and therefore,

$$\begin{aligned}\int_S H(y,\sigma)\,d\sigma&=\int_S\int_0^{\infty}\left\{\dot{\gamma}(0,y)\eta\big(r\sigma-\dot{\gamma}(0,y)\big)+\psi\big(r\sigma-\dot{\gamma}(0,y)\big)\right\}r^{n-1}\,dr\,d\sigma\\&=\int_{\mathbb{R}^n}\left\{\dot{\gamma}(0,y)\eta\big(z-\dot{\gamma}(0,y)\big)+\psi\big(z-\dot{\gamma}(0,y)\big)\right\}dz=0\end{aligned}$$

where the last equality follows from the fact that $\int_{\mathbb{R}^n}\psi=\int_{\mathbb{R}^n}\eta=0$. $\square$

Although the kernel defining the operator T_1 satisfies the Calderón–Zygmund conditions, this operator is not exactly of their type because the domain of integration in the definition of $T_{1,\varepsilon}$ is $\varepsilon<|x-y|\leqslant C_3d(x)$ instead of $\varepsilon<|x-y|$. However, we show in the next lemma that the continuity of T_1 follows from the general theory of Calderón and Zygmund.

We will make use of the Hardy–Littlewood maximal function which we denote Mf. Also, we will use again a Whitney decomposition of Ω (see Section 2 for its definition and properties). To simplify notation we call d_Q the diameter of a cube Q.

Lemma 4.5. *The operator*

$$T_1f=\lim_{\varepsilon\to 0}T_{1,\varepsilon}f\quad\text{with } T_{1,\varepsilon}f(x)=\int_{\varepsilon<|x-y|\leqslant C_3d(x)}K_1(x,y)f(y)\,dy$$

defines a bounded operator in $L^p(\Omega)$ for all $1<p<\infty$, and the convergence holds in the L^p norm.

Proof. Let $\widetilde{T}_\varepsilon$ be the operator defined by

$$\widetilde{T}_\varepsilon f = \int\limits_{\varepsilon<|x-y|} K_1(x,y) f(y)\,dy.$$

Recall that $K_1(x,y) = H(y, x-y)$ and then, the adjoint operator of $\widetilde{T}_\varepsilon$ is given by

$$\widetilde{T}_\varepsilon^* g(y) = \int\limits_{\varepsilon<|x-y|} H(y, x-y) g(x)\,dx.$$

Now, Lemma 4.4 shows that the kernel of this operator satisfies the conditions of Theorem 2 of [3] and therefore,

$$\widetilde{T}^* g = \lim_{\varepsilon\to 0} \widetilde{T}_\varepsilon^* g$$

with convergence in L^p, and $\widetilde{T}^*$ is bounded in L^p, for $1 < p < \infty$. Moreover, the norms of $\widetilde{T}_\varepsilon^*$ as operators in L^p are bounded uniformly in ε. As mentioned in [3, p. 291], the same results follow by duality for the operators $\widetilde{T}_\varepsilon$.

Consequently, if for a constant $\delta > 0$ we define

$$T_{1,\varepsilon,\delta} f(x) = \int\limits_{\varepsilon<|x-y|\leqslant\delta} K_1(x,y) f(y)\,dy, \tag{4.19}$$

we obtain

$$\|T_{1,\varepsilon,\delta} f\|_{L^p(\Omega)} \leqslant C \|f\|_{L^p(\Omega)} \tag{4.20}$$

with a constant C independent of ε and δ, and

$$\lim_{\varepsilon\to 0} T_{1,\varepsilon,\delta} f =: T_{1,\delta} f \in L^p(\Omega) \tag{4.21}$$

with convergence in the L^p norm. Indeed, this follows immediately from the results given above by writing

$$T_{1,\varepsilon,\delta} f(x) = \widetilde{T}_\varepsilon f - \widetilde{T}_\delta f.$$

Let W be a Whitney decomposition of Ω and choose a constant c small enough such that:

(1) for any $Q \in W$ and any $x \in Q$, $c d_Q < C_3 d(x)$;
(2) if $x \in Q$ and $|x-y| \leqslant c d_Q$ then $y \in Q^*$.

Now, given a cube $Q \in W$, suppose that $\varepsilon < cd_Q$. Then for $x \in Q$, we can write

$$T_{1,\varepsilon} f(x) = \int\limits_{\varepsilon<|x-y|\leqslant cd_Q} K_1(x,y) f(y)\,dy + \int\limits_{cd_Q<|x-y|\leqslant C_3 d(x)} K_1(x,y) f(y)\,dy$$

and therefore, in view of (2) and using the notation given in (4.19) we have, for $x \in Q$,

$$T_{1,\varepsilon} f(x) = T_{1,\varepsilon,cd_Q}(\chi_{Q^*} f)(x) + \int\limits_{cd_Q<|x-y|\leqslant C_3 d(x)} K_1(x,y) f(y)\,dy \tag{4.22}$$

where χ_{Q^*} is the characteristic function of Q^*. But, recalling that $K_1(x,y) \leqslant C/|x-y|^n$ and that, for $x \in Q$, $d_Q \sim d(x)$, it is easy to see that

$$\left| \int\limits_{cd_Q<|x-y|\leqslant C_3 d(x)} K_1(x,y) f(y)\,dy \right| \leqslant CMf(x)$$

for any $x \in Q$. In particular, if $f \in L^p(\Omega)$, the second term in the right-hand side of (4.22) is a function of $L^p(Q)$. Therefore, it follows from (4.21) that $T_{1,\varepsilon} f$ converges in the $L^p(Q)$ norm to a function $T_1 f$. Moreover, using (4.20), we obtain from (4.22) that

$$\int\limits_Q \left|T_{1,\varepsilon} f(x)\right|^p dx \leqslant C \left\{ \int\limits_{Q^*} \left|f(x)\right|^p dx + \int\limits_Q \left|Mf(x)\right|^p dx \right\}. \tag{4.23}$$

On the other hand, if $\varepsilon \geqslant cd_Q$, the same argument shows that, for $x \in Q$,

$$\left|T_{1,\varepsilon} f(x)\right| \leqslant CMf(x)$$

and so, (4.23) is true for all ε. Therefore, summing over all $Q \in W$, using the boundedness in L^p of the Hardy–Littlewood maximal operator, and recalling that $\sum_Q \chi_{Q^*}(x) \leqslant C$ for some C depending only on the dimension, we obtain

$$\|T_{1,\varepsilon} f\|_{L^p(\Omega)} \leqslant C \|f\|_{L^p(\Omega)}$$

with a constant C independent of ε.

To finish the proof, it only remains to prove that $T_{1,\varepsilon} f$ converges to $T_1 f$ in $L^p(\Omega)$. If for $j \in \mathbb{N}$ we call W_j the subset of W of all the cubes of side length less than 2^{-j}, we have that the measure of $\bigcup_{Q\in W_j} Q^*$ tends to zero when j tends to ∞. Therefore, in view of (4.23) and the fact that $\|T_{1,\varepsilon} f\|_{L^p(Q)} \to \|T_1 f\|_{L^p(Q)}$ in $L^p(Q)$ for every $Q \in W$, we can make $\|T_{1,\varepsilon} f\|_{L^p(\bigcup_{Q\in W_j})}$ and $\|T_1 f\|_{L^p(\bigcup_{Q\in W_j})}$ (and consequently $\|T_1 f - T_{1,\varepsilon} f\|_{L^p(\bigcup_{Q\in W_j})}$) smaller than any given positive number by taking j large enough. Then, the proof concludes by observing that the cubes in $W \setminus W_j$ are a finite number and using that, for those cubes, $T_{1,\varepsilon} f \to T_1 f$ in $L^p(Q)$. □

Finally, we have to prove the continuity of the operator corresponding to K_2. Moreover, the next lemma shows in particular that the integral

$$\int K_2(x, y) f(y)\, dy$$

is absolutely convergent for almost every x when $f \in L^p$ and so, we can work directly with the operator $T_2 f = \lim_{\varepsilon \to 0} T_{2,\varepsilon}$.

Lemma 4.6. *There exists a constant $C = C(K, \delta, n, \psi)$ such that*

$$\left|T_2 f(x)\right| \leqslant C M f(x).$$

Proof. From the definition of K_2 it is easy to see that

$$\left|K_2(x, y)\right| \leqslant C \min\left\{\frac{1}{|x-y|^n}, \frac{1}{d(y)^n}\right\}$$

where C depends only on n and the L^∞ norm of ψ. Now, we can write

$$T_2 f(x) = \int\limits_{|x-y| \leqslant d(x)/2} K_2(x, y) f(y)\, dy + \int\limits_{d(x)/2 < |x-y| \leqslant C_3 d(x)} K_2(x, y) f(y)\, dy. \tag{4.24}$$

To bound the first part, observe that if $|x - y| \leqslant d(x)/2$, then $d(x)/2 \leqslant d(y)$ and therefore, using that

$$\left|K_2(x, y)\right| \leqslant C \frac{1}{d(y)^n},$$

we obtain

$$\left|\int\limits_{|x-y| \leqslant d(x)/2} K_2(x, y) f(y)\, dy\right| \leqslant \frac{C}{(d(x)/2)^n} \int\limits_{|x-y| \leqslant d(x)/2} \left|f(y)\right| dy \leqslant C M f(x).$$

Now, the other term of (4.24) can be bounded in an analogous way using that

$$\left|K_2(x, y)\right| \leqslant C \frac{1}{|x-y|^n}$$

and therefore the lemma is proved. □

Summing up all our results we obtain our main theorem:

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain with respect to x_0 and with constant L. Given $f \in L^p(\Omega)$, $1 < p < \infty$, such that $\int_\Omega f = 0$, the vector function*

$$\mathbf{u}(x) = \int_\Omega G(x, y) f(y)\, dy$$

with $G = (G_1, \dots, G_n) : \overline{\Omega} \times \Omega \to \mathbb{R}^n$ defined as in (3.3)*, verifies that $\mathbf{u} \in W_0^{1,p}(\Omega)^n$ and*

$$\operatorname{div} \mathbf{u} = f \quad \text{in } \Omega.$$

Moreover, there exists a constant $C = C(L, d(x_0), \operatorname{diam}(\Omega), n, \omega, p)$ such that

$$\|\mathbf{u}\|_{W^{1,p}(\Omega)^n} \leqslant C\|f\|_{L^p(\Omega)}. \tag{4.25}$$

Proof. First, using the bound for G given in (3.5) we obtain, by an application of the Young inequality, that $\mathbf{u} \in L^p(\Omega)^n$ and

$$\|\mathbf{u}\|_{L^p(\Omega)^n} \leqslant C\|f\|_{L^p(\Omega)} \tag{4.26}$$

with $C = C(\delta, K, n, \omega, \operatorname{diam}(\Omega))$.

From Lemma 3.6 we already know that $\operatorname{div} \mathbf{u} = f$. Now, the estimate (4.25) follows from Lemmas 4.1, 4.3, 4.6 and 4.5, and (4.26), recalling that, from Lemma 2.1, we know that the constants K and δ depend on L, $d(x_0)$ and $\operatorname{diam}(\Omega)$.

It only remains to show that $\mathbf{u} \in W_0^{1,p}(\Omega)^n$. But, the bound (3.7) gives that for any $0 < \alpha \leqslant 1$,

$$\left|\mathbf{u}(x)\right| \leqslant C d(x)^\alpha \int_\Omega \frac{f(y)}{|x - y|^{n-1+\alpha}}\, dy. \tag{4.27}$$

Now, suppose first that $p > n$ and let q be the dual exponent of p. If we take $\alpha < 1 - n/p$ then $q(n - 1 + \alpha) < n$ and then, using the Hölder inequality in (5.3) we obtain

$$\left|u(x)\right| \leqslant C d(x)^\alpha \|f\|_{L^p(\Omega)}$$

with $C = C(\delta, K, n, \omega, \operatorname{diam}(\Omega), p)$. In particular, $\mathbf{u}$ is continuous at the boundary. But, in [12] it is proved that for an arbitrary open set Ω, if a function is continuous, vanishes on $\partial\Omega$ and belongs to $W^{1,p}(\Omega)$, then it belongs to $W_0^{1,p}(\Omega)$. Therefore, we conclude the proof in the case $p > n$.

Finally, for any $1 < p < \infty$, take a sequence $f_m \in L^\infty(\Omega)$ such that $f_m \to f$ in $L^p(\Omega)$ and let

$$\mathbf{u}_m(x) = \int_\Omega G(x, y) f_m(y)\, dy.$$

Then, from (4.25) applied to $f - f_m$ it follows that $\mathbf{u}_m \to \mathbf{u}$ in $W^{1,p}(\Omega)^n$. But we already know that $\mathbf{u}_m \in W_0^{1,p}(\Omega)^n$ and therefore, $\mathbf{u} \in W_0^{1,p}(\Omega)^n$ and the theorem is proved. □

An important consequence of our result is the validity of the Korn inequality on bounded John domains. Although the argument used to prove this fact is well known, we recall it in the next theorem for the sake of completeness.

We will use the following standard notation. For $\mathbf{v} \in W^{1,p}(\Omega)^n$, $D\mathbf{v}$ denotes the matrix of first derivatives of $\mathbf{v}$ and $\varepsilon(\mathbf{v})$ its symmetric part (i.e., the strain tensor), namely,

$$\varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Theorem 4.2 *(Korn inequality). Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain. Then, there exists a constant C depending only on Ω such that*

$$\|D\mathbf{v}\|_{L^p(\Omega)^{n\times n}} \leqslant C\left\{\|\mathbf{v}\|_{L^p(\Omega)^n} + \|\varepsilon(\mathbf{v})\|_{L^p(\Omega)^{n\times n}}\right\}. \tag{4.28}$$

Proof. It is not difficult to see that the following inequality is a consequence of the result proved in Theorem 4.1,

$$\|f\|_{L^p(\Omega)} \leqslant C\left\{\|f\|_{W^{-1,p}(\Omega)} + \|\nabla f\|_{W^{-1,p}(\Omega)^n}\right\}$$

with a constant depending only on Ω. Therefore (4.28) follows by using this inequality and the well-known identity

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial \varepsilon_{ik}(v)}{\partial x_j} + \frac{\partial \varepsilon_{ij}(v)}{\partial x_k} - \frac{\partial \varepsilon_{jk}(v)}{\partial x_i}. \qquad \square$$

5. The converse for domains satisfying the separation property

A natural question is whether the condition of being a John domain is also necessary for the existence of continuous right inverses of the divergence. In this section we prove that a bounded domain $\Omega \subset \mathbb{R}^n$ which satisfies the separation property introduced in [2] is a John domain if and only if the divergence operator acting on $W_0^{1,p}(\Omega)^n$ admits a continuous right inverse for some p such that $1 < p < n$. In particular, the result applies to planar simply connected domains, indeed, it was proved in [2] that these domains satisfy the separation property.

Given p, we denote with p' its dual exponent and, if p is such that $1 < p < n$, we call p^* the "critical exponent," namely, $p^* = pn/(n-p)$.

It is easy to check that, if $1 < p < n$, then $(p^*)' < n$ and

$$\left[(p^*)'\right]^* = p'. \tag{5.1}$$

Lemma 5.1. *Let* $1 < p < n$. *If* Ω *admits a continuous right inverse of* $\operatorname{div} : W_0^{1,q}(\Omega)^n \to L_0^q(\Omega)$ *for* $q := (p^*)'$, *then the Sobolev–Poincaré inequality for* p *holds in* Ω, *namely, there exists a constant* C *such that*

$$\|f\|_{L^{p^*}(\Omega)} \leqslant C\|\nabla f\|_{L^p(\Omega)^n} \tag{5.2}$$

for all $f \in W^{1,p}(\Omega) \cap L_0^p(\Omega)$.

Proof. Given $f \in W^{1,p}(\Omega) \cap L_0^p(\Omega)$, let $g \in L^q(\Omega)$. From our hypothesis we know that there exists $\mathbf{u} \in W_0^{1,q}(\Omega)^n$ such that

$$\operatorname{div} \mathbf{u} = g - g_\Omega \quad \text{in } \Omega$$

and

$$\|\mathbf{u}\|_{W^{1,q}(\Omega)^n} \leqslant C\|g\|_{L^q(\Omega)} \tag{5.3}$$

where g_Ω denotes the average of g over Ω. We have

$$\int_\Omega fg = \int_\Omega f(g - g_\Omega) = \int_\Omega f \operatorname{div} \mathbf{u} = -\int_\Omega \nabla f \cdot \mathbf{u} \leqslant \|\nabla f\|_{L^p(\Omega)^n} \|\mathbf{u}\|_{L^{p'}(\Omega)^n}.$$

Now, since $\mathbf{u} \in W_0^{1,q}(\Omega)^n$, we know that

$$\|\mathbf{u}\|_{L^{q^*}(\Omega)^n} \leqslant C\|\mathbf{u}\|_{W^{1,q}(\Omega)^n}.$$

Indeed, since the extension by zero of $\mathbf{u}$ belongs to $W^{1,q}(\mathbb{R}^n)$, this inequality follows by a standard imbedding theorem. But, from (5.1) we know that $p' = q^*$ and so, using (5.3) we obtain

$$\int_\Omega fg \leqslant \|\nabla f\|_{L^p(\Omega)^n} \|\mathbf{u}\|_{W^{1,q}(\Omega)^n} \leqslant C\|\nabla f\|_{L^p(\Omega)^n} \|g\|_{L^q(\Omega)}$$

for any $g \in L^q(\Omega)$, and therefore the proof concludes recalling that $q = (p^*)'$. $\square$

Now, our result is a consequence of Lemma 5.1 and the following theorem which was proved in [2] (we refer to this paper for the separation property).

Theorem 5.1. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain satisfying the separation property. The Sobolev–Poincaré inequality* (5.2) *holds in* Ω *for some* p *such that* $1 < p < n$ *if and only if* Ω *is a John domain.*

Then, we have

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the separation property. Then, Ω admits a continuous right inverse of* $\operatorname{div}: W_0^{1,q}(\Omega)^n \to L_0^q(\Omega)$ *for some q such that $1 < q < n$ if and only if Ω is a John domain.*

Proof. In view of Theorem 4.1, it only remains to show that if Ω admits a continuous right inverse of $\operatorname{div}: W_0^{1,q}(\Omega)^n \to L_0^q(\Omega)$ for some q such that $1 < q < n$, then it is a John domain.

But this follows immediately from Lemma 5.1 and Theorem 5.1 observing that if $1 < q < n$, then $q = (p^*)'$ for $p = (q^*)'$. $\square$

In particular, for the case of planar domains we obtain the following result.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then, Ω admits a continuous right inverse of* $\operatorname{div}: W_0^{1,q}(\Omega)^2 \to L_0^q(\Omega)$ *for some q such that $1 < q < 2$ if and only if Ω is a John domain.*

Proof. The result is a consequence of the fact that a simply connected planar domain satisfies the separation property (see [2]). $\square$

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