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Reconstructing C^*-Algebras from Their Murray von Neumann Orders

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1. INTRODUCTION [8, 9, 14]

All C^*-algebras considered in this paper will be *separable* and *unital*. For any C^*-algebra A, we let $L(A)$ denote the set of Murray von Neumann equivalence classes of projections in A, equipped with the well-known comparison relation: a class $[p]$ is weaker (or, smaller) than a class $[q]$ if p is equivalent to a subprojection of q. We call $L(A)$ the *Murray von Neumann poset* of A. The CAR algebra and the universal UHF algebra are two nonisomorphic C^*-algebras whose Murray von Neumann posets are both order-isomorphic to the unit interval of $\mathbf{Q}$. We are interested in situations where, as a poset alone, $L(A)$ determines A, up to isomorphism. Kaplansky [16, Sect. 5] considered a similar problem for the Banach space $C(X)$ of real-valued continuous functions over a compact Hausdorff space X. By a *Boolean* space we mean a totally disconnected compact Hausdorff space. By the *spectrum* of A we mean the set of primitive ideals of A equipped with the hull-kernel topology. Our first result is the following

THEOREM 1. *If two liminary C^*-algebras with Boolean spectrum have order-isomorphic Murray von Neumann posets, then they are isomorphic.*

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A natural problem is now to characterize the Murray von Neumann posets of liminary C^*-algebras with Boolean spectrum. Recall that the *center* $B(L)$ of a bounded distributive lattice L (i.e., a distributive lattice with smallest element 0 and largest element 1) is the set of complemented elements in L. By prim L we denote the set of prime ideals of L ordered by inclusion. We let minprim L be the set of minimal prime ideals of L. Given a set S, we let card S denote the cardinality of S.

THEOREM 2. *L is the Murray von Neumann poset of some (hence, of a unique) liminary C^*-algebra with Boolean spectrum iff L is a countable bounded distributive lattice satisfying the following two additional conditions:*

(I) prim L *is a disjoint union of finite maximal chains, and*

(II) *for every $f \in L$ and $r \in \mathbf{Q}$, there is $b \in B(L)$ such that for each* $J \in$ minprim L

$$b \in J \quad \textit{iff} \quad r = \frac{\operatorname{card}\{K \in \operatorname{prim} L \mid J \subseteq K \textit{ and } f \notin K\}}{\operatorname{card}\{K \in \operatorname{prim} L \mid J \subseteq K\}}.$$

By a result of Bratteli and Elliott [3], each liminary C^*-algebra with Boolean spectrum is an *AF algebra*, i.e., [2], the norm closure of the union of an ascending sequence of finite-dimensional C^*-algebras, all with the same unit. Elliott [10] classified AF algebras in terms of their Murray von Neumann poset *plus* a partially defined addition given by summing equivalence classes of orthogonal projections.

Our final result concerns AF algebras A with comparability of projections, i.e., with totally ordered $L(A)$. The finite-dimensional case has already been settled by the previous theorems.

THEOREM 3. *Assume that A is an infinite-dimensional AF algebra with comparability of projections. Let J be the maximal ideal of A. Then the following two conditions are equivalent:*

(i) *A/J is isomorphic to $\mathbf{C}$, and each primitive quotient of A contains a smallest nonzero equivalence class of projections;*

(ii) *A is uniquely determined by $L(A)$ among all AF algebras.*

2. ON l-GROUPS AND MV ALGEBRAS [1, 4, 5]

Given a (proper) l-ideal J in an abelian l-group G, we write $J \in \operatorname{prim} G$ if G/J is totally ordered. We write $J \in \max G$ if J is maximal. We say that G is *hyperarchimedean* if prim $G = \max G$. For each $n = 1, 2, \ldots$, we let $\mathbf{Z}\,1/n$ denote the additive group of integral multiples of $1/n$ with the natural order. By an isomorphism in the category of abelian l-groups with strong unit we mean a unit-preserving lattice-group isomorphism.

PROPOSITION 2.1. *Let G be an abelian l-group with a distinguished strong unit u. Suppose that for every $J \in \operatorname{prim} G$ there is $k = 1, 2, \ldots$ such that $(G/J, u/J) \cong (\mathbf{Z}\, 1/k, 1)$. Then G is hyperarchimedean. Further, (G, u) is isomorphic to an l-group H of rational-valued functions with finite range over the Boolean space $X = \max G$ with the constant function 1 as the strong unit, and any two different points of X are separated by a function of H. For each $f \in H$ and $r \in \mathbf{Q}$ there is a Boolean element $b \in H$, i.e., an element $0 \leqslant b \leqslant 1$ with $(b + b) \wedge 1 = b$, such that for all $x \in X$, $b(x) = 0$ iff $f(x) = r$.*

Proof. Since G/J is a subgroup of $\mathbf{R}$, J is maximal [1, 2.6.7], and G is hyperarchimedean by [1, 14.1.2]. The isomorphism of G with a separating l-group H of rational-valued functions on the compact Hausdorff space $X = \operatorname{prim} G = \max G$ follows from [1, 13.2.6], together with our hypothesis about each G/J. The space X is totally disconnected, because G is hyperarchimedean and has a strong unit [1, 14.1.5]. Identifying G with H, we have the canonical identification $x \in X \to J_x \in \max G$, where $J_x = \{f \in G \mid f(x) = 0\}$. Accordingly, we identify $f(x)$ with f/J_x. Suppose that range(f) is infinite (absurdum hypothesis). By continuity and compactness, range(f) is a closed and bounded subset of $\mathbf{R}$. Let ρ be an accumulation point of range(f). Then $\rho \in \text{range}(f)$, whence $\rho \in \mathbf{Q}$, say $\rho = a/b$, for some integers a, b. Set $g = |bf - a| \wedge 1$. Then, for each $\varepsilon > 0$, range(g) has infinite intersection with the closed interval $[0, \varepsilon]$. This contradicts the fact that G is hyperarchimedean [1, 14.1.2(vi)]. To conclude the proof, let $r = m/n$. Set $h = |nf - m| \wedge 1$. Then $h^{-1}(0) = f^{-1}(r)$. For some integer $k > 0$, $b = kh \wedge 1$ will be the required Boolean element. Q.E.D.

An *MV algebra* $D = (D, 0, 1, {}^*, \oplus, \cdot)$ is a commutative semigroup $(D, \oplus)$ such that $x \oplus 0 = x$, $x \oplus 1 = 1$, $0^* = 1$, $1^* = 0$, $x \cdot y = (x^* \oplus y^*)^*$, and $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$. Replacing y in the last equation by 0 and by 1, we obtain $x^{**} = x$, and $1 = x^* \oplus x$, respectively. Thus, by [19, 2.6], the present definition agrees with Chang's original definition [4, p. 468]. We let prim D and max D, respectively, denote the set of prime and the set of maximal ideals of D, as defined in [5]. Given an abelian l-group G with strong unit $u = 1$, the functor Γ equips the unit interval $[0, 1] = \{x \in G \mid 0 \leqslant x \leqslant 1\}$ with the operations $x^* = 1 - x$, $x \oplus y = 1 \wedge (x + y)$, and $x \cdot y = 0 \vee (x + y - 1)$. Γ also restricts to $[0, 1]$ every morphism of $(G, 1)$. By [19, 3.9], Γ is a categorical equivalence between abelian l-groups with strong unit, and MV algebras. The map $J \to J \cap [0, 1]$ is a homeomorphism of prim G onto prim $\Gamma(G, 1)$. As proved in [4, 1.4 and 1.11], for every MV algebra D, the operations $x \vee y = (x^* \oplus y)^* \oplus y$ and

$x \wedge y = (x^* \vee y^*)^*$ make D into a distributive lattice $L(D)$, in which 0 is the smallest element, and 1 is the largest. We denote by $B(D)$ the set of *Boolean* elements of D, i.e., those elements x such that $x \oplus x = x$. By [4, 1.17], $B(D)$ is an MV subalgebra of D which is also a Boolean algebra; the lattice operations $\vee$ and $\wedge$ coincide over $B(D)$ with the MV operations $\oplus$ and $\cdot$, respectively; in addition, $B(D) = B(L(D))$.

DEFINITION. An MV algebra A is *liminary* if A/J is finite for all $J \in \operatorname{prim} A$.

PROPOSITION 2.2. *Every liminary MV algebra is isomorphic to an MV algebra A of rational-valued continuous functions over the Boolean space $X = \max A = \operatorname{prim} A$. Any two distinct points of X are separated by an element of $B(A)$. Every function of A has a finite range. For each $f \in A$ and $r \in \mathbf{Q}$ there is $b \in B(A)$ such that for all $J \in \operatorname{prim} A$, $b \in J$ iff $f(J) = r$.*

Proof. By [19, Sect. 3], A can be written as $\Gamma(G, 1)$ for a unique (up to isomorphism) l-group G with strong unit 1. By the preservation properties of the Γ functor, the map $J \to J \cap [0, 1]$ is a one–one correspondence between l-ideals of G and ideals of A. Under this correspondence, prime (resp., maximal) l-ideals of G are mapped into prime (resp., maximal) ideals of A. A straightforward computation shows that $\Gamma(G/J, 1/J) \cong \Gamma(G, 1)/(J \cap [0, 1])$. Thus, $(G, 1)$ verifies the hypotheses of Proposition 2.1, and its Boolean elements are exactly the Boolean elements of A. The desired conclusions now easily follow by identifying G with an l-group of rational-valued functions over prim G, as in Proposition 2.1, on noting that by definition of Γ, $A = \{f \in G \mid 0 \leqslant f \leqslant 1\}$. Q.E.D.

Remark. By Proposition 2.2, we may identify every liminary MV algebra A with a separating MV algebra of rational-valued continuous functions over the space $X = \max A = \operatorname{prim} A$. The map $y \to J_y = \{f \in A \mid f(y) = 0\}$ identifies points of X and maximal ideals of A. For any $y \in X$ there is a unique integer $n = n_y \geqslant 1$ such that A/J_y is isomorphic to the chain $\{0, 1/n, ..., (n-1)/n, 1\}$ with the natural MV operations.

PROPOSITION 2.3. *Let A be a liminary MV algebra with underlying lattice L. For each $x \in X$ and $r \in \{0, 1/n_x, ..., (n_x - 1)/n_x, 1\}$, set $J_{xr} = \{f \in A \mid f(x) \leqslant r\}$. It follows that*

(i) *J_{xr} is a prime ideal of L;*

(ii) *for each $J \in \operatorname{prim} L$, there is precisely one pair (x, r) with $J = J_{xr}$.*

Proof. (i) This is trivial.

(ii) Let $B=B(L)$. Then $I=J\cap B$ is a maximal ideal of B; in symbols, $I\in\max B$. Let $V_I\subseteq X=\operatorname{prim} A$ be given by $V_I=\bigcap\{b^{-1}(0)\mid b\in I\}$. Given $b_1, ..., b_k$ in I, there is a point $w\in X$ such that $(b_1\vee\cdots\vee b_k)(w)=0$, and since all the functions in A are continuous, we have that V_I is the intersection of a family of closed sets with the finite intersection property. The compactness of X implies that $V_I\neq\emptyset$. By maximality of I, together with the separation property of Proposition 2.2, V_I is a singleton, say $V_I=\{x\}$. Further, for each $b\in B$, we have that $b\in I$ iff $b(x)=0$. Since A is liminary, we may define $r=\max\{f(x)\mid f\in J\}\in\mathbf{Q}$, because $f(x)\in\{0, 1/n_x, 2/n_x, ..., (n_x-1)/n_x, 1\}$ for each $f\in A$.

To conclude the proof, it remains to be shown that $J=J_{xr}$. To this purpose, note that if $g\notin J_{xr}$ (i.e., $g(x)>r$), then $g\notin J$ by the definition of r. Conversely, if $g\in J_{xr}$, say $g(x)=s\leqslant r$, choose $f\in J$ with $f(x)=r$. By Proposition 2.2, there is $b\in B$ such that for all $y\in X$, $b(y)=0$ iff $g(y)=s$ and $f(y)=r$. Then $b(x)=0$, whence $b\in I\subseteq J$, and $b\vee f\in J$. The inequality $b\vee f\geqslant g$ holds true over the set $W=b^{-1}(0)$, because $f(z)=r\geqslant s$ over W; the inequality is also true over the complementary set $X\setminus W$, because $b=1$ over $X\setminus W$. It follows that $g\in J$. Therefore, $J=J_{xr}$, as required. Q.E.D.

PROPOSITION 2.4. *Let A be a liminary MV algebra. Then for all $x, y\in X$, $r\in\{0, 1/n_x, ..., (n_x-1)/n_x, 1\}$, and $s\in\{0, 1/n_y, ..., (n_y-1)/n_y, 1\}$ we have*

(i) *if $r<s$ then J_{xr} is strictly contained in J_{xs};*

(ii) *if $x\neq y$ then J_{xr} is incomparable with J_{ys};*

(iii) *the set* minprim L *of minimal prime ideals of L coincides with* prim A.

Proof. This follows from Proposition 2.3 and the separation property in Proposition 2.2. Q.E.D.

Remark. A more general result than Proposition 2.4(iii) was proved in [20].

PROPOSITION 2.5. *Assume that A is a liminary MV algebra with underlying lattice L. Then for all $f\in A$ and $x\in X$ we have* $f(x)=\operatorname{card}\{J\in\operatorname{prim} L\mid J_{x0}\subseteq J \text{ and } f\notin J\}/\operatorname{card}\{J\in\operatorname{prim} L\mid J_{x0}\subseteq J\}$.

Proof. This is by direct inspection. Q.E.D.

PROPOSITION 2.6. *Let A be a liminary MV algebra. Then its underlying lattice $L=L(A)$ is bounded and distributive, and satisfies the following additional conditions:*

(I) *prime ideals in L occur in finite maximal pairwise disjoint chains;*

(II) *for each $f \in L$ and $r \in \mathbf{Q}$, there is an element b in the center B of L such that for all $J \in \operatorname{minprim} L$, we have $b \in J$ iff $r = \operatorname{card}\{K \in \operatorname{prim} L \mid J \subseteq K$ and $f \notin K\}/\operatorname{card}\{K \in \operatorname{prim} L \mid J \subseteq K\}$.*

Proof. This is immediate from Propositions 2.2–2.5. Q.E.D.

3. The Lattices of Liminary C^*-Algebras with Boolean Spectrum [14]

Recall that for any distributive lattice L, we let prim L denote the set of prime ideals of L ordered by inclusion. We let minprim L denote the set of minimal prime ideals of L. We denote by B (or, by $B(L)$ if there is danger of confusion) the center of L, and by max B the family of maximal ideals of B. In light of Proposition 2.6, in this section we consider bounded distributive lattices L, satisfying the following two conditions:

(I) prim L is a disjoint union of finite maximal chains;

(II) for all $f \in L$ and $r \in \mathbf{Q}$, there is $b \in B$ such that for all $J \in \operatorname{minprim} L$, $b \in J$ iff $r = f^{\circ}(J) = \operatorname{card}\{K \in \operatorname{prim} L \mid J \subseteq K$ and $f \notin K\}/\operatorname{card}\{K \in \operatorname{prim} L \mid J \subseteq K\}$.

PROPOSITION 3.1. *Suppose that L is a bounded distributive lattice satisfying Conditions (I) and (II). Then the map $f \to f^{\circ}$ is a lattice embedding of L into the lattice $\mathbf{Q}^{\operatorname{minprim} L}$ with natural pointwise operations.*

Proof. By the Birkhoff Stone theorem [14, p. 75], the map is one–one. The proof that for each $K \in \operatorname{minprim} L$, $(f \wedge g)^{\circ}(K) = f^{\circ}(K) \wedge g^{\circ}(K)$ uses primality of K. Q.E.D.

LEMMA 3.2. *Suppose that L is a bounded distributive lattice.*

(i) *If $J \in \operatorname{prim} L$ and $b \in B \cap J$, then b belongs to every prime $K \subseteq J$.*

(ii) *Given $f \in L$ and $b \in B$, assume that f and b belong to the same minimal primes of L. Then b is the smallest Boolean majorant of f.*

(iii) *If, in addition, L satisfies Conditions (I) and (II), then for each $f \in L$ there is a unique $b = b_f \in B$ such that b and f belong to the same minimal primes of L; moreover, for each g and h in L, $b_{g \wedge h} = b_g \wedge b_h$.*

Proof. (i) This is trivial.

(ii) We first prove that $b \geqslant f$. Otherwise, by the Birkhoff Stone theorem there is $J \in \operatorname{prim} L$ such that $b \in J$ and $f \notin J$; by (i), J may be assumed to be a minimal prime, thus contradicting our hypothesis. We now prove that b is the smallest majorant of f. By way of contradiction, suppose that $d \in B$, $d \geqslant f$, but $d \geqslant b$ does not hold. Again by the Birkhoff Stone theorem, let J be a prime ideal with $d \in J$ and $b \notin J$; by (i), J may be assumed to be minimal, whence by assumption $f \notin J$, contradicting $f \leqslant d$.

(iii) The existence of b_f is ensured by setting $r = 0$ in Condition (II), and using Condition (I); uniqueness follows from (ii). To conclude the proof, note that for all $J \in \text{minprim } L$, $b_{g \wedge h} \in J$ iff $g \wedge h \in J$ iff either $g \in J$ or $h \in J$ iff either $b_g \in J$ or $b_h \in J$ iff $b_g \wedge b_h \in J$. Now apply (ii). Q.E.D.

PROPOSITION 3.3. *Suppose that* L *is a bounded distributive lattice satisfying Conditions* (I) *and* (II) *above. Then we have*:

(i) *each* $I \in \max B$ *generates a minimal prime ideal of* L;

(ii) *each* $J \in \text{minprim } L$ *is generated by exactly one* $I \in \max B$, *namely, by* $I = J \cap B$.

Proof. (i) Let J denote the ideal generated by I. From Lemma 3.2(ii)–(iii), it follows that for all $f \in L$, $f \in J$ iff $b_f \in I$; thus, J is prime. Let P denote the minimal prime ideal below J, as given by Condition (I). Then $P \cap B \in \max B$, whence $P \cap B = J \cap B$. Since J is generated by $I = I \cap B \subseteq J \cap B$, it follows that $J \cap B$ generates J. Since $P \cap B$ generates some prime ideal contained in P, $P \cap B$ generates P. We conclude that $P = J$.

(ii) Clearly, $I = J \cap B$ is a maximal Boolean ideal. By (i), I generates a minimal prime $P \subseteq J$, whence $P = J$ by the assumed minimality of J. Uniqueness of I follows from its maximality. Q.E.D.

Remark. The map $f \to b_f$ is considered in [7] under the name of Boolean multiplicative closure. In that paper it is proved that lattices having Boolean multiplicative closures automatically satisfy Conditions (i) and (ii) of Proposition 3.3. Bounded distributive lattices satisfying Conditions (I) and (II) are a subclass of the lattices considered in [6]. The counterexample given in [6, p. 370] shows that bounded distributive lattices fulfilling (I) and satisfying (i) and (ii) of Proposition 3.3 need not satisfy Condition (II).

PROPOSITION 3.4. *Suppose that* L *is a bounded distributive lattice satisfying Conditions* (I) *and* (II). *Then for each* $f \in L$, *the function* f° *has a finite range.*

Proof. First, recalling Proposition 3.1, we may identify L with a lattice of rational-valued functions over minprim L. Using the canonical bijection from minprim L onto max B given by Proposition 3.3, we see that Condition (II) amounts to requiring continuity of each f: minprim $L \to \mathbf{Q}$, where minprim L is now equipped with the Stone topology of max B, and

$\mathbf{Q}$ is equipped with the discrete topology. We conclude the proof by recalling the compactness of max B. Q.E.D.

Remark. Consider the following condition, where $\mathbf{N} = \{0, 1, 2, ...\}$ denotes the set of natural numbers:

(II*) for all $f \in L$ and $n \in \mathbf{N}$ there is $b \in B(L)$ such that for each $J \in \text{minprim } L$, $b \in J$ iff $f^{\circ\circ}(J) = n$, where $f^{\circ\circ}(J) = \text{card}\{K \in \text{prim } L \mid J \subseteq K$ and $f \notin K\}$.

In any bounded distributive lattice L satisfying (I), Condition (II*) amounts to requiring the continuity of each function $f^{\circ\circ}$: minprim $L \to \mathbf{N}$. By compactness, each $f^{\circ\circ}$ has a finite range. Letting $f = 1$ in (II*), we see that the denominator of f° in (II) is continuous, whence Condition (II*) implies that each function f° in (II) is continuous, and (II*) is stronger than (II). It is not hard to see that (II*) is *strictly* stronger than (II): as a matter of fact, bounded distributive lattices fulfilling (I) and (II*) are precisely the underlying lattices of finite products of Post MV algebras of finite order, while—by Proposition 2.6, together with Proposition 3.5—lattices fulfilling (I) and (II) are the underlying lattices of liminary MV algebras. Recall that for $n \geq 2$, A is said to be a *Post MV algebra of order* n if A is isomorphic to the MV algebra of all continuous functions from a Boolean space into the n-element MV algebra $\{0, 1/(n-1), 2/(n-1), ..., (n-2)/(n-1), 1\}$, with the natural pointwise operations and the discrete topology. Clearly, the liminary MV algebras strictly include the Post MV algebras of finite order.

PROPOSITION 3.5. *Suppose that L is a bounded distributive lattice satisfying Conditions* (I) *and* (II). *Let A denote the MV algebra generated by L in the set $\mathbf{Q}^{\text{minprim } L}$ with the natural pointwise MV operations. Then*:

(i) *each element of A is already in L, and L coincides with $L(A)$, the underlying lattice of A*;

(ii) *A is liminary*;

(iii) *if D is an MV algebra whose underlying lattice $L(D)$ is order-isomorphic to $L(A)$, then D is isomorphic to A.*

Proof. (i) Using Proposition 3.3, equip the set $X = \text{minprim } L$ with the Stone topology of max B. On identifying each element of $f \in L$ with a rational-valued continuous function over minprim L, by Proposition 3.4, A will be a subalgebra of the MV algebra of all continuous rational-valued functions with finite range defined over X (because the set of all such functions forms an MV algebra containing L). Then it is immediate to see that Condition (II) holds for all functions $g \in A$, whence, for every $r \in \mathbf{Q}$, there is $b \in B$ such that for every $K \in X$, $b \in K$ iff $g(K) = r$. We now show that whenever a function f is a member of L, then so is its negation $f^* = 1 - f$. Suppose that $r = m/n \in \text{range}(f)$. There is a clopen $U \subseteq X$ such that $U = f^{-1}(r)$. For every $Y \in U$ there exists a function $g \in L$ such that $g(Y) = 1 - f(Y) = 1 - r$. By continuity, this equality holds in a clopen neighbourhood V of Y. A compactness argument shows that there are clopen sets $V_1, ..., V_k$ and functions $g_1, ..., g_k \in L$ such that $U = V_1 \cup \cdots \cup V_k$ and $g_i = 1 - r$ over V_i, for each $i = 1, ..., k$. Let b_i denote the element of $B(L)$ which is equal to 1 over V_i and is equal to 0 over $X \setminus V_i$. Then over U we have $f^* = (g_1 \wedge b_1) \vee \cdots \vee (g_k \wedge b_k)$. Repeating the argument for each $s \in \text{range}(f)$, we may write $f^* = (g_1 \wedge b_1) \vee \cdots \vee (g_h \wedge b_h)$ for suitable $g_1, ..., g_h \in L$ and $b_1, ..., b_h \in B(L)$. Since L is a lattice, f^* is an element of L. A similar argument shows that if f and g are in L, then so also is $f \oplus g$. Therefore, A and L have the same elements. To prove that $L(A)$ and L are the same lattice, given any two functions $f, g \in L$ and a point $K \in X$, observe that the pointwise lattice operations of L, as well as those of $L(A)$, are expressible in exactly the same way in terms of the MV operations of A as follows: $(f \vee g)(K) = ((f^* \oplus g)^* \oplus g)(K)$, and $(f \wedge g)(K) = (f^* \vee g^*)^*(K)$.

(ii) Let K be a prime ideal of A. Then the underlying lattice of the quotient MV algebra A/K is totally ordered. If A/K were infinite (absurdum hypothesis), then we would have an infinite ascending sequence of prime ideals of $L(A)$, all greater than K. By (i), $L = L(A)$ would then violate Condition (I), a contradiction. Thus, A is liminary.

(iii) (Compare with [18, last Theorem, p. 42].) As in (ii), D is liminary, and we may apply the results of Section 2. In particular, by Proposition 2.4(iii), we have prim D = minprim L = prim A. If A and D are not isomorphic MV algebras (absurdum hypothesis), there are functions f and g in L, together with some $K \in$ minprim L such that either $(f^{*_A})(K) \neq (f^{*_D})(K)$, or $(f \oplus_A g)(K) \neq (f \oplus_D g)(K)$. By Proposition 2.5, the underlying lattices of the MV quotients A/K and D/K are equal to the same finite chain $R = \{0, 1/n, ..., (n-1)/n, 1\}$. Since, by [4, 1.4(vi)], the $*$ operation is order-reversing, it follows that $(f^{*_A})(K) = (f^{*_D})(K)$, and hence, the operations $*^A$ and $*^D$ must coincide. In order to prove that $\oplus_A$ coincides with $\oplus_D$, let us display the elements of R as follows: $0 = r_0 < r_1 < r_2 < \cdots < r_{n-1} < r_n = 1$. We already know that for every MV algebra $B = (B, 0, 1, *, \oplus, \cdot)$ such that $L(B) = R$, the $*$ operation is uniquely determined by R, specifically, $r_i^* = r_{n-i}$, for each $i = 0, ..., n$. To show that $\oplus$ is uniquely determined by R, given $r_i \in R$, if $i = n$, $r_i \oplus r_1 = 1$; if $i < n$, by [4, 3.15] we have that $r_i \oplus (r_i^* \cdot r_{i+1}) = r_{i+1}$, and, a fortiori, $r_i^* \cdot r_{i+1} \geqslant r_1$. Furthermore, we have the inequality $r_i \oplus r_1 \geqslant r_{i+1}$, for otherwise, by [4, 1.8], $r_i \oplus r_1 = r_i \oplus 0 < 1$, whence by [4, 3.13], $r_1 = 0$, which is impossible. Therefore we can write $r_i \oplus r_1 \leqslant r_i \oplus (r_i^* \cdot r_{i+1}) = r_{i+1} \leqslant r_i \oplus r_1$.

It is now easy to see that $r_i \oplus r_j = r_{\min(n, i+j)}$ for all $i, j = 0, 1, ..., n$, and hence, B is uniquely determined by R, as required. (Compare with [18, first Theorem, p. 35].) In particular, the $\oplus$ operation of the quotient MV algebra A/K agrees with that of D/K, and hence $\oplus_A$ coincides with $\oplus_D$. Since in every MV algebra the $\cdot$ operation is definable in terms of the operations $*$ and $\oplus$, we have proved that the MV algebras A and D are isomorphic, thus providing the required contradiction to our absurdum hypothesis. Q.E.D.

4. Proof of Theorems [9, 10, 13]

Proof of Theorem 1. Let A and B be liminary C^*-algebras with Boolean spectrum. Since by our standing hypothesis, A and B are unital, each primitive quotient of A and B is finite-dimensional [8, 4.7.14(b)]. By [3, Theorem, p. 80, Step (i)], A and B are AF algebras. By Elliott's classification theorem [10], we may write, in the notation of [13], $(K_0(A), [A]) = (G, u)$, and $(K_0(B), [B]) = (H, v)$, for some countable partially ordered abelian groups G and H with the Riesz interpolation property (i.e., the sum of two intervals is an interval), and with strong unit u and v, respectively.

By an ideal J of G we mean a directed subgroup of G with the following property: whenever $x \leqslant y \leqslant z$ and $x, z \in J$ then $y \in J$. This is equivalent to the definition given in [12]. As a consequence of the Riesz property, the intersection of two ideals of G is an ideal. A prime ideal of G is an ideal $p \neq G$ such that, whenever the intersection of two ideals I and J is contained in p, then either I or J is contained in p. By the spectrum of G, in symbols, Spec G, we mean the space of all prime ideals of G, with the Jacobson, or Zariski, topology, in which an open set is the set of all prime ideals not containing some ideal of G. It follows from [11, third paragraph, p. 43], that Spec G is homeomorphic to the primitive spectrum of A. Since, by assumption, the latter is a Hausdorff space, then so is the former.

In particular, the intersection of two compact open sets in Spec G is compact. As remarked above, every primitive quotient of A is finite-dimensional, and hence, has comparability of projections. Then the preservation properties of K_0 for exact sequences [9, 9.1], ensure that for every prime ideal p of G, the quotient G/p is totally ordered—indeed G/p is isomorphic to the additive group of integers with the natural order.

Applying to G the main theorem of [12], we obtain that G is lattice-ordered. Another application of the preservation properties of K_0 shows that G verifies the hypotheses of Proposition 2.1, the spectrum of G coinciding in this case with the topological space prim G.

Similar conclusions hold for H. Set $R = \Gamma(G, u)$ and $S = \Gamma(H, v)$. Then the preservation properties of Γ established in [19, Sect. 3] ensure that R

and S are countable liminary MV algebras. By Proposition 2.6, $L(R)$ and $L(S)$ are countable bounded distributive lattices satisfying Conditions (I) and (II). Further, by [10], $L(R)$ and $L(S)$ are order-isomorphic to the Murray von Neumann posets of A and B, respectively. Since by hypothesis, $L(R)$ is order-isomorphic to $L(S)$, by Proposition 3.5, R is isomorphic to S. Since, by [19, 3.9], Γ is a categorical equivalence, it follows that (G, u) is isomorphic to (H, v). By Elliott's classification theorem, A is isomorphic to B. Q.E.D.

Proof of Theorem 2. ($\rightarrow$) This follows from the above proof of Theorem 1.

($\leftarrow$) Given a countable bounded distributive lattice L satisfying Conditions (I) and (II), by Propositions 3.1 and 3.4 we may identify L with a lattice of rational-valued functions over minprim L, each function having a finite range. By Proposition 3.5, there is (up to isomorphism) a unique MV algebra R such that $L = L(R)$. Moreover, R is countable and liminary. By [19, 3.9], we may write $R = \Gamma(G, u)$ for exactly one abelian l-group G with unit u. Further, G is countable, and each prime quotient G/J is of the form $\mathbf{Z}\, 1/k$, with 1 as the strong unit, for suitable $k = k_J = 1, 2, \ldots$. As a consequence of Elliott's classification theorem [10], together with the Effros Handelman Shen theorem [9, 3.1], we may write $(G, u) = (K_0(A), [A])$ for a unique AF algebra A; furthermore, A is unital and separable, and the preservation properties of K_0 for exact sequences ensure that A is liminary. By [11, third paragraph, p. 43], the spectrum of A is homeomorphic to prim $K_0(A)$. By Proposition 2.1, the latter concides with max $K_0(A)$, and is Boolean. Thus, L is the Murray von Neumann poset of some (actually, of a unique) liminary unital separable C^*-algebra A with Boolean spectrum. Q.E.D.

Proof of Theorem 3. (ii) $\rightarrow$ (i). Assuming that (i) fails for A, we prove that (ii) fails.

Case 1. Some primitive quotient of A contains no smallest nonzero equivalence class of projections.

Then by Elliott's classification, together with the preservation properties of K_0 for exact sequences, $K_0(A)$ is a countable totally ordered abelian group having a densely ordered (prime) quotient. Let

$$\{0\} = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_\beta \subseteq J_{\beta+1} \subseteq \cdots \subseteq J_\mu \subseteq K_0(A)$$

be an ascending chain of convex subgroups of $K_0(A)$, indexed by the ordinals smaller than the countable ordinal $\mu + 1$, satisfying the following conditions:

(a) for all $\beta < \mu$, the quotient $J_{\beta+1}/J_\beta$ is isomorphic to $\mathbf{Z}$, the additive group of integers with natural order;

(b) whenever β is a limit ordinal $\leqslant \mu$, $J_\beta = \bigcup_{\gamma < \beta} J_\gamma$;

(c) $K_0(A)/J_\mu$ is dense (equivalently, $K_0(A)/J_\mu$ has no smallest element > 0).

It follows from Maltsev's analysis [17, VII, Sect. 1] that a chain satisfying (a), (b), and (c) exists and is unique. Let now S be the direct sum of μ copies of $\mathbf{Z}$, and $S \oplus \mathbf{Q}$ be the direct sum of S with the additive group of rationals. Equip $S \oplus \mathbf{Q}$ with the lexicographic order: thus, by definition, for any element $k = (k_0, k_1, ..., k_\beta, k_{\beta+1}, ..., r) \in S \oplus \mathbf{Q}$, we say that $k > 0$ if either $r > 0$, or $r = 0$ and $k_\alpha > 0$, where α is the largest ordinal such that $k_\alpha \neq 0$ (this order is well defined, since only finitely many terms of k are nonzero). Since any two countable dense total orders without endpoints are isomorphic, it follows that the order of $S \oplus \mathbf{Q}$ is isomorphic to the order of $K_0(A)$. Without loss of generality, $L(A)$ is order-isomorphic to the interval $[0, u]$ of $S \oplus \mathbf{Q}$, where $u = (0, 0, ..., 0, ..., 1)$. We now exhibit two nonisomorphic AF algebras A' and A'' such that both $L(A')$ and $L(A'')$ are order-isomorphic to $[0, u]$. Using Elliott's classification, let A' be defined by $(K_0(A'), [A']) = (S \oplus \mathbf{Q}, u)$. Let now $S \oplus \mathbf{D}$ denote the lexicographic direct sum of S with the additive group $\mathbf{D}$ of dyadic rationals (namely, the rationals of the form $a/2^n$, with $a \in \mathbf{Z}$ and $n = 0, 1, ...$), again with the strong unit u. The unit interval of $S \oplus \mathbf{D}$ is order-isomorphic to $[0, u]$, but $(S \oplus \mathbf{D}, u)$ and $(S \oplus \mathbf{Q}, u)$ are not isomorphic as totally ordered groups with strong unit: for, u is divisible by 3 in $S \oplus \mathbf{Q}$, and u is not divisible by 3 in $S \oplus \mathbf{D}$. By Elliott's classification, the AF algebra A'' given by $(K_0(A''), [A'']) = (S \oplus \mathbf{D}, u)$ is not isomorphic to A', while both $L(A'')$ and $L(A')$ are order-isomorphic to $L(A)$. Therefore, (ii) fails for A.

Case 2. A/J is not isomorphic to $\mathbf{C}$.

We may assume that each primitive quotient of A has a smallest nonzero equivalence class of projections. Since K_0 preserves exact sequences, every quotient of $K_0(A)$ has a smallest element > 0. In the terminology of [15, p. 290], $K_0(A)$ is *ω-discrete*. By [15, 4.6], $K_0(A)$ may be identified, as a totally ordered group, with the lexicographic direct sum T of $\beta + 1$ copies of $\mathbf{Z}$, for some countable ordinal β. Since A is infinite-dimensional, $\beta \geqslant 1$. Further, the image K of J under K_0 is the maximal (proper) convex subgroup of $K_0(A)$, and, by our current assumption, we can write $(K_0(A)/K, [A]/K) = (\mathbf{Z}, k)$, for some integer $k > 1$. It follows that $L(A)$ is order-isomorphic to the interval $[0, u]$ of T, where $u = (0, 0, ..., 0, ..., k)$. We exhibit two nonisomorphic AF algebras A' and A'' with both $L(A')$ and $L(A'')$ order-isomorphic to $L(A)$. We define A' by $(K_0(A'), [A']) = (T, u)$. Let $a = (1, 0, ..., 0, ..., 0) \in T$ be the smallest element > 0. Then the interval $[0, u + a]$ is order-isomorphic to $[0, u]$. However, the totally ordered groups with strong unit (T, u) and $(T, u + a)$ are not isomorphic, because

u is divisible by k, and $u+a$ is not. Using Elliott's classification together with [11], the AF algebra A'' given by $(K_0(A''), [A'']) = (T, u+a)$ is not isomorphic to A', while both $L(A'') = [0, u+a]$ and $L(A') = [0, u]$ are order-isomorphic to $L(A)$. We conclude that (ii) fails for A also in the present case, as required to complete the proof that (ii) → (i).

(i) → (ii). By Elliott's classification, it follows that $(G, u) = (K_0(A), [A])$ is a countable totally ordered abelian group with strong unit u. Moreover, as in the above Case 2, we have that every quotient G/I has a smallest element >0. By [15, 4.6], for some countable ordinal β, the totally ordered group G may be identified with the lexicographic direct sum of $\beta+1$ copies of $\mathbf{Z}$. Since by assumption A is infinite-dimensional, it follows that $\beta \geqslant 1$. By definition of lexicographic order, an element $k = (k_0, k_1, ..., k_\alpha, ..., k_\beta) \in G$ is >0 iff $k_\alpha > 0$, where α is the largest ordinal such that $k_\alpha \neq 0$. Since $A/J \cong \mathbf{C}$, the image K of J under K_0 is the maximal (proper) convex subgroup of G, and $(G/K, u/K) \cong (\mathbf{Z}, 1)$. It is no loss of generality to assume that $u = (0, 0, ..., 0, ..., 1)$. Suppose (absurdum hypothesis) that an AF algebra A' is not isomorphic to A, while $L(A')$ is order-isomorphic to $L(A)$. Then, in particular, $L(A')$ is totally ordered. Since in the partially ordered group $(G', u') = (K_0(A'), [A'])$, sums of intervals are intervals, and u' is a strong unit, an easy induction argument shows that G' is totally ordered. For each $n \in \mathbf{Z}$, the interval $[nu', (n+1)u']$ is order-isomorphic to $[0, u']$, whence the order of G' is isomorphic to the order of G. Maltsev's analysis [17, VII, Sect. 1] shows that each quotient of G' has a smallest element >0. Again by [15, 4.6], G' may be identified with the lexicographic direct sum of $\gamma+1$ copies of $\mathbf{Z}$, for some countable ordinal γ. Since the orders of G and G' are the same, it follows that $\gamma = \beta$, and hence we may identify G and G' as totally ordered groups. The strong unit u' of G' has the form $u' = (h_0, h_1, ..., h_\alpha, ..., h_\beta)$, where the h's are integers and $h_\beta \geqslant 1$. Since by hypothesis $L(A') = [0, u']$ is order-isomorphic to $L(A) = [0, u]$, it follows that $h_\beta = 1$. Let now $d = u' - u = (h_0, h_1, ..., h_\alpha, ..., 0)$. Note that d belongs to the maximal convex subgroup K of G. Each element x of $G = G'$ has a unique decomposition $x = y + z$, where $y \in K$ and $z = (0, 0, ..., 0, ..., m)$. The map $y + z \to (y + md) + z$ is an isomorphism of the totally ordered group G onto G' sending u into u'. Thus, $(G, u) \cong (G', u')$ and, by Elliott's classification, $A \cong A'$, a contradiction.

The proof of Theorem 3 is now complete. Q.E.D.

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